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TEICHMÜLLER'S MODULSATZ AND THE VARIATION OF THE DIRICHLET INTEGRAL

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Abstract—We show that changing the level curve of a harmonic function with the classical Hadamard variation with a small parameter entails a change in the Dirichlet integral of the function which is quadratic in the parameter. As a corollary, we supplement the well-known theorem of Teichmüller about the sum of moduli of doubly connected domains into which an annulus is subdivided by a continuum that differs little from a concentric circle.

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1. Introduction

Consider the annulus $B = \{z : s < |z| < t\}$ with $0 < s < t < \infty$ and denote by $\operatorname{mod} D$ the modulus of a doubly connected domain $D \subset \mathbb{C}$; in particular,

$$\operatorname{mod} B = \frac{1}{2\pi} \log \frac{t}{s}.$$

Take some continuum γ that separates B into disjoint doubly connected domains B_1 and B_2. Grötzsch's Lemma shows that

$$\Delta(B, \gamma) := \operatorname{mod} B - \operatorname{mod} B_1 - \operatorname{mod} B_2 \tag{1}$$

is nonnegative and vanishes only in the case that $\gamma = \{z : |z| = r\}$ for an arbitrary r with $s < r < t$. In 1938 Teichmüller established [1] the following: If

$$\Delta(B, \gamma) \le \delta$$

for $\delta > 0$ sufficiently small then there is $C < \infty$, independent of B and δ, such that

$$\frac{\sup\{|z| : z \in \gamma\}}{\inf\{|z| : z \in \gamma\}} \le 1 + C\sqrt{\delta \log \frac{1}{\delta}}.$$

This proposition is known in the literature as Teichmüller's Modulsatz; see [2, Proposition 9.5; 3, Corollary 2.34; 4, Theorem 4.1], as well as [5, Chapter VI, Section 6, "narrow Modulsatz"]). Teichmüller pointed out [1] the accuracy of his estimate understood in the sense that for every $\varepsilon > 0$ there exists a continuum γ_ε avoiding some concentric circle in B by ε, while

$$\Delta(B, \gamma_\varepsilon) \le \delta(\varepsilon), \quad \delta(\varepsilon) \asymp \frac{\varepsilon^2}{\log \frac{1}{\varepsilon}} \quad \text{as } \varepsilon \to 0.$$

Bertilsson gave [3, Example 2.26] an explicit form of such continuum in the "dual problem"; see also [4, Chapter V, Exercise 10]. The factor $1/\bigl(\log \frac{1}{\varepsilon}\bigr)$ appears because one point of the continuum $\gamma(\varepsilon)$ approaches the circle as $\varepsilon \to 0$ one order in ε slower than the others. It is natural to suppose that a more uniform convergence of γ_ε to the circle would ensure that

$$\Delta(B, \gamma_\varepsilon) = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0. \tag{2}$$

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Indeed, [6] observes that (2) holds whenever we obtain γ_ε from a concentric circle via some Hadamard deformation, defined in general as follows. Take a smooth curve γ in $\mathbb{C}$ and a real twice continuously differentiable function φ on γ. Given a sufficiently small $\varepsilon > 0$, define the "deformation" of γ as

$$\delta n(z) := \varepsilon\varphi(z) + O(\varepsilon^2) \tag{3}$$

such that γ goes into the curve $\gamma_\varepsilon = \{z_\varepsilon = z + \delta n(z) i\, dz/|dz| : z \in \gamma\}$. Here $\delta n(z)$ is a twice continuously differentiable function on γ and $O(\varepsilon^2)$ admits on γ a uniform estimate.[1)] In the case (3) we can prove (2) using Hadamard's variational formula for the Dirichlet integral [7, (A3.11); 6, (2.2)]. In [6] we used (2) substantially to obtain a fine property of the Green's energy of a discrete charge.

In this note we give a direct proof of a more general result than (2). Moreover, we pass from doubly connected domains to arbitrary ones. In this regard, instead of comparing the moduli of annular domains, we study the behavior of the Dirichlet integral of a harmonic of function when its level curves changes via the deformation in (3). Now we proceed to precise statements.

