Numerical-Analytical Study of Linked Orbits in the Restricted Elliptic Doubly Averaged Three-Body Problem

M. A. Vashkov'yak*

Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, 125047 Russia

*e-mail: vashkov@keldysh.ru

Received October 6, 2020; revised October 25, 2020; accepted November 6, 2020

Abstract—The restricted elliptic doubly averaged three-body problem is considered. The terms up to the second order inclusive with respect to the orbital eccentricity of the perturbing body are retained in the expansion of the perturbing function. The elements of the elliptical orbit for the perturbed body are deemed arbitrary. The special, so-called linked orbits of a negligible-mass body are investigated by numerically integrating the averaged equations in Keplerian elements. For such orbits the points of their intersection with the orbital plane of the perturbing body are on different sides of this orbit. The evolution of hypothetical and some real cometary orbits is described in the simplest Sun-Jupiter-comet model; their differences from the corresponding orbits in the circular problem have been revealed.

Keywords: restricted elliptic three-body problem, double averaging, linked orbits, numerical integration

DOI: 10.1134/S0038094621020064

INTRODUCTION AND FORMULATION OF THE PROBLEM

Studies of the long-term evolution of orbits in the restricted elliptic three-body problem are generally carried out in the doubly averaged formulation. The integrable case of a *circular* orbit of the perturbing body is widely used. The studies of this case initiated by the renowned scientists H. von Zeipel (von Zeipel, 1910) and N.D. Moiseev (Moiseev, 1945) were elaborated significantly by M.L. Lidov (Lidov, 1961, 1962) and I. Kozai (Kozai, 1962). These studies are described in detail in the monographs by Shevchenko (2017) and Ito and Ohtsuka (2019). Note that the extensive paper by H. von Zeipel has become deservedly famous and is reflected in the above scientific-historical study by Ito and Ohtsuka (2019) only due to the reference to it in Baily and Emel'yanenko (1996) in connection with the study of the evolution of one of the types of cometary orbits. The works by Moiseev, Lidov, and Kozai performed later were both the results of a qualitative study of the averaged three-body problem and the necessity of investigating the orbital dynamics of artificial satellites of planets and the dynamics of asteroids.

In his extensive paper Zeipel (1910) identified and qualitatively studied three main cases of the orbit location for the perturbed body in the doubly averaged circular problem: the inner one, the outer one, and the case of intersecting, in particular, the so-called linked orbits. The spatial location of linked orbits is characterized by the fact that one of the points of intersection

of the orbit of a negligible-mass body with the plane of the orbit of the perturbing body is located inside it, while the other one is outside. Such a classification in the restricted circular doubly averaged three-body problem for *uniformly close orbits*, along with an analysis of the conditions for their intersection, was proposed by Lidov and Ziglin (1974). The topology of two linked and unlinked Keplerian orbits of all those types was described in detail by Kholshevnikov and Titov (2007).

Note that the phrase “like the rings of a chain” proposed in the monograph by Ito and Ohtsuka (2019) as the English analog of the French term “comme les anneaux d’une chaîne” used by von Zeipel (1910) is more sensible and geometrically understandable than the term “linked orbits”. The linked orbits in the restricted *elliptic* doubly averaged three-body problem are the subject of our study.

Consider the motion of particle P of negligible mass under the attraction of a central point S of mass m and a perturbing point J of mass $m_1 \ll m$ moving relative to S in an elliptical orbit with a semimajor axis a_1 and eccentricity e_1 . Let us introduce a rectangular coordinate system $Oxyz$ with the origin at point S whose reference plane xOy coincides with the orbital plane of point J . Let the Ox axis be directed to the pericenter of the orbit of point J , the Oy axis be in the direction of its motion from the pericenter in the reference plane, and the Oz axis complements the coordinate system to a right-handed one. The perturbed orbit of point P is characterized by osculating Keple-

rian elements: the semimajor axis a , the eccentricity e , the inclination i , the argument of pericenter ω , and the longitude of the ascending node Ω . In the chosen coordinate system the perturbing point J has coordinates x_1, y_1 , and $z_1 = 0$.

The secular part W of the complete perturbing function is used to investigate the orbital evolution of point P :

$$W(a, e, i, \omega, \Omega, a_1, e_1) = \frac{fm_1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\Delta(\lambda, \lambda_1)} d\lambda_1 d\lambda. \quad (1)$$

Here, $\Delta = |\mathbf{r} - \mathbf{r}_1| = \sqrt{(x - x_1)^2 + (y - y_1)^2 + z^2}$ is the distance between the perturbed and perturbing points, λ and λ_1 are the mean longitudes of these points, and f is the gravitational constant. The absence of low-order commensurabilities between the mean motions of points J and P is assumed. In the function W a_1 and e_1 act as parameters of the evolution problem.

The first integrals of the equations of perturbed motion in elements are

$$a = \text{const}, \quad W(a, e, i, \omega, \Omega, a_1, e_1) = \text{const}, \quad (2)$$

while one more first integral exists in the case of $e_1 = 0$ (Moiseev, 1945):

$$(1 - e^2) \cos^2 i = c_1 = \text{const}. \quad (3)$$

THE AVERAGED PERTURBING FUNCTION AND EVOLUTION EQUATIONS

Another expression, equivalent to (1), for the function W via well-known formulas is also commonly used in analytical studies:

$$W = \frac{1}{2\pi} \int_0^{2\pi} (1 - e \cos E) V(E) dE, \quad (4)$$

$$V(E) = \frac{fm_1}{2\pi} \int_0^{2\pi} \frac{d\lambda_1}{\Delta(\lambda_1, E)},$$

where V is the attractive force function of the elliptical Gaussian ring simulating the averaged influence of the perturbing point and E is the eccentric anomaly of point P .

