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DECOMPOSITIONS IN SEMIRINGS

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Abstract: We prove that each element of a complete atomic l-semiring has a canonical decomposition. We also find some sufficient conditions for the decomposition to be unique that are expressed by first-order sentences. As a corollary, we obtain a theorem of Avgustinovich–Frid which claims that each factorial language has the unique canonical decomposition.

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1. Introduction

In [1], Avgustinovich and Frid proved that each factorial language can be presented as the catenation of irreducible factorial languages; moreover, this representation is unique and optimal in a sense (the decompositions were called in [1] *canonical*; see [1, p. 151] and cf. Definition 7 of this article). Later, these authors established in [2] that all the irreducible components of the canonical decomposition of a regular factorial language are regular themselves. We also refer to the paper by Frid [3] on the related matters.

In this paper, we present an algebraic point of view at the matter. Namely, we show that each element of an ordered semiring with certain properties has the canonical decomposition (see Theorem 1). Furthermore, we find the two conditions, either of them expressed by a first-order sentence, that guarantee the uniqueness of canonical decompositions in ordered semirings (see Theorem 2). Then we demonstrate in Section 5 that the set of factorial languages over a fixed finite alphabet forms an ordered semiring with all required properties. This yields the above-mentioned theorem by Avgustinovich and Frid (see Corollary 2).

Our lattice-theoretic terminology is in accordance with [4, 5]. Our semiring terminology is in accordance with [4, 6].

2. The Basic Notions

We present in this section the necessary definitions and auxiliary results that will be used later.

2.1. l-Semirings.

Definition 1. An algebra $\mathscr{M} = \langle M; \cdot, 1\rangle$ is a *monoid* if the following are satisfied:
(i) $a(bc) = (ab)c$ for all $a, b, c \in M$;
(ii) $a \cdot 1 = 1 \cdot a = a$ for all $a \in M$.
The monoid $\mathscr{M}$ is *commutative* if $ab = ba$ for all $a, b \in M$.

Definition 2 [6, p. 1]. An algebra $\mathscr{R} = \langle R; +, \cdot, 0, 1\rangle$ is a *semiring* if the following are satisfied:
(i) $\langle R; +, 0\rangle$ is a commutative monoid;
(ii) $\langle R; \cdot, 1\rangle$ is a monoid;
(iii) $a(b + c)d = abd + acd$ for all $a, b, c, d \in R$;
(iv) $a0 = 0a = 0$ for all $a \in R$.

If $\langle R; \cdot, 1\rangle$ is a monoid, $X \subseteq R$, and $r \in R$ then we put $rX = \{rx \mid x \in X\}$ and $Xr = \{xr \mid x \in X\}$.

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Definition 3 [6, p. 217]. A semiring $\mathscr{R} = \langle R; +, \cdot, 0, 1\rangle$ is a *lattice-ordered semiring* or just an *l-semiring* if we can define the lattice operations of join $\vee$ and meet $\wedge$ on R with the properties:

(i) $a + b = a \vee b$ for all $a, b \in R$;

(ii) $ab \leq a \wedge b$ for all $a, b \in R$.

Here $a \leq b$ means that $a \vee b = b$ or, equivalently, $a + b = b$. If the lattice $\langle R; +, \wedge\rangle$ is distributive then so we call the l-semiring $\mathscr{R}$.

In what follows, we consider lattice-ordered semirings in the signature $\{+, \cdot, \wedge, 0, 1\}$.

Definition 4. We say that an l-semiring $\langle R; +, \cdot, \wedge, 0, 1\rangle$ is *complete* if $\langle R; +, \wedge\rangle$ is a complete lattice with the following properties:

(i) $p(\Sigma X)r = \Sigma pXr$ for all $p, r \in R$ and $X \subseteq R$;

(ii) $p(\bigwedge X)r = \bigwedge pXr$ for all $p, r \in R$ and all $X \subseteq R$.

We note that our definition of complete l-semiring differs from the definition of complete lattice-ordered semiring given in [6, p. 227].

Lemma 1. *If $\mathscr{R} = \langle R; +, \cdot, \wedge, 0, 1\rangle$ is an l-semiring then the operation $\cdot$ respects the ordering $\leq$. Moreover, $a \leq 1$ for all $a \in R$.*

Proof. Suppose that $a_i \leq b_i$ in $\mathscr{R}$ where $i < 2$. This means that $a_0a_1 \leq a_0a_1 + a_0b_1 = a_0(a_1 + b_1) = a_0b_1 \leq a_0b_1 + b_0b_1 = (a_0 + b_0)b_1 = b_0b_1$; i.e., the operation $\cdot$ indeed respects $\leq$.