Given a finite domain B in the plane $\mathbb{C}$ whose boundary consists of analytic arcs and closed analytic Jordan curves, consider a nonconstant function u continuous on $\overline{B}$, harmonic on B, and satisfying the boundary conditions of the mixed Dirichlet problem [4, Theorem B.4]. More exactly, on some closed arcs (curves) Γ_1 of the boundary of B it takes constant values, while on the remaining parts Γ_2 of the boundary of B the normal derivative $\partial u/\partial n$ of u vanishes; the latter set can be empty. Consider some collection $\{\gamma\}$ consisting of finitely many disjoint closed Jordan arcs or closed Jordan curves in B lying on (possibly distinct) level curves of u[2)]. To each curve $\gamma \in \{\gamma\}$ associate the curve γ_ε with $\varepsilon > 0$ obtained from γ via the deformation in (3), where φ is a real twice continuously differentiable function defined on the union $\bigcup\gamma$, while $\varphi \not\equiv 0$ on $\bigcup\gamma$ and the support of $\delta n(z)$ avoids the endpoints of $\gamma \in \{\gamma\}$. Henceforth $\cup$ and Σ stand for the union and the sum over all curves $\gamma \in \{\gamma\}$. Assume that ε is so small that all curves γ_ε are pairwise disjoint and lie in B. Suppose that the function u_ε is continuous on $\overline{B}$, harmonic on $B_\varepsilon := B \setminus \bigcup\gamma_\varepsilon$, satisfies the boundary conditions for u on ∂B, and on each curve γ_ε takes the constant value equal to the value of u on the curve γ corresponding under the deformation in (3). Put

$$I(v, \Omega) = \iint\limits_{\Omega} |\nabla v|^2\, dxdy.$$

Theorem 1. *Under the above conditions we have the asymptotic equality*

$$I(u_\varepsilon, B_\varepsilon) - I(u, B) \asymp \varepsilon^2 \quad \text{as } \varepsilon \to 0. \tag{4}$$

Observe that the left-hand side in (4) is nonnegative by the Dirichlet principle.

The proof of (4) rests substantially on Kellogg's results about the behavior of partial derivatives of a harmonic function on the boundary of its domain of definition [8].

We confine ourselves to the case that u and u_ε are potential functions for generalized condensers [9]. It is clear from the proof of Theorem 1 that (4) also holds if we replace the boundary conditions for these functions by the existence and continuity of their first partial derivatives in a neighborhood of ∂B.

In connection with (2) and (4), the assumption comes up that

$$\Delta(B, \gamma_\varepsilon) \asymp \varepsilon^2 \quad \text{as } \varepsilon \to 0 \tag{5}$$

is valid. However, $\Delta(B, \gamma_\varepsilon) = 0$ for $\delta n(z) \equiv c\varepsilon$, where c is a constant. The author is aware of examples of concrete deformations (3) for which (5) indeed holds. Possibly, $\delta n(z) \equiv c\varepsilon$ is the unique case for which this fails.

The final part of this article gives a corollary to Theorem 1 in the case that B is a circular annulus, see the inequality in (9). We show that this corollary also yields (2).

[1)] In contrast to the original [7, § 3], we introduce in (3) the obvious additional term $O(\varepsilon^2)$ useful in applications.

[2)] This means curves on which u takes constant values.

2. Proof of Theorem 1

We may assume that the boundary of B consists of analytic Jordan curves. Consider the function

$$f_\varepsilon = \frac{u - u_\varepsilon}{\varepsilon}$$

on B_ε and some function f which is harmonic on $B \setminus \bigcup \gamma$, continuous on $\overline{B}$, and satisfies the boundary conditions

$$f = 0 \text{ on } \Gamma_1, \quad \frac{\partial f}{\partial n} = 0 \text{ on } \Gamma_2, \quad f(z) = \varphi(z)\frac{\partial u}{\partial n}(z),\ z \in \gamma\ \forall \gamma \in \{\gamma\}.$$