Since for the orbits under consideration the distance r can be both less than and greater than r_1 as they evolve, the commonly used expansions of the inverse distance $1/\Delta$ into series in Legendre polynomials are inapplicable. Therefore, in this paper we will use the analytical expression for the function V , though with a limited accuracy up to e_1^2 inclusive, from Vashkov'yak (1986) for a nearly coplanar system of N of weakly elliptical Gaussian rings, but valid for any relation between r and r_1 . Taking into account the orientation of the introduced coordinate system and assuming that $N = 1, i_1 = \omega_1 = \Omega_1 = k_1 = u_1 = v_1 = 0$, and $h_1 = e_1$ in Eqs. (6)–(8) of this paper, we will obtain

$$V(E) = \frac{fm_1}{\sqrt{a_1^2 + r^2}} \Phi(x, y, z), \quad (5)$$

and, for coherence, we will permit ourselves to also reproduce the basic simplified computational formulas from this paper by supplementing them with new analytical relations.

The function Φ dependent on the rectangular coordinates x, y, z is expressed via hypergeometric Gaussian functions F :

$$\Phi = (1 - \varepsilon + \mu) F\left(\frac{1}{4}, \frac{3}{4}; 1; \zeta\right) + \left(v - \frac{1}{2}\varepsilon\right) F\left(\frac{5}{4}, \frac{3}{4}; 2; \zeta\right), \quad (6)$$

Functional series of various structures, depending on the numerical value of the argument ζ ($|\zeta| < 1$), are used for their calculation, so that

$$\Phi = \begin{cases} \sum_{n=0}^{\infty} \left[1 + \mu - \frac{3(2n+1)}{2(n+1)}\varepsilon + \frac{4n+1}{n+1}v \right] B_n \zeta^n, & |\zeta| < \zeta^*; \\ \frac{1}{\pi\sqrt{2}} \sum_{n=0}^{\infty} \{ [H_n - \ln(1 - \zeta)] [1 + \mu + 4(4n+1)v - (8n+3)\varepsilon] + 8(\varepsilon - 2v) \} B_n (1 - \zeta)^n, & |\zeta| > \zeta^*. \end{cases} \quad (7)$$

Here,

$$\begin{aligned} \varepsilon &= \frac{a_1 e_1 x}{a_1^2 + r^2}, \quad \mu = \varepsilon^2 \left(2 + \frac{a_1^2 + z^2}{2\rho^2} \right) - \frac{a_1^2 e_1^2 y^2}{(a_1^2 + r^2)\rho^2}, \\ v &= \frac{3}{2}\varepsilon^2 - \frac{a_1^2 e_1^2}{2(a_1^2 + r^2)}, \quad r^2 = \rho^2 + z^2, \\ \rho^2 &= x^2 + y^2, \quad \zeta = \frac{4a_1^2}{(a_1^2 + r^2)^2} \\ &\times [\rho^2 + 2\varepsilon(a_1^2 - \rho^2 + z^2) + \theta], \\ \theta &= e_1^2 \left\{ a_1^2 - x^2 \left[\frac{6a_1^2(a_1^2 - \rho^2 + z^2)}{(a_1^2 + r^2)^2} + \frac{a_1^2 + r^2}{\rho^2} \right] \right. \\ &\left. + y^2 \left(\frac{a_1^2 + z^2}{\rho^2} - \frac{4a_1^2}{a_1^2 + r^2} \right) \right\}. \end{aligned} \quad (8)$$

The constant coefficients B_n and H_n are defined by the recurrence relations

$$\begin{aligned} B_n &= \frac{(4n-3)(4n-1)}{16n^2} B_{n-1}, \quad B_0 = 1, \\ H_n &= H_{n-1} + \frac{2(3-8n)}{n(4n-3)(4n-1)}, \quad H_0 = 6\ln 2, \end{aligned} \quad (9)$$

while the empirical value of ζ^* is taken to be 0.5.

Remark. The function V depends only on the squares of the coordinates y and z , while the coordinate x enters into this function both quadratically and linearly (into the numerator of ε and via it into ζ). The existence of two planar particular solutions, $y = 0$ and $z = 0$, in the *singly* averaged (only over λ_1) evolution problem follows from such a double symmetry of the function V with respect to y and z .

The rectangular coordinates x, y, z are expressed via E by the well-known formulas for unperturbed Keplerian motion

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} \cos E - e \\ (\cos E - e)p_2 + \sqrt{1-e^2}\sin E \\ p_3 \end{pmatrix}, \quad (10)$$

$$\begin{aligned} p_1 &= \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i, \\ p_2 &= \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i, \quad p_3 = \sin\omega\sin i, \\ q_1 &= -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i, \\ q_2 &= -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i, \quad q_3 = \cos\omega\sin i. \end{aligned} \quad (11)$$

Below, it will be convenient to introduce a new independent variable, a “dimensionless time” τ , according to the formula

$$\tau = \frac{m_1 a}{m a_1} n(t - t_0), \quad (12)$$

where $n = \frac{\sqrt{fm}}{a^{3/2}}$ is the mean motion of point P , and a normalized perturbing function

$$w = \frac{a_1}{fm_1} W = \text{const}. \quad (13)$$

To describe the evolution of orbits, we will use the Lagrange equations in elements with the function w that is their first and unique integral:

$$\begin{aligned} \frac{de}{d\tau} &= -\frac{\sqrt{1-e^2}}{e} \frac{\partial w}{\partial \omega}, \quad \frac{di}{d\tau} = \frac{\cot i}{\sqrt{1-e^2}} \frac{\partial w}{\partial \omega} - \frac{\text{cosec} i}{\sqrt{1-e^2}} \frac{\partial w}{\partial \Omega}, \\ \frac{d\omega}{d\tau} &= \frac{\sqrt{1-e^2}}{e} \frac{\partial w}{\partial e} - \frac{\cot i}{\sqrt{1-e^2}} \frac{\partial w}{\partial i}, \\ \frac{d\Omega}{d\tau} &= \frac{\text{cosec} i}{\sqrt{1-e^2}} \frac{\partial w}{\partial i}. \end{aligned} \quad (14)$$