Given $a \in R$, we have $a = a \cdot 1 \leq a \wedge 1 \leq 1$. $\square$

An element r of an ordered semiring $\langle R; +, \cdot, 0, 1, \leq\rangle$ is *prime* if $ab \leq r$ implies that $a \leq r$ or $b \leq r$ for all $a, b \in R$; and r is $\wedge$-*prime* if $a \wedge b \leq r$ implies that $a \leq r$ or $b \leq r$ for all $a, b \in R$. Moreover, $r \in R\backslash\{1\}$ is *irreducible* if $ab = r$ implies that $a = r$ or $b = r$ for all $a, b \in R$. It is clear that each prime element is irreducible. An element $r \in R\backslash\{1\}$ is *completely irreducible* if $ab = r$ implies that $\{a, b\} = \{r, 1\}$ for all $a, b \in R$.

Given a semiring $\langle R; +, \cdot, 0, 1\rangle$ and a set $X \subseteq R$, let $[X]$ denote the submonoid of $\langle R; \cdot, 1\rangle$ generated by X.

Definition 5. Let $\mathscr{R} = \langle R; +, \cdot, \wedge, 0, 1\rangle$ be a complete l-semiring. Then $\mathscr{R}$ is *atomic* if there is a subset $A \subseteq R\backslash\{1\}$ consisting of the prime elements satisfying the condition

$$\text{for each } r \in R\text{, there is } X \subseteq [A] \text{ such that } \bigwedge X = r.$$

The elements of A are called *atoms* of the l-semiring $\mathscr{R}$ in this case.

Clearly, $\bigwedge[A] = 0$ in terms of Definition 5. Also each complete distributive algebraic lattice turns to a complete atomic l-semiring under the assumption that the operations $\cdot$ and $\wedge$ agree. Some more involved example of a complete atomic l-semiring appears in Section 5.

Lemma 2. *If $r = \bigwedge X$ and $\varnothing \neq X \subseteq [A]$ then $r \neq 1$.*

Proof. As $X \neq \varnothing$, there are $n < \omega$ and $a_0, \dots, a_n \in A$ such that $a_0 \cdots a_n \in X$. We see in this case that $r \leq a_0 \cdots a_n \leq a_0 \wedge \cdots \wedge a_n \leq a_0 < 1$ by Lemma 1. $\square$

The following is a generalization of Lemma 6 of [1] for semirings:

Lemma 3. *Let $\langle R; +, \cdot, \wedge, 0, 1\rangle$ be a complete atomic l-semiring with a finite set $A \subseteq R$ of atoms and let $\{r_i \mid i < \omega\} \subseteq R\backslash\{1\}$. Then there are $n_0 < \omega$ and a nonempty set $B \subseteq A$ such that*

$$\bigwedge_{m<\omega} r_{n+m} \cdots r_n = \bigwedge[B]$$

for all $n \geq n_0$.

Proof. Put

$$B = \{a \in A \mid \text{the set } \{n < \omega \mid r_n \leq a\} \text{ is infinite}\}.$$

It follows from the definition of A that $B \neq \varnothing$. We choose the least $n_0 < \omega$ which satisfies the condition

$$\text{if } r_n \leq a \text{ for some } n \geq n_0 \text{ then } a \in B.$$

Such n_0 exists by the definition of the set B and the finiteness of A. Furthermore, for all $b_0, \dots, b_k \in B$ and $n \geq n_0$, there are $m_0, \dots, m_k < \omega$ such that $n \leq m_k < \dots < m_0 < \omega$ and $r_{m_i} \leq b_i$ for all $i \leq k$. Therefore, $r_{m_0} \cdots r_{m_k} \leq b_0 \cdots b_k$ by Lemma 1. Thus, by Definition 3(ii) $r_{n+m} \cdots r_n \leq b_0 \cdots b_k$, where $m < \omega$ is such that $n \leq m_k < \dots < m_0 \leq n + m$. This implies that

$$\bigwedge_{m<\omega} r_{n+m} \cdots r_n \leq \bigwedge [B].$$

Further, according to Definition 5, for all $n \geq n_0$ and $m < \omega$, there is $X_{mn} \subseteq [A]$ such that $r_{n+m} \cdots r_n = \bigwedge X_{mn}$. It is clear that if $x \leq a \in A$ for some $x \in X_{mn}$ then $r_{n+m} \cdots r_n \leq x \leq a$, whence $r_{n+k} \leq a$ for some $k \leq m$ in view of the primality of a. The definition of n_0 yields $a \in B$. We conclude therefore that $X_{mn} \subseteq [B]$; i.e.,

$$\bigwedge [B] \leq \bigwedge \Big\{ \bigwedge X_{mn} \mid m < \omega \Big\} = \bigwedge_{m<\omega} r_{n+m} \cdots r_n.$$

Thus, the desired equality

$$\bigwedge [B] = \bigwedge_{m<\omega} r_{n+m} \cdots r_n$$

holds. $\square$

The next statement is a generalization of Lemma 7 of [1] for semirings.