Henceforth, differentiation is with respect to the positively oriented normal to the corresponding curve. In view of the uniform continuity of f, for every real $\delta > 0$ and arbitrary curve $\gamma \in \{\gamma\}$ we have

$$|f(z) - f(z_\varepsilon)| < \delta, \quad z \in \gamma,$$

for ε sufficiently small. Taylor's formula yields

$$f_\varepsilon(z_\varepsilon) = \varphi(z)\frac{\partial u}{\partial n}(z) + O(\varepsilon) = f(z) + O(\varepsilon) \quad \text{as } \varepsilon \to 0;$$

furthermore, $O(\varepsilon)$ is uniform in $z \in \gamma$. Hence, $|f_\varepsilon(z_\varepsilon) - f(z_\varepsilon)| \le \delta$ for all $z_\varepsilon \in \gamma_\varepsilon$ and ε sufficiently small. The maximum principle for harmonic functions and Hopf's Lemma imply that $|f_\varepsilon(z) - f(z)| \le \delta$ for all $z \in B_\varepsilon$, and consequently, on an arbitrary compact subset of $B \setminus \bigcup \gamma$ for ε small. Thus, f_ε together with partial derivatives converge to f as $\varepsilon \to 0$ uniformly inside $B \setminus \bigcup \gamma$.

Associate to each arc γ_ε with $\gamma \in \{\gamma\}$ a doubly connected domain Q_{γ_ε} with one boundary component γ_ε and the other some closed analytic Jordan curve. Associate to the closed curve γ_ε with $\gamma \in \{\gamma\}$ two disjoint doubly connected domains $Q^+_{\gamma_\varepsilon}$ and $Q^-_{\gamma_\varepsilon}$ with one boundary component γ_ε and the other a closed analytic Jordan curve. Assume that Q_{γ_ε}, $Q^+_{\gamma_\varepsilon}$, and $Q^-_{\gamma_\varepsilon}$ are disjoint and the closures of the domains lie in B. By Kellogg's Theorem [8, Theorem 1] we conclude that f_ε has continuous first partial derivatives on the closures of Q_{γ_ε}, $Q^+_{\gamma_\varepsilon}$, and $Q^-_{\gamma_\varepsilon}$. We will need the normal derivative $\partial f_\varepsilon/\partial n$ on the boundaries of the domains to be bounded uniformly in ε.

The boundary value problem for harmonic functions, in our case for the function f_ε, is reduced in [8] to integral equations. We can express the solution f_ε, in $Q^+_{\gamma_\varepsilon}$ for definiteness, as the sum of a double layer potential W_ε and a single layer potential V_ε [8, Section 3]. The partial derivatives of the potentials on the boundary of $Q^+_{\gamma_\varepsilon}$ are integrals over the boundary of $Q^+_{\gamma_\varepsilon}$ of some functions depending continuously on the domain $Q^+_{\gamma_\varepsilon}$ (with respect to ε), as well as, in the case of W_ε, on the derivative $\partial f_\varepsilon/\partial s$ of the boundary value of f_ε along the tangent to the boundary of $Q^+_{\gamma_\varepsilon}$; see [8, pp. 111, 114, 120, and Section 6]. It is clear from the above that $\partial f_\varepsilon/\partial s$ is continuous and bounded uniformly in ε both on the curve γ_ε by definition and on $(\partial Q^+_{\gamma_\varepsilon}) \setminus \gamma_\varepsilon$ by the uniform convergence on $B \setminus \bigcup \gamma$ of the partial derivative of f_ε. In view of the expression (22) of [8], we conclude that the first partial derivatives of f_ε are bounded on $\partial Q^+_{\gamma_\varepsilon}$ uniformly in ε. We verify similarly that the first partial derivatives of f_ε on $\partial Q^-_{\gamma_\varepsilon}$ are bounded uniformly in ε. In the case of Q_{γ_ε} these derivatives are also bounded, which we can easily verify by mapping Q_{γ_ε} conformally onto a Jordan domain and applying previous arguments to the corresponding superposition; the support of $\delta n(z)$ does not contain the endpoints of γ.

Henceforth we denote the curve $\gamma \in \{\gamma\}$ also by γ^+, whereas the same curve with the opposite direction by γ^-. Similarly, $\gamma^+_\varepsilon = \gamma_\varepsilon$ and γ^-_ε is the curve opposite to γ^+_ε. Applying the Green's formula,[3)]

[3)]In our case this formula is valid because of the restrictions on the growth of the gradients of u and u_ε in a neighborhood of the endpoints of Γ_2; cf. [4, p. 455; 9, p. 306].