The existence of stationary solutions of these equations is possible if the conditions

$$\frac{de}{d\tau} = \frac{di}{d\tau} = \frac{d\omega}{d\tau} = \frac{d\Omega}{d\tau} = 0 \text{ are fulfilled.}$$

PARTIAL DERIVATIVES OF THE NORMALIZED FUNCTION w WITH RESPECT TO THE ELEMENTS

Generally, for arbitrary orbits of point P a solution of Eqs. (14) can be found apparently only by a numerical method, while the process of calculations can be controlled by the constancy of the function w along this solution. In Vashkov'yak (1986) the partial derivatives of the function w with respect to the elements were calculated by a difference method. Here, we use a combined method, in which the quadratures

$$\begin{aligned} \frac{\partial w}{\partial e} &= \frac{1}{2\pi} \int_0^{2\pi} \left[(1 - e \cos E) \frac{\partial \tilde{V}}{\partial e} - \tilde{V} \cos E \right] dE, \\ \left\| \frac{\partial w}{\partial i} \right\| &= \frac{1}{2\pi} \int_0^{2\pi} (1 - e \cos E) \left\| \frac{\partial \tilde{V}}{\partial i} \right\| dE \\ \left\| \frac{\partial w}{\partial \omega} \right\| &= \frac{1}{2\pi} \int_0^{2\pi} (1 - e \cos E) \left\| \frac{\partial \tilde{V}}{\partial \omega} \right\| dE \\ \left\| \frac{\partial w}{\partial \Omega} \right\| &= \frac{1}{2\pi} \int_0^{2\pi} (1 - e \cos E) \left\| \frac{\partial \tilde{V}}{\partial \Omega} \right\| dE \end{aligned} \quad (15)$$

are found numerically by the Gaussian method, while the derivatives of the normalized function

$$\tilde{V} = \frac{a_1}{fm_1} V = \frac{a_1}{\sqrt{a_1^2 + r^2}} \Phi(x, y, z) \quad (16)$$

are found analytically. For the completeness of the set of formulas, we give the necessary expressions to calculate the derivatives of the function $\tilde{V}$ with respect to the orbital elements:

$$\begin{pmatrix} \frac{\partial \tilde{V}}{\partial e} \\ \frac{\partial \tilde{V}}{\partial i} \\ \frac{\partial \tilde{V}}{\partial \omega} \\ \frac{\partial \tilde{V}}{\partial \Omega} \end{pmatrix} = \frac{\partial \tilde{V}}{\partial x} \begin{pmatrix} \frac{\partial x}{\partial e} \\ \frac{\partial x}{\partial i} \\ \frac{\partial x}{\partial \omega} \\ \frac{\partial x}{\partial \Omega} \end{pmatrix} + \frac{\partial \tilde{V}}{\partial y} \begin{pmatrix} \frac{\partial y}{\partial e} \\ \frac{\partial y}{\partial i} \\ \frac{\partial y}{\partial \omega} \\ \frac{\partial y}{\partial \Omega} \end{pmatrix} + \frac{\partial \tilde{V}}{\partial z} \begin{pmatrix} \frac{\partial z}{\partial e} \\ \frac{\partial z}{\partial i} \\ \frac{\partial z}{\partial \omega} \\ \frac{\partial z}{\partial \Omega} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial \tilde{V}}{\partial x} \\ \frac{\partial \tilde{V}}{\partial y} \\ \frac{\partial \tilde{V}}{\partial z} \end{pmatrix} = \frac{a_1}{\sqrt{a_1^2 + r^2}} \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{pmatrix} - \frac{\Phi}{a_1^2 + r^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (17)$$

$$\begin{pmatrix} \frac{\partial x}{\partial e} \\ \frac{\partial x}{\partial y} \\ \frac{\partial x}{\partial z} \\ \frac{\partial x}{\partial \Omega} \end{pmatrix} = -a \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \frac{e \sin E}{\sqrt{1 - e^2}} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial x}{\partial i} \\ \frac{\partial x}{\partial y} \\ \frac{\partial x}{\partial z} \\ \frac{\partial x}{\partial \Omega} \end{pmatrix} = a \begin{pmatrix} (\cos E - e) \sin \omega + \sqrt{1 - e^2} \cos \omega \sin E \\ (\cos E - e) \cos \omega + \sqrt{1 - e^2} \sin \omega \sin E \\ -\sin \omega \\ \cos \omega \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{\partial x}{\partial \omega} \\ \frac{\partial x}{\partial y} \\ \frac{\partial x}{\partial z} \\ \frac{\partial x}{\partial \Omega} \end{pmatrix} = a \begin{pmatrix} (\cos E - e) \frac{q_1}{q_2} - \sqrt{1 - e^2} \sin E \frac{p_1}{p_2} \\ (\cos E - e) \frac{q_1}{q_3} - \sqrt{1 - e^2} \sin E \frac{p_1}{p_3} \\ (\cos E - e) \frac{q_1}{0} - \sqrt{1 - e^2} \sin E \frac{p_1}{0} \\ (\cos E - e) \frac{q_1}{0} - \sqrt{1 - e^2} \sin E \frac{p_1}{0} \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial x}{\partial \Omega} \\ \frac{\partial x}{\partial y} \\ \frac{\partial x}{\partial z} \\ \frac{\partial x}{\partial \Omega} \end{pmatrix} = a \begin{pmatrix} (\cos E - e) \frac{-p_2}{p_1} + \sqrt{1 - e^2} \sin E \frac{-q_2}{q_1} \\ (\cos E - e) \frac{-p_2}{p_1} + \sqrt{1 - e^2} \sin E \frac{-q_2}{q_1} \\ (\cos E - e) \frac{-p_2}{p_1} + \sqrt{1 - e^2} \sin E \frac{-q_2}{q_1} \\ (\cos E - e) \frac{-p_2}{p_1} + \sqrt{1 - e^2} \sin E \frac{-q_2}{q_1} \end{pmatrix}, \quad (18)$$