Lemma 4. *Let $\langle R; +, \cdot, \wedge, 0, 1 \rangle$ be a complete atomic l-semiring with finitely many atoms and let $\{p_i, r_{i+1} \mid i < \omega\} \subseteq R$.*

(i) *If $p_i = p_{i+1} r_{i+1}$ for all $i < \omega$ then there is $n < \omega$ such that $p_n = p_{n+k}$ for all $k < \omega$.*

(ii) *If $p_i = r_{i+1} p_{i+1}$ for all $i < \omega$ then there is $n < \omega$ such that $p_n = p_{n+k}$ for all $k < \omega$.*

Proof. We will demonstrate (i), as (ii) has some symmetric proof that uses a claim symmetric to the claim of Lemma 3.

If there is $n < \omega$ such that $r_{n+k} = 1$ for all $k < \omega$ then the desired statement is obvious. Therefore, it suffices to consider the case when $\{r_{i+1} \mid i < \omega\} \subseteq R \backslash \{1\}$. Indeed, $p_i = p_{i+1} r_{i+1} \leq p_{i+1}$ for all $i < \omega$. According to Lemma 3, there are a finite set of atoms $B \subseteq R$ and $n_0 < \omega$ such that

$$\bigwedge_{m<\omega} r_{n+m} \cdots r_n = \bigwedge [B]$$

for each $n \geq n_0$. We fix a particular integer $n \geq n_0$ and put $a = \sum_{i<\omega} p_i$ and $b = \bigwedge [B]$ (we recall that we consider a complete l-semiring). Given $k < \omega$, we see by Lemma 1 that

$$p_n = p_{n+k+1} r_{n+k+1} \cdots r_{n+1} \leq \Big(\sum_{i<\omega} p_i \Big) r_{n+k+1} \cdots r_{n+1} = a r_{n+k+1} \cdots r_{n+1},$$

whence

$$\begin{aligned} p_n &\leq \bigwedge_{k<\omega} (a r_{n+k+1} \cdots r_{n+1}) = a \Big(\bigwedge_{k<\omega} r_{n+k+1} \cdots r_{n+1} \Big) = ab = \Big(\sum_{i<\omega} p_i \Big) b \\ &= \Big(\sum_{i<\omega} p_{n+i+1} \Big) b = \sum_{i<\omega} p_{n+i+1} b \leq \sum_{i<\omega} p_{n+i+1} r_{n+i} \cdots r_{n+1} = p_n. \end{aligned}$$

Therefore, $p_n = ab$ for all $n \geq n_0$ and the desired statement follows. $\square$

2.2. Factorial languages. Given arbitrary languages K, L, and L_i, with $i \in I$, over an alphabet Σ, we consider the operations

$$\bigcap_{i\in I} L_i = \{\alpha \in \Sigma^* \mid \alpha \in L_i \text{ for all } i \in I\};$$

$$\bigcup_{i\in I} L_i = \{\alpha \in \Sigma^* \mid \alpha \in L_i \text{ for some } i \in I\};$$

$$KL = \{\alpha\beta \mid \alpha \in K, \beta \in L\}; \quad L^* = \bigcup\{L^n \mid n < \omega\}.$$

Let λ denote the *empty word* and let $|\alpha|$ denote the *length* of $\alpha \in \Sigma^*$; in particular, $|\lambda| = 0$.

Definition 6. A language $L \subseteq \Sigma^*$ over an alphabet Σ is *factorial* if $\alpha\beta\gamma \in L$ implies that $\beta \in L$ for all $\alpha, \gamma \in \Sigma^*$.

A language $L \subseteq \Sigma^*$ is *prefixal* if L contains all nonempty prefixes of each of its words; in other words, $\alpha\beta \in L$ implies that $\alpha \in L$ for each nonempty $\alpha \in \Sigma^*$ and $\beta \in \Sigma^*$. A language $L \subseteq \Sigma^*$ is *suffixal* if L contains all nonempty suffices of each of its words; in other words, $\beta\alpha \in L$ implies that $\alpha \in L$ for all nonempty $\alpha \in \Sigma^*$ and $\beta \in \Sigma^*$.