we obtain

$$I(u_\varepsilon, B_\varepsilon) - I(u, B) = I(u_\varepsilon, B_\varepsilon) - I(u, B_\varepsilon)$$

$$-2\sum\Big[\int\limits_{\gamma_\varepsilon^+}(u-u_\varepsilon)\frac{\partial u}{\partial n}\,|dz| + \int\limits_{\gamma_\varepsilon^-}(u-u_\varepsilon)\frac{\partial u}{\partial n}\,|dz|\Big]$$

$$= I(u_\varepsilon, B_\varepsilon) + I(u, B_\varepsilon) + 2\int\limits_{\partial B_\varepsilon}(u_\varepsilon - u + u)\frac{\partial u}{\partial n}\,|dz| = I(u-u_\varepsilon, B_\varepsilon)$$

$$= -\int\limits_{\partial B_\varepsilon}(u-u_\varepsilon)\frac{\partial(u-u_\varepsilon)}{\partial n}\,|dz|$$

(the boundary of B is oriented in the positive direction)

$$= -\sum\Big[\int\limits_{\gamma_\varepsilon^+}(u-u_\varepsilon)\frac{\partial(u-u_\varepsilon)}{\partial n}\,|dz| + \int\limits_{\gamma_\varepsilon^-}(u-u_\varepsilon)\frac{\partial(u-u_\varepsilon)}{\partial n}\,|dz|\Big].$$

Among the above relations we highlight the two equalities

$$I(u_\varepsilon, B_\varepsilon) - I(u, B) = -\sum\Big[\int\limits_{\gamma_\varepsilon^+}(u-u_\varepsilon)\frac{\partial(u-u_\varepsilon)}{\partial n}\,|dz| + \int\limits_{\gamma_\varepsilon^-}(u-u_\varepsilon)\frac{\partial(u-u_\varepsilon)}{\partial n}\,|dz|\Big], \tag{6}$$

$$I(u_\varepsilon, B_\varepsilon) - I(u, B) = I(u-u_\varepsilon, B_\varepsilon). \tag{7}$$

Appreciating the above information about f_ε, we arrive at the estimate

$$\Big|\int\limits_{\gamma_\varepsilon^\pm}(u-u_\varepsilon)\frac{\partial(u-u_\varepsilon)}{\partial n}\,|dz|\Big| = \varepsilon^2\Big|\int\limits_{\gamma_\varepsilon^\pm} f_\varepsilon\frac{\partial f_\varepsilon}{\partial n}\,|dz|\Big| = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

Consequently, (6) yields

$$I(u_\varepsilon, B_\varepsilon) - I(u, B) = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

To prove the inverse relation with (6) would require a sharper estimate for the derivative $\partial f_\varepsilon/\partial n$, and consequently a deeper analysis of the proof of Theorem 1 in [8]. It is simpler to observe that under the hypotheses of Theorem 1 there is a subarc $\gamma_0 \subset \gamma$, $\gamma \in \{\gamma\}$ on which $\varphi(z) \neq 0$, while Hopf's Lemma yields $\partial u/\partial n \neq 0$ on γ_0. Consequently, $f \not\equiv 0$ in $B \setminus \bigcup \gamma$. Thus, $B \setminus \bigcup \gamma$ includes a closed disk E with $I(f, E) \neq 0$. For ε sufficiently small the disk E lies in B_ε and

$$I(u-u_\varepsilon, B_\varepsilon) \geq I(u-u_\varepsilon, E) = \varepsilon^2 I(f_\varepsilon, E) = \varepsilon^2 I(f, E) + o(\varepsilon^2) \geq \varepsilon^2\frac{I(f,E)}{2}.$$

With (7) this yields

$$\varepsilon^2 = O(I(u_\varepsilon, B_\varepsilon) - I(u, B)) \quad \text{as } \varepsilon \to 0.$$

The proof of Theorem 1 is complete.