$$r_1 = \sin i \sin \Omega, \quad r_2 = -\sin i \cos \Omega, \quad r_3 = \cos i.$$

$$\begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{pmatrix} = \frac{\partial \Phi}{\partial \varepsilon} \begin{pmatrix} \frac{\partial \varepsilon}{\partial x} \\ \frac{\partial \varepsilon}{\partial y} \\ \frac{\partial \varepsilon}{\partial z} \end{pmatrix} + \frac{\partial \Phi}{\partial \zeta} \begin{pmatrix} \frac{\partial \zeta}{\partial x} \\ \frac{\partial \zeta}{\partial y} \\ \frac{\partial \zeta}{\partial z} \end{pmatrix} + \frac{\partial \Phi}{\partial \mu} \begin{pmatrix} \frac{\partial \mu}{\partial x} \\ \frac{\partial \mu}{\partial y} \\ \frac{\partial \mu}{\partial z} \end{pmatrix} + \frac{\partial \Phi}{\partial \nu} \begin{pmatrix} \frac{\partial \nu}{\partial x} \\ \frac{\partial \nu}{\partial y} \\ \frac{\partial \nu}{\partial z} \end{pmatrix}, \quad (19)$$

$$\frac{\partial \varepsilon}{\partial x} = \frac{a_1 e_1}{(a_1^2 + r^2)^2} (a_1^2 - x^2 + y^2 + z^2), \quad \frac{\partial \varepsilon}{\partial y} = -\frac{2a_1 e_1 xy}{(a_1^2 + r^2)^2}, \quad \frac{\partial \varepsilon}{\partial z} = \frac{2a_1 e_1 xz}{(a_1^2 + r^2)^2},$$

$$\frac{\partial \mu}{\partial x} = \varepsilon \frac{\partial \varepsilon}{\partial x} \left(4 + \frac{a_1^2 + z^2}{\rho^2} \right) + \frac{x}{\rho^4} \times \left[\frac{2a_1^2 e_1^2 y^2}{(a_1^2 + r^2)^2} (a_1^2 + 2x^2 + 2y^2 + z^2) - \varepsilon^2 (a_1^2 + z^2) \right],$$

$$\frac{\partial \mu}{\partial y} = \varepsilon \frac{\partial \varepsilon}{\partial y} \left(4 + \frac{a_1^2 + z^2}{\rho^2} \right) + \frac{y}{\rho^4} \times \left[\frac{2a_1^2 e_1^2 y^2}{(a_1^2 + r^2)^2} (a_1^2 + 2x^2 + 2y^2 + z^2) - \varepsilon^2 (a_1^2 + z^2) \right] \times \frac{2a_1^2 e_1^2 y}{(a_1^2 + r^2) \rho^2}, \quad (20)$$

$$\frac{\partial \mu}{\partial z} = \varepsilon \frac{\partial \varepsilon}{\partial z} \left(4 + \frac{a_1^2 + z^2}{\rho^2} \right) + \frac{z}{\rho^2} \left[\varepsilon^2 + \frac{2a_1^2 e_1^2 y^2}{(a_1^2 + r^2)^2} \right],$$

$$\begin{pmatrix} \frac{\partial \nu}{\partial x} \\ \frac{\partial \nu}{\partial y} \\ \frac{\partial \nu}{\partial z} \end{pmatrix} = 3\varepsilon \begin{pmatrix} \frac{\partial \varepsilon}{\partial x} \\ \frac{\partial \varepsilon}{\partial y} \\ \frac{\partial \varepsilon}{\partial z} \end{pmatrix} + \frac{a_1^2 e_1^2}{(a_1^2 + r^2)^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$\begin{aligned}
 \frac{\partial \zeta}{\partial x} &= \frac{4a_1^2}{(a_1^2 + r^2)^2} \left[2(a_1^2 - \rho^2 + z^2) \left(\frac{\partial \varepsilon}{\partial x} - \frac{4\varepsilon x}{a_1^2 + r^2} \right) + 2x(1 - 2\varepsilon) - \frac{4x}{a_1^2 + r^2} (\rho^2 + \theta) + \frac{\partial \theta}{\partial x} \right], \\
 \frac{\partial \zeta}{\partial y} &= \frac{4a_1^2}{(a_1^2 + r^2)^2} \left[2(a_1^2 - \rho^2 + z^2) \left(\frac{\partial \varepsilon}{\partial y} - \frac{4\varepsilon y}{a_1^2 + r^2} \right) + 2y(1 - 2\varepsilon) - \frac{4y}{a_1^2 + r^2} (\rho^2 + \theta) + \frac{\partial \theta}{\partial y} \right], \\
 \frac{\partial \zeta}{\partial z} &= \frac{4a_1^2}{(a_1^2 + r^2)^2} \left[2(a_1^2 - \rho^2 + z^2) \left(\frac{\partial \varepsilon}{\partial z} - \frac{4\varepsilon z}{a_1^2 + r^2} \right) + 4\varepsilon z - \frac{4z}{a_1^2 + r^2} (\rho^2 + \theta) + \frac{\partial \theta}{\partial z} \right], \\
 \frac{\partial \theta}{\partial x} &= 2e_1^2 x \left[-\frac{6a_1^2}{(a_1^2 + r^2)^2} (a_1^2 - \rho^2 + z^2) \left(1 - \frac{2x^2}{a_1^2 + r^2} \right) + \frac{2a_1^2}{(a_1^2 + r^2)^2} (3x^2 + 2y^2) \right. \\
 &\quad \left. - \frac{y^2}{\rho^4} (2a_1^2 - \rho^2 + 2z^2) - \frac{x^2}{\rho^2} \right], \\
 \frac{\partial \theta}{\partial y} &= 4e_1^2 y \left\{ \frac{x^2}{\rho^4} (a_1^2 + z^2) + \frac{a_1^2}{a_1^2 + r^2} \left[\frac{6x_1^2}{(a_1^2 + r^2)^2} (a_1^2 - \rho^2 + z^2) + \frac{3x^2 + 2y^2}{a_1^2 + r^2} - 2 \right] \right\}, \\
 \frac{\partial \theta}{\partial z} &= 2e_1^2 z \left\{ \frac{y^2 - x^2}{\rho^2} + \frac{2a_1^2}{(a_1^2 + r^2)^2} \left[2y^2 - 3x^2 + \frac{6x_1^2}{a_1^2 + r^2} (a_1^2 - \rho^2 + z^2) \right] \right\}.
 \end{aligned} \tag{21}$$