The set of all factorial languages over an alphabet Σ we will denote by $\mathscr{F}_\Sigma$ or just by $\mathscr{F}$ when there is no confusion. The following has a straightforward proof:

Lemma 5. *Let $n > 0$ and let $A, B, A_i \subseteq \Sigma^*$, with $i \in I$, be factorial languages. Then so are $\bigcup_{i\in I} A_i$, $\bigcap_{i\in I} A_i$, and AB.*

Given a language $A \subseteq \Sigma^*$, put

$$F(A) = \{\xi \in \Sigma^* \mid \xi \neq \lambda,\ \alpha\xi\beta \in A \text{ for some } \alpha, \beta \in \Sigma^*\};$$

$$P(A) = \{\xi \in \Sigma^* \mid \xi \neq \lambda,\ \alpha\xi \in A \text{ for all } \alpha \in A\};$$

$$S(A) = \{\xi \in \Sigma^* \mid \xi \neq \lambda,\ \xi\alpha \in A \text{ for all } \alpha \in A\}.$$

It is clear that $F(A)$ is the least factorial language containing A. Moreover, if A is a factorial language, then $P(A)$ is a prefixal language and $S(A)$ is a suffixal language.

Lemma 6 [1, Lemmas 5 and 5′]. *Let $A, B \subseteq \Sigma^*$ be factorial languages.*
(i) *$F(A\backslash XA)B = F(AB\backslash XAB)$ for every prefixal language $X \subseteq \Sigma^*$.*
(ii) *$AF(B\backslash BX) = F(AB\backslash ABX)$ for every suffixal language $X \subseteq \Sigma^*$.*

3. Existence of Decompositions in *l*-Semirings

Definition 7. A representation $r = r_0 \cdots r_n$, where $n < \omega$, of an element r of an ordered semiring $\langle R; +, \cdot, \leq, 0, 1\rangle$ as a product of elements of R is a *canonical decomposition* for r if the following are satisfied:
(i) r_i is irreducible for each $i \leq n$;
(ii) $r = r_0 \cdots r_{i-1} x r_{i+1} \cdots r_n$ implies that $x \leq r_i$ for all $i \leq n$.

From Definition 7 we obtain

Lemma 7. *Let $\langle R; +, \cdot, \wedge, 0, 1\rangle$ be an ordered semiring. If $r = p_0 \cdots p_n$ is a canonical decomposition of $r \in R$ then $r' = p_k \cdots p_m$ for all k and m such that $1 \leq k \leq m \leq n$.*

Lemma 8. *Let $\langle R; +, \cdot, \wedge, 0, 1\rangle$ be a complete l-semiring and let $r \in R\backslash\{1\}$. Then either r is an irreducible element or there is a decomposition $r = r_0 r_1$ which satisfies condition* (ii) *of Definition* 7.

Proof. If r is not irreducible then there are $x, y \in R\backslash\{r\}$ such that $r = xy$. According to Definition 4(i), the set

$$A = \{z \in R \mid r = zy\}$$

contains the greatest element r_0 with respect to $\leq$ and

$$B = \{z \in R \mid r = r_0 z\}$$

contains the greatest element r_1 with respect to $\leq$. Therefore, $r = r_0 r_1$ and $r \notin \{r_0, r_1\}$ as $r < x \leq r_0$ and $r < y \leq r_1$. Thus, $1 \notin \{r_0, r_1\}$. If $r = r_0 z$ then $z \leq r_1$ by the choice of r_1. If $r = z r_1$ then $r = r + r = z r_1 + r_0 r_1 = (z + r_0) r_1$. Using the choice of r_0 and r_1 together with Lemma 1, we have

$$r = r_0 y \leq (z + r_0) y \leq (z + r_0) r_1 = r;$$

i.e., $r = (z + r_0) y$. We conclude that $z \leq z + r_0 \leq r_0$. Thus, condition (ii) of Definition 7 is satisfied too. □

In the proof of the following theorem, we use some idea of the proof of the existence of canonical decompositions of factorial languages from Avgustinovich and Frid (cf. [1, p. 156]).