3. Moduli and Capacities

To an arbitrary doubly connected domain $D \subset \mathbb{C}$ with nondegenerate boundary components E_0 and E_1, associate the condenser $C = (E_0, E_1)$ whose capacity equals $\operatorname{cap} C = I(\omega, D)$. Here ω is the "potential function" of C continuous on $\overline{D}$, harmonic on D, vanishing on E_0, and equal to 1 on E_1. It is well known that

$$\operatorname{cap} C = \frac{1}{\operatorname{mod} D};$$

for more detail on capacity, see [9]. Put

$$B(\tau_1,\tau_2)=\{z:\tau_1<|z|<\tau_2\},\quad T(\tau)=\{z:|z|=\tau\}.$$

The circle $\gamma=T(r)$ is a level curve of the function

$$u(z)=\frac{\log(|z|/t)}{\log(s/t)},\quad 0<s<r<t<\infty.$$

The curve γ_ε resulting from γ by the deformation in (3) partitions the annulus $B=B(s,t)$ into disjoint doubly connected domains B_1 and B_2; assume that $T(s)\subset\partial B_1$. In the case $\varphi\not\equiv 0$ Theorem 1 yields

$$I(u_\varepsilon,B_1\cup B_2)-I(u,B)\asymp\varepsilon^2\quad\text{as }\varepsilon\to 0,\tag{8}$$

where u_ε is a function continuous on $\overline{B}$, harmonic on $B_1\cup B_2$, equal to 1 on $T(s)$ and to 0 on $T(t)$, and

$$u_\varepsilon=\delta:=\frac{\log(r/t)}{\log(s/t)}\text{ on }\gamma_\varepsilon.$$

The function u is potential for the condenser $C=(T(t),T(s))$, while $(u_\varepsilon-\delta)/(1-\delta)$ is the potential function for the condenser $C_1=(\gamma_\varepsilon,T(s))$ and u_ε/δ is the potential function for the condenser $C_2=(T(t),\gamma_\varepsilon)$. Thus, (8) becomes

$$0\le(1-\delta)^2\operatorname{cap}C_1+\delta^2\operatorname{cap}C_2-\operatorname{cap}C\asymp\varepsilon^2\quad\text{as }\varepsilon\to 0.$$

In terms of moduli this inequality looks like

$$0\le\frac{\operatorname{mod}^2 B_1^*}{\operatorname{mod}B_1}+\frac{\operatorname{mod}^2 B_2^*}{\operatorname{mod}B_2}-\operatorname{mod}B\asymp\varepsilon^2\quad\text{as }\varepsilon\to 0,\tag{9}$$

where $B_1^*=B(s,r)$ and $B_2^*=B(r,t)$, while the deformation of (3) satisfies the condition $\varphi\not\equiv 0$.

Verify that (9) implies (2). We may assume that $\varphi\not\equiv 0$. Denote by B_1' and B_2' the circular annuli $B(s,r(\varepsilon))$ and $B(r(\varepsilon),t)$ whose areas in the logarithmic metric $(2\pi|z|)^{-1}|dz|$ are equal respectively to the areas of the domains B_1 and B_2 in the same metric. It is obvious that $r(\varepsilon)=r+c\varepsilon+O(\varepsilon^2)$, where c is some constant depending on the function φ. Rengel's Lemma [9, Section 5.5] yields

$$\operatorname{mod}B_1\le\operatorname{mod}B_1',\quad\operatorname{mod}B_2\le\operatorname{mod}B_2'.$$

Subtracting from (9) the equality

$$\frac{\operatorname{mod}^2 B_1^*}{\operatorname{mod}B_1'}+\frac{\operatorname{mod}^2 B_2^*}{\operatorname{mod}B_2'}-\operatorname{mod}B=O(\varepsilon^2)\quad\text{as }\varepsilon\to 0,$$

we obtain

$$\frac{\operatorname{mod}^2 B_1^*}{\operatorname{mod}B_1}-\frac{\operatorname{mod}^2 B_1^*}{\operatorname{mod}B_1'}+\frac{\operatorname{mod}^2 B_2^*}{\operatorname{mod}B_2}-\frac{\operatorname{mod}^2 B_2^*}{\operatorname{mod}B_2'}\le C\varepsilon^2,$$

where C is some constant. Hence,

$$\operatorname{mod}B_k'-\operatorname{mod}B_k=O(\varepsilon^2)\quad\text{as }\varepsilon\to 0,\ k=1,2.$$

Consequently,

$$0\le\operatorname{mod}B-\operatorname{mod}B_1-\operatorname{mod}B_2=\operatorname{mod}B_1'-\operatorname{mod}B_1+\operatorname{mod}B_2'-\operatorname{mod}B_2=O(\varepsilon^2)\quad\text{as }\varepsilon\to 0,$$

which means the validity of (2). Similarly we can establish the inequalities supplementing Theorem 4.2 and Corollary 4.3 of [4].

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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