$$\begin{aligned}
 \frac{\partial \Phi}{\partial \varepsilon} &= \begin{cases} -\frac{3}{2} \sum_{n=0}^{\infty} \frac{2n+1}{n+1} B_n \zeta^n, & |\zeta| < \zeta^*, \\ \frac{1}{\pi\sqrt{2}} \sum_{n=0}^{\infty} \{8 - (8n+3)[H_n - \ln(1-\zeta)]\} B_n (1-\zeta)^n, & |\zeta| > \zeta^*. \end{cases} \\
 \frac{\partial \Phi}{\partial \zeta} &= \begin{cases} \sum_{n=1}^{\infty} \left[1 + \mu - \frac{3(2n+1)}{2(n+1)} \varepsilon + \frac{4n+1}{n+1} \nu \right] n B_n \zeta^{n-1}, & |\zeta| < \zeta^*, \\ \frac{1}{\pi\sqrt{2}} \sum_{n=0}^{\infty} \{[1 - (8n+3)\varepsilon + \mu + 4(4n+1)\nu][1 - n(H_n - \ln(1-\zeta)) - 8n(\varepsilon - 2\nu)]\} B_n (1-\zeta)^{n-1}, & |\zeta| > \zeta^* \end{cases} \\
 \frac{\partial \Phi}{\partial \mu} &= \begin{cases} \sum_{n=0}^{\infty} B_n \zeta^n, & |\zeta| < \zeta^*, \\ \frac{1}{\pi\sqrt{2}} \sum_{n=0}^{\infty} [H_n - \ln(1-\zeta)] B_n (1-\zeta)^n, & |\zeta| > \zeta^*, \end{cases} \\
 \frac{\partial \Phi}{\partial \nu} &= \begin{cases} \sum_{n=0}^{\infty} \frac{4n+1}{n+1} B_n \zeta^n, & |\zeta| < \zeta^*, \\ \frac{4}{\pi\sqrt{2}} \sum_{n=0}^{\infty} \{(4n+1)[H_n - \ln(1-\zeta)] - 4\} B_n (1-\zeta)^n, & |\zeta| > \zeta^*. \end{cases}
 \end{aligned} \tag{22}$$

Relations (7)–(11) and (15)–(22) represent a complete set of formulas to calculate the right-hand sides of the evolution equations (14).

In view of the properties of the function V (see the remark in the previous section), system (14) has two particular solutions or integrable cases.

Case 1. If $\sin i = 0$, then the orbital plane of point P coincides with the orbital plane of the perturbing point J . By symmetry, it turns out that $di/d\tau = 0$. However, in this planar solution at any a , a_1 , and e_1 , in addition to

regular orbits, there exist irregular ones intersecting (but not linked) with the orbit of point J (Vashkov'yak, 1982).

Case 2. If $\cos i = 0$ and $\sin \Omega = 0$, then in the elliptic problem under consideration the orbital plane of point P is orthogonal to the orbital plane of the perturbing point J and passes through its line of apsides. By symmetry, it turns out that $di/d\tau = 0$ and $d\Omega/d\tau = 0$. The above formulas of the quadratic approximation in e_1

allow this to be also verified directly. Indeed, for $\cos i = 0$ and $\sin \Omega = 0$ we have

$$\begin{aligned}\delta &= \text{sign}(\cos \Omega) = \pm 1, \\ p_1 &= \delta \cos \omega, \quad p_2 = 0, \quad p_3 = \sin \omega, \\ q_1 &= -\delta \sin \omega, \quad q_2 = 0, \quad q_3 = \cos \omega, \\ r_1 &= 0, \quad r_2 = -\delta, \quad r_3 = 0, \\ x &= a\delta \left[(\cos E - e) \cos \omega - \sin \omega \sqrt{1 - e^2} \sin E \right], \\ y &= 0, \\ z &= a \left[(\cos E - e) \sin \omega + \cos \omega \sqrt{1 - e^2} \sin E \right], \\ \left\| \frac{\partial w}{\partial i} \right\| &= \frac{1}{2\pi} \int_0^{2\pi} (1 - e \cos E) \left\| \frac{\partial \tilde{V}}{\partial \Omega} \right\| dE.\end{aligned}\quad (23)$$

In this case,

$$\begin{aligned}\left\| \frac{\partial \tilde{V}}{\partial i} \right\| &= \frac{\partial \tilde{V}}{\partial y} \left\| \frac{\partial y}{\partial i} \right\| = \frac{\partial \tilde{V}}{\partial y} \left\| -\delta z \right\| = \frac{a_1}{\sqrt{a_1^2 + r^2}} \frac{\partial \Phi}{\partial y} \left\| -\delta z \right\| \\ &= \frac{a_1}{\sqrt{a_1^2 + r^2}} \left(\frac{\partial \Phi}{\partial e} \frac{\partial e}{\partial y} + \frac{\partial \Phi}{\partial \zeta} \frac{\partial \zeta}{\partial y} + \frac{\partial \Phi}{\partial \mu} \frac{\partial \mu}{\partial y} + \frac{\partial \Phi}{\partial v} \frac{\partial v}{\partial y} \right) \left\| -\delta z \right\|.\end{aligned}\quad (24)$$

It is easy to verify that at $y = 0$ the derivatives of the functions e , ζ , μ , and v with respect to y become zero, so that $\frac{\partial w}{\partial i} = \frac{\partial w}{\partial \Omega} = 0$ and $\frac{di}{d\tau} = \frac{d\Omega}{d\tau} = 0$.