Theorem 1. *Let $\mathscr{R} = \langle R; +, \cdot, \wedge, 0, 1\rangle$ be a complete atomic l-semiring with finitely many atoms. Then each $p \in R\backslash\{1\}$ has a canonical decomposition.*

Proof. If p is an irreducible element then $p = p$ is a canonical decomposition of p. Otherwise by Lemma 8, there are nonunit $p_1, r_1 \in R\backslash\{p\}$ such that the decomposition $p = p^0 = p_0 = p_1 r_1$ satisfies condition (ii) of Definition 7. If p_1 is an irreducible element then $p_1 = p_1$ is a canonical decomposition of p_1. Otherwise by Lemma 8, there are nonunit $p_2, r_2 \in R\backslash\{p_1\}$ such that the decomposition $p_1 = p_2 r_2$ satisfies condition (ii) of Definition 7. Repeating this argument, we obtain some set $\{p_i, r_{i+1} \mid i < \omega\} \subseteq R\backslash\{1\}$ such that, for all $i < \omega$, the decomposition $p_i = p_{i+1} r_{i+1}$ satisfies condition (ii) of Definition 7. By Lemma 4(i), there is a least natural $n < \omega$ such that $p_n = p_{n+1}$. This means that p_n is irreducible; moreover, $n > 0$ as p is reducible.

Let $p_n = a_1$ and let p^1 denote the least element of

$$\{z \in R \mid p^0 = a_1 z\}.$$

Then $r_n \cdots r_1 \leq p^1$ and the decomposition $p = p^0 = a_1 p^1$ satisfies condition (ii) of Definition 7. Indeed, if $p^1 = 1$ then $p^0 = a_1 = p_n$ is an irreducible element; a contradiction. If $p^0 = a p^1$ for some $a \in R$ then

$$p^0 = p^0 + p^0 = a p^1 + a_1 p^1 = (a + a_1) p^1.$$

If $a \nleq a_1$ then $a + a_1 > a_1 = p_n$, whence $(a + a_1) r_n > p_n r_n = p_{n-1}$ by the choice of p_n. Repeating the same argument, we infer that $(a + a_1) r_n \cdots r_i > p_{i-1}$ for all $i \in \{1, \dots, n\}$ by the choice of p_i. Hence, $p^0 = (a + a_1) p^1 \geq (a + a_1) r_n \cdots r_1 > p_0 = p^0$ by the choice of p_1 and Lemma 1, which is impossible. The contradiction obtained shows that $a \leq a_1$. Moreover, by the choice of p^1, we conclude that $p^0 \neq a_1 z$ for all $z \nleq p^1$. Therefore, the decomposition $p^0 = a_1 p^1$ satisfies condition (ii) of Definition 7.

If p^1 is a reducible element then we apply the above argument to p^1 to find a_2 and $p^2 \in R\backslash\{p^1\}$ such that a_2 is an irreducible element and the decomposition $p^1 = a_2 p^2$ satisfies condition (ii) of Definition 7. Therefore, $p = a_1 a_2 p^2$. It is clear that $p^0 \neq a p^1 = a a_2 p^2$ for all $a \nleq a_1$. If $p = a_1 a p^2$ for some $a \in R$ then

$$p^0 = p^0 + p^0 = a_1 a_2 p^2 + a_1 a p^2 = a_1 (a + a_2) p^2.$$

If $a \nleq a_2$ then $a + a_2 > a_2$, whence $(a + a_2) p^2 > a_2 p^2 = p^1$ by the choice of a_2. It follows that $p^0 = a_1 (a + a_2) p^2 > a_1 p^1 = p^0$ by the choice of p^1, which is impossible. The contradiction obtained shows that $a \leq a_2$. If $p = a_1 a_2 z$ for some $z \in R$ then

$$p^0 = p^0 + p^0 = a_1 a_2 p^2 + a_1 a_2 z = a_1 a_2 (p^2 + z).$$

If $z \nleq p^2$ then $p^2 + z > p^2$, whence $a_2 (p^2 + z) > a_2 p^2 = p^1$ by the choice of p^2. It follows that $p^0 = a_1 a_2 (p^2 + z) > a_1 p^1 = p^0$ by the choice of p^1, which is impossible. This contradiction demonstrates that $z \leq p^2$. Therefore, the decomposition $p^0 = a_1 a_2 p^2$ satisfies condition (ii) of Definition 7.