Equations (14) simplified for this case of orthogonal-apsidal orbits take the form

$$\frac{de}{d\tau} = -\frac{\sqrt{1-e^2}}{e} \frac{\partial w}{\partial \omega}, \quad \frac{d\omega}{d\tau} = \frac{\sqrt{1-e^2}}{e} \frac{\partial w}{\partial e}.\quad (25)$$

A general qualitative study of this case by taking into account the possible intersections of the orbits of points P and J was carried out by a numerical-analytical method in Vashkov'yak (1984) for arbitrary a , a_1 , and e_1 . In this paper greater attention is given to linked orbits and, in particular, to the stationary solutions of Eqs. (25) existing at $\omega_0 = 0, \pi$. It can be shown that, in this case, $\frac{\partial w}{\partial \omega} = 0$ and the stationary eccentricities themselves are determined as the roots of the transcendental equation

$$\frac{\partial w(a, a_1, e_1, e_0, \omega_0 = 0, \pi; \Omega_0 = 0, \pi)}{\partial e} = 0.\quad (26)$$

ON THE EVOLUTION OF SOME HYPOTHETICAL AND REAL COMETARY ORBITS IN THE SOLAR SYSTEM MODEL

First we will turn to the linked orbits of point P highly inclined to the reference plane. In the integra-

ble case of orthogonal-apsidal orbits, a numerical solution of Eq. (26) makes it possible to find the stationary values of the eccentricity e_0 at given $\omega_0 = 0^\circ, 180^\circ, \Omega_0 = 180^\circ$ and fixed parameters a, a_1, e_1 . In addition, it can be shown that the values of e_0 depend on ω_0 and Ω_0 only through the combination $\delta_1 = \text{sign}(\cos \omega_0 \cos \Omega_0) = \pm 1$. However, the difference in e_0 at different signs of δ_1 is fairly small and $\sim e_1^2$.

At an inclination and longitude of the ascending node different from those adopted in case 2, all of the orbital elements will change with time. Our numerical integration of the evolution system (14) by the Runge–Kutta method at $a_1 = 5.2$ AU, $e_1 = 0.048$, and the mass ratio m/m_1 in the Sun–Jupiter system makes it possible to estimate such changes for fictitious (or hypothetical) cometary orbits. Tables 1 and 2 give such an estimate in the time interval $t^* = 500000$ years for orbits with a semimajor axis $a = 10a_1 = 52$ AU, $e_0(\delta_1 = 1) = 0.9890$, $e_0(\delta_1 = -1) = 0.9905$. These tables present

$$\begin{aligned}\Delta_1 &= \max |e(t^*) - e_0|, \quad \Delta_2 = \max |i(t^*) - i_0|, \\ \Delta_3 &= \max |\omega(t^*) - \omega_0|, \quad \Delta_4 = \max |\Omega(t^*) - \Omega_0|.\end{aligned}$$

Table 1 was compiled for three values of $i_0 = 90^\circ, 85^\circ, 95^\circ$. In the exact solution of Eqs. (14) $i_0 = 90^\circ, \Omega_0 = 0^\circ, 180^\circ$ the deviations are zero and, therefore, the corresponding rows are omitted. At $\Omega_0 = 0$ and 180° the results for $i_0 = 95^\circ$ identical to the corresponding data for $i_0 = 85^\circ$ are also omitted.

Table 1. Maximum changes of the elements e, i, ω, Ω in a time interval of 500000 years for $i_0 = 90^\circ, 90^\circ \pm 5^\circ$

i_0 , deg	Ω_0 , deg	Δ_1	Δ_2 , deg	Δ_3 , deg	Δ_4 , deg
90	60	0.001	24.3	1.0	21.9
90	120	0.002	20.8	1.1	20.3
85	0	0.0003	2.2	0.3	10.3
85	60	0.0004	24.0	0.8	12.3
85	120	0.0017	22.5	1.2	12.4
85	180	0.0001	2.0	0.2	7.5
95	60	0.0022	23.8	0.6	31.6
95	120	0.0028	18.7	1.4	28.2

Table 2. The same as Table 1, but for $i_0 = 90^\circ \pm 30^\circ$

i_0 , deg	Ω_0 , deg	Δ_1	Δ_2 , deg	Δ_3 , deg	Δ_4 , deg
60	0	0.010	1.4	9.8	60.0
60	60	0.003	19.8	7.3	35.1
60	120	0.004	29.0	8.3	26.5
60	180	0.005	15.1	7.9	42.6
120	60	0.013	9.1	9.6	74.3
120	120	0.010	2.9	10.2	63.0

Fig. 1. Polar diagram or projection of the phase trajectory onto the $(e \cos \omega, e \sin \omega)$ plane for case I ($e_0 = 0.55$, $i_0 = 40^\circ.78$).

Fig. 2. Same as Fig. 1, but for case II ($e_0 = 0.4$, $i_0 = 32^\circ.31$).

The initial deviations in i_0 and Ω_0 from their equilibrium values at $t = t^*$ lead to insignificant changes in the shape of the orbit (Δ_1, Δ_3), but to a noticeable change in its orientation (Δ_2 reaches 24° , Δ_4 is about 32°).

Table 2 was compiled for more significant initial deviations of the orbit from the orthogonal one, $i_0 = 90^\circ \pm 30^\circ$. At $\Omega_0 = 0^\circ$ and 180° it also contains the results for $i_0 = 120^\circ$ identical to the corresponding data for $i_0 = 60^\circ$.