Repeating this argument, we obtain some set $\{a_i, p^{i+1} \mid i < \omega\} \subseteq R\backslash\{1\}$ such that a_i is irreducible for all $i < \omega$, $p^i = a_{i+1} p^{i+1}$, and the decomposition $p^0 = a_1 \cdots a_i p^i$ satisfies condition (ii) of Definition 7. By Lemma 4(ii), there is a least natural $m < \omega$ such that $p^m = p^{m+1}$; i.e., the element p^m is irreducible. Therefore, each member of the decomposition $p = a_1 \cdots a_m p^m$ is irreducible and all conditions of Definition 7 are satisfied. Hence, $p = a_1 \cdots a_m p^m$ is a canonical decomposition. □

4. Sufficient Conditions for the Uniqueness of Decompositions in *l*-Semirings

We consider the sentence (C_0) presenting the universal closure of the following formula with free variables $\{x_0, x_1, y_0, y_1\}$ in the language $\{+, \cdot\}$:

$$[x_0x_1 = y_0y_1] \ \to \ [x_0 \le y_0] \ \& \ [y_1 \le x_1] \ \vee \ [y_0 \le x_0] \\ \& \ [x_1 \le y_1] \ \vee \ [x_0 = y_0] \ \vee \ [x_1 = y_1].$$

We also consider the sentence (C_1) presenting the universal closure of the following formula with free variables $\{x, x_0, x_1, y_0, y_1, z\}$ in the language $\{+, \cdot\}$:

$$[xx_0x_1 = xy_0y_1] \ \& \ [xx_0 \le xy_0] \ \& \ [y_1 \le x_1] \ \longrightarrow \ \exists y \exists z[xx_0x_1 = xyzx_1] \\ \& \ [y_0 \le y] \ \& \ [xy_0 = xy] \ \& \ [x_0 \le yz] \ \& \ [y_1 \le zx_1].$$

As usual, we abbreviate $x + y = y$ as $x \le y$.

Theorem 2. *Let* $\mathscr{R} = \langle R; +, \cdot, \wedge, 0, 1\rangle$ *be an l-semiring in which* (C_0) *and* (C_1) *hold. Then each element in* $\mathscr{R}$ *has at most one canonical decomposition.*

Proof. Given $r \in R\backslash\{1\}$ having a canonical decomposition, let $n(r)$ denote the least number of elements in such decomposition for r. We prove by induction on $n(r)$ that each $r \in R$ has at most one canonical decomposition. If $n(r) = 1$ for some element $r \in R$ then r is irreducible. Thus, if $r = r_0 \cdots r_m$ is a canonical decomposition with $m > 0$ then $r \in \{r_0, \dots, r_m\}$. Suppose that $r = r_i$ for some $i < m$. By Lemma 7, $r' = rr_{i+1}$ is a canonical decomposition. We have $r = r_0 \cdots r_m \le r_ir_{i+1} = rr_{i+1} = r' \le r$ whence $r = r'$ and $r = rr_{i+1}$ is a canonical decomposition. As $r = r \cdot 1$, we conclude that $1 \le r_{i+1} \le 1$ whence $r_{i+1} = 1$ which is a contradiction with the definition of canonical decomposition. If $r = r_m$ then we see as above that $r = r_{m-1}r$ is a canonical decomposition whence $r_{m-1} = 1$, which is again impossible. Therefore, $m = 1$ and $r = r_1$. Suppose now that the statement of Theorem 2 holds for each $r \in R\backslash\{1\}$ with $n(r) \le i$, where $i > 0$.

Let $r = p_1 \cdots p_n = r_1 \cdots r_m$ be canonical decompositions of $r \in R\backslash\{1\}$, where $n = n(r) = i + 1$ and $m < \omega$. Then $1 < i + 1 = n \le m$ whence $m > 1$. We put $p_0 = r_0 = 1$.

Claim 1. $p_j = r_j$ *for all* $j \le n$.

Proof. Induct on $j \le n$. As $p_0 = r_0 = 1$, the desired statement is true for $j = 0$. Suppose that $p_j = r_j$ for all $j \le k < n$ and show that $p_{k+1} = r_{k+1}$. We put

$$\delta(x_0) = p_1 \cdots p_kp_{k+1}, \quad \delta(x_1) = p_{k+2} \cdots p_n;$$

$$\delta(y_0) = p_1 \cdots p_kr_{k+1}, \quad \delta(y_1) = r_{k+2} \cdots r_m.$$

As $\delta(x_0)\delta(x_1) = r = \delta(y_0)\delta(y_1)$ by the induction hypothesis, the premise of (C_0) holds in $\mathscr{R}$ under the interpretation δ. By assumption, the conclusion of (C_0) also holds in $\mathscr{R}$ under this interpretation. Therefore, the following four cases are possible:

Case 1: $\delta(x_1) = \delta(y_1) = r'$. The two subcases are possible:

Case 1.1: $r' = 1$. In this case, $r = \delta(x_0) = \delta(y_0)$ whence $n = k + 1 = m$ and $r = p_1 \cdots p_kp_{k+1} = r_1 \cdots r_kr_{k+1}$ is a canonical decomposition of r. By the definition of canonical decomposition, $p_{k+1} \le r_{k+1} \le p_{k+1}$; i.e., $p_{k+1} = r_{k+1}$ which is our desired conclusion.