In this case, the inclination also remains close to the initial one, but the deviations of the remaining elements are tens of degrees, reaching approximately 75° .

Below, we will consider the orbits of point P linked with the orbit of point J , but with an arbitrary spatial orientation. The evolution of such orbits is usually studied using numerical methods even in the integrable doubly averaged *circular* problem ($e_1 = 0$) due to the absence of a rigorous analytical expression for the averaged perturbing function. The families of phase trajectories in the $(e \cos \omega, e \sin \omega)$ plane constructed as equipotential contours of the function W are presented in Ito and Ohtsuka (2019, Section 5.8, Fig. 24). These families correspond to the hypothetical linked orbits of point P for three pairs of values of the ratio a/a_1 and the constant of the integral c_1 . In all three cases considered

$$\text{I}(a/a_1 = 0.9, c_1 = 0.4); \quad \text{II}(a_1/a = 0.7, c_1 = 0.6); \\ \text{III}(a_1/a = 0.8, c_1 = 0.313)$$

there exist stationary center-type singular points and closed periodic trajectories enclosing them in the phase plane.

The ellipticity of the orbit of the perturbing point J , naturally, leads to qualitative changes of the families of trajectories in the circular problem. Since the equa-

tions for e and ω are not decoupled from the remaining ones, as is the case at $e_1 = 0$, due to the absence of the integral c_1 in the elliptic problem, the evolution of these elements can be traced only in projection of the phase trajectory onto the $(e \cos \omega, e \sin \omega)$ or (ω, e) plane. In this paper we numerically integrated system (14) for $a_1 = 5.2$ AU, $e_1 = 0.048$, and the mass ratio m/m_1 in the Sun-Jupiter system. The difference of this ratio from the one adopted in the calculations by Ito and Ohtsuka (2019) for the Sun-(Earth + Moon) system does not affect the structure of the phase trajectories, but leads only to a change in the time scale.

For a comparison with the results of the circular problem in cases I, II, and III, out of all families of integral curves in the circular problem we chose the trajectories with $\omega_0 = 180^\circ$ and initial $e_0 = 0.55, 0.4$, and 0.65 , respectively. The initial inclinations were calculated from e_0 and c_1 as $i_0 = \arccos \sqrt{\frac{c_1}{1-e_0^2}}$, while the initial longitude of the ascending node Ω_0 was taken to be zero.

Figures 1–3 show the trajectories for cases I, II, and III, respectively, and fragments of the polar diagrams with plotted numerical values of the angles ω and radii e . The circles and triangles mark the initial and final points, respectively. The time intervals are 100000 years for cases I, II and 500000 years for case III. The dashed lines are the so-called “separatrices”, which are not integral curves and correspond to the intersections of the orbits of points P and J . They are defined by the equations

$$\frac{a(1-e^2)}{a_1(1 \pm e_1)} \pm e \cos \omega - 1 = 0.$$

Fig. 3. Same as Fig. 1, but for case III ($e_0 = 0.65$, $i_0 = 42^\circ.59$).

In all three cases, the closed periodic trajectories of the circular problem are modified and become non-periodic, nevertheless, retaining an oscillatory pattern and remaining in the regions of linked orbits. The amplitude of these oscillations can both decrease (Figs. 1, 3) and increase (Fig. 2) with time.

However, the eccentricity of Jupiter's orbit can also lead to qualitative changes in the behavior of the trajectory. Figure 4 shows a chaotic trajectory that begins in the ω libration region at $\omega_0 = 180^\circ$, then passes into the circulation region, and returns to the libration region, but relative to $\omega = 0$. The initially linked orbit of point P intersects the orbit of the perturbing point J as it evolves, exiting from one linking region, then traverses the region of unlinked orbits and enters another linking region.

Interestingly, real cometary orbits almost orthogonal to the ecliptic, with the above types of evolution, are also found within the model under consideration (Sun–Jupiter–comet). If we make a selection by perihelion distance $q < 2$ AU and inclination $85^\circ < i < 95^\circ$ on the set of cometary orbits presented in the JPL database (https://ssd.jpl.nasa.gov/sbdb_query.cgi#x), then only ten orbits remain in this sample. The evolution of seven of them is reduced to successive intersections of Jupiter's orbit with passages of the linking regions. Figures 5–7 give an idea of the evolution of the remaining three orbits. In contrast to the previous figures, fragments of the (ω, e) plane are shown here not in the polar coordinate system, but in the rectangular one, which is more convenient for highly elliptical orbits. The up-to-date (initial) values of the elements for our numerical integration were taken from the above JPL database. The initial and final values in the $(\omega - e)$ plane are marked by the circles and triangles, respectively.

Fig. 4. Example of a chaotic trajectory with a change of the regimes of variations in the argument of perihelion and with intersections of the orbit of the perturbing point in a time interval of 55000 years for case I, but at $e_0 = 0.475$ and $i_0 = 44^\circ.052$.

Figure 5 presents the variations in orbital elements for comet C/1955 L1 (Mrkos) in a time interval of 1 Myr. The motion of the phase point begins in the region of orbital linking and ω libration, while the corresponding stationary value of the eccentricity calculated by solving Eq. (26) is $e_0 = 0.9818$. The phase point has no time to make one revolution relative to the libration center, while the trajectory passes through the “separatrix” corresponding to the intersection of the cometary orbit with Jupiter's orbit. After the relatively short time interval of ~ 200000 years, there occur the second intersection of these orbits and the entry into another linking region with a ω libration center offset by 180° from the initial one. In this case, the initially prograde motion of the comet becomes retrograde already in 100000 years, while the inclination, increasing monotonically, reaches about 115° .

Figure 6 presents the variations in orbital elements for comet C/1861 J1 (Great comet) in a time interval of 3 Myr. The motion of the phase point begins and remains in the region of orbital linking and ω libration, while the corresponding stationary value of the eccentricity is $e_0 = 0.9820$. In this case, the initially prograde motion of the comet becomes retrograde in 2 Myr, while the inclination reaches about 132° at a minimum of 31° .

Figure 7 presents the variations in orbital elements for comet 122P/de Vico in a time interval of 3 Myr. The motion of the phase point also begins and remains in the region of orbital linking and ω libration, while the corresponding stationary value of the eccentricity

Fig. 5. Variations in the argument of perihelion and orbital eccentricity for comet C/1955 L1 (Mrkos) in a time interval of 1 Myr in a simplified model (Sun–Jupiter–comet).

Fig. 6. Same as in Fig. 5, but for comet C/1861 J1 (Great comet) and a time interval of 3 Myr.

is $e_0 = 0.9630$. The motion changes with time from prograde to retrograde and vice versa. The extreme inclinations are 43° and 138° .

In conclusion, it should be recalled that all the above examples of orbital evolution and, in particular,

the examples of libration of the argument of perihelion were constructed within the adopted model (Sun–Jupiter–comet). However, the real motions of comets in the Solar System can differ from their model motions even qualitatively. These differences are due to the neglect of the influence of the remaining plan-

Fig. 7. Same as Fig. 6, but for comet 122P/de Vico.

ets, except for Jupiter, and the averaged model of the problem. For example, in the rigorous solution of the complete system of non-averaged differential equations the libration of the argument of perihelion can change to its circulation. The orbit of comet 122P/de Vico is an example of chaotic orbital evolution in a real cometary medium, but this property is found only in the non-averaged Solar System model including both several perturbing bodies (Baily et al. 1992) and one body (Ito and Ohtsuka, 2019; Section 5.8, Fig. 25).

Regarding the methodology of the work described in this paper, note that owing to the studies performed relatively recently and reflected in the monographs and the paper by Kondrat'ev (2007, 2012) and Antonov et al. (2008), the representation of the force function for an elliptical Gaussian ring was obtained in a closed form without expansions in terms of any parameters. However, its practical use involves well-known difficulties and may in future be the subject of a special study.

OPEN ACCESS

This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this license, visit <http://creativecommons.org/licenses/by/4.0/>.

REFERENCES

- Antonov, V.A., Nikiforov, I.I., and Kholshevnikov, K.V., *Elementy teorii gravitatsionnogo potentsiala i nekotorye sluchai ego yavnogo vyrazheniya* (Elements of the Theory of Gravitational Potential and Some Cases of Its Explicit Expression), St. Petersburg: Izd. S.-Peterb. Univ., 2008.
- Bailey, M.E. and Emel'yanenko, V.V., Dynamical evolution of Halley-type comets, *Mon. Not. R. Astron. Soc.*, 1996, vol. 278, pp. 1087–1110.
- Bailey, M.E., Chambers, J.E., and Hahn, G., Origin of sungrazers: a frequent cometary end-state, *Astron. Astrophys.*, 1992, vol. 257, pp. 315–322.
- Ito, T. and Ohtsuka, K., The Lidov-Kozai oscillation and Hugo von Zeipel, *Environ. Earth Planets*, 2019, vol. 7, no. 1, pp. 1–113.
- Kholshevnikov, K.V. and Titov, V.B., *Zadacha dvukh tel* (Two-Body Problem), St. Petersburg: S.-Peterb. Gos. Univ., 2007.
- Kondrat'ev, B.P., *Teoriya potentsiala. Nove metody i zadachi s resheniyami* (Potential Theory. New Methods and Problems with Solutions), Moscow: Mir, 2007.
- Kondratyev, B.P., Potential of a Gaussian ring. A new approach, *Sol. Syst. Res.*, 2012, vol. 46, no. 5, pp. 352–362.
- Kozai, Y., Secular perturbations of asteroids with high inclination and eccentricity, *Astron. J.*, 1962, vol. 67, pp. 591–598.
- Lidov, M.L., Evolution of the orbits of artificial satellites of planets under the influence of gravitational disturbances of external bodies, *Iskustv. Sputniki Zemli*, 1961, no. 8, pp. 5–45.
- Lidov, M.L., The evolution of orbits of artificial satellites of planets under the action of gravitational perturbations of external bodies, *Planet. Space Sci.*, 1962, no. 9, pp. 719–759.
- Lidov, M.L. and Ziglin, S.L., The analysis of restricted circular twice-averaged three body problem in the case of close orbits, *Celest. Mech.*, 1974, vol. 10, no. 2, p. 151.
- Moiseev, N.D., Some basic simplified schemes of celestial mechanics obtained by averaging the restricted circular three-point problem. 2. On the averaged versions of the spatial restricted circular three-point problem, *Tr. Gos. Astron. Inst. im. Shternberga*, 1945, vol. 15, no. 1, pp. 100–117.
- Shevchenko, I., *The Lidov-Kozai Effect – Applications in Exoplanet Research and Dynamical Astronomy*, Astrophysics and Space Science Library, Dordrecht: Springer, 2017, vol. 441.
- Vashkov'yak, M.A., Evolution of orbits in the two-dimensional restricted elliptical twice-averaged three-body problem, *Cosmic Res.*, 1982, vol. 20, no. 3, pp. 236–244.
- Vashkov'yak, M.A., Integrable cases of the restricted twice-averaged three-body problem, *Cosmic Res.*, 1984, vol. 22, no. 3, pp. 260–267.
- Vashkov'yak, M.A., An investigation of the evolution of some asteroid orbits, *Cosmic Res.*, 1986, vol. 24, no. 3, pp. 255–266.
- von Zeipel, H., Sur l'application des séries de M. Lindstedt à l'étude du mouvement des comètes périodiques, *Astron. Nachrichten*, 1910, vol. 183, pp. 345–418. <https://doi.org/10.1002/asna.19091832202>

Translated by V. Astakhov