Case 1.2: $r' \neq 1$. In this case, $r' = p_{k+2} \cdots p_n = r_{k+2} \cdots r_m$ is a canonical decomposition of r' by Lemma 7. As $n(r') \le n - (k + 1) < n$, we conclude by our induction hypothesis made in the beginning of the proof of Theorem 2 that $n = m$ and $p_j = r_j$ for all $j \in \{k + 2, \dots, n\}$. Therefore, $r = p_1 \cdots p_kp_{k+1}p_{k+2} \cdots p_n = p_1 \cdots p_kr_{k+1}p_{k+2} \cdots p_n$ are canonical decompositions of r. By the definition of canonical decomposition, we obtain $p_{k+1} \le r_{k+1} \le p_{k+1}$; i.e., $p_{k+1} = r_{k+1}$, which is our desired conclusion.

Case 2: $\delta(x_0) = \delta(y_0) = r'$. By Lemma 7, $r' = p_1 \cdots p_k p_{k+1} = p_1 \cdots p_k r_{k+1}$ is a canonical decomposition of r'. By the definition of canonical decomposition, we conclude that $p_{k+1} \leq r_{k+1} \leq p_{k+1}$; i.e., $p_{k+1} = r_{k+1}$, which is our desired conclusion.

Case 3: $\delta(x_0) \leq \delta(y_0)$ and $\delta(y_1) \leq \delta(x_1)$. We put

$$\gamma(x) = p_1 \cdots p_k, \quad \gamma(x_0) = p_{k+1}, \quad \gamma(x_1) = \delta(x_1) = p_{k+2} \cdots p_n,$$

$$\gamma(y_0) = r_{k+1}, \quad \gamma(y_1) = \delta(y_1) = r_{k+2} \cdots r_m.$$

In this case

$$\gamma(x)\gamma(x_0)\gamma(x_1) = \gamma(x)\gamma(y_0)\gamma(y_1),$$

$$\gamma(x)\gamma(x_0) = \delta(x_0) \leq \delta(y_0) = \gamma(x)\gamma(y_0),$$

$$\gamma(y_1) = \delta(y_1) \leq \delta(x_1) = \gamma(x_1);$$

i.e., the premise of (C_1) holds in $\mathscr{R}$ under the interpretation γ. By assumption, the conclusion of (C_1) also holds in $\mathscr{R}$ under this interpretation. Therefore, there are $b, c \in R$ such that

$$\gamma(y_0) \leq b, \quad \gamma(x)\gamma(y_0) = \gamma(x)b, \quad r = \gamma(x)bc\gamma(x_1) = p_1 \cdots p_k bc p_{k+2} \cdots p_n,$$

$$\gamma(x_0) \leq bc, \quad \gamma(y_1) \leq c\gamma(x_1).$$

By Lemma 7 and our induction hypothesis, $\gamma(x)\gamma(y_0) = p_1 \cdots p_k r_{k+1}$ is a canonical decomposition. Thus, the first two conditions above imply $b \leq r_{k+1} = \gamma(y_0) \leq b$; i.e., $b = r_{k+1} = \gamma(y_0)$. Furthermore, $r = p_1 \cdots p_n$ is a canonical decomposition. Hence, $r = \gamma(x)bc\gamma(x_1) = p_1 \cdots p_k bc p_{k+2} \cdots p_n$ and $p_{k+1} = \gamma(x_0) \leq bc$ imply that $p_{k+1} = bc = r_{k+1}c$. As p_{k+1} is irreducible, we conclude that $p_{k+1} \in \{r_{k+1}, c\}$. The following two subcases are therefore possible:

Case 3.1: $p_{k+1} = r_{k+1}$. This is our desired conclusion.

Case 3.2: $\gamma(x_0) = p_{k+1} = c$. In this case

$$\gamma(x)\gamma(y_1) \geq \gamma(x)\gamma(y_0)\gamma(y_1) = r = \gamma(x)\gamma(x_0)\gamma(x_1) = \gamma(x)c\gamma(x_1) \geq \gamma(x)\gamma(y_1),$$

whence $r = \gamma(x)\gamma(y_1) = r_1 \cdots r_k r_{k+2} \cdots r_m$, which contradicts the canonicity of decomposition $r = r_1 \cdots r_m$. Thus, this subcase is impossible.

Case 4: $\delta(y_0) \leq \delta(x_0)$ and $\delta(x_1) \leq \delta(y_1)$. This case is symmetric to Case 3.
The proof of Claim 2 is complete. □

By Claim 1, $r = p_1 \cdots p_n = p_1 \cdots p_n r_{n+1} \cdots r_m$ are canonical decompositions. If $n < m$ then

$$r = p_1 \cdots p_n \geq p_1 \cdots p_n r_{n+1} \geq p_1 \cdots p_n r_{n+1} \cdots r_m = r$$

whence $r = p_1 \cdots p_n r_{n+1}$ is a canonical decomposition by Lemma 7. As $r = p_1 \cdots p_n \cdot 1$, we conclude that $1 \leq r_{n+1} \leq 1$ whence $r_{n+1} = 1$, which is a contradiction with the definition of canonical decomposition. This contradiction shows that $n = m$. The proof is complete. □

We can now present the main result of this article:

Theorem 3. *Each nonunit element of a complete atomic l-semiring* $\mathscr{R}$ *with finitely many atoms which satisfies* (C_0) *and* (C_1) *has the unique canonical decomposition.*

Proof. Follows from Theorems 1 and 2. □

5. An Application

In this section, we present an application of Theorem 3 to the semiring of factorial languages over a fixed finite alphabet.

Proposition 1. *The structure* $\langle \mathscr{F}_\Sigma; \cap, \cdot, \cup, \Sigma^*, \{\lambda\}\rangle$ *is a complete distributive l-semiring for an arbitrary alphabet* Σ.

Proof. As the operation $\cap$ of intersection of languages is obviously associative and commutative and $X \cap \Sigma^* = X$ for all $X \subseteq \Sigma$, the algebraic structure $\langle \mathscr{F}_\Sigma; \cap, \Sigma^*\rangle$ is a commutative monoid. Furthermore, as the operation $\cdot$ of catenation is associative and $L\{\lambda\} = \{\lambda\}L = L$ for all $L \subseteq \Sigma^*$, the algebraic structure $\big\langle \mathscr{F}_\Sigma; \cdot, \{\lambda\}\big\rangle$ is a monoid. Therefore, conditions (i)–(ii) of Definition 2 are satisfied. Moreover, in view of the equality $L\Sigma^* = \Sigma^* L = \Sigma^*$, where $L \subseteq \Sigma^*$ is an arbitrary language, condition (iv) of Definition 2 is also satisfied.

We note that the partial order $\leq$ on $\mathscr{F}_\Sigma$ is the reverse set-theoretic inclusion. The lattice $\langle \mathscr{F}_\Sigma; \cap, \cup, \Sigma^*, \{\lambda\}\rangle$ is obviously complete and distributive. Condition (i) of Definition 3 is obviously satisfied. As $A \cup B \subseteq AB$ for factorial languages A and B, condition (ii) of Definition 3 is satisfied. Moreover, it is not hard to see that condition (ii) of Definition 4 is also satisfied.

We show that condition (i) of Definition 4 (in particular, condition (iii) of Definition 2) is satisfied. Indeed, let $R, S \in \mathscr{F}_\Sigma$ and $\mathscr{X} \subseteq \mathscr{F}_\Sigma$. It suffices to show that

$$\bigcap R\mathscr{X}S = \bigcap\{RXS \mid X \in \mathscr{X}\} \subseteq R\Big(\bigcap \mathscr{X}\Big)S.$$

To this end, take $\alpha \in \bigcap\{RXS \mid X \in \mathscr{X}\}$. This means that, for each $X \in \mathscr{X}$, there are words $\beta_X \in R$, $\gamma_X \in X$, and $\delta_X \in S$ such that $\alpha = \beta_X\gamma_X\delta_X$. Let β be the longest prefix of α which belongs to R. Similarly, let δ be the longest suffix of α belonging to S. The two cases are possible:

Case 1: $|\beta| + |\delta| > |\alpha|$. In this case, $\alpha = \beta\delta'$, where δ' is a suffix of δ. We have in particular that $\delta' \in S$. As $\lambda \in \bigcap \mathscr{X}$, we conclude that $\alpha \in R\big(\bigcap \mathscr{X}\big)S$.

Case 2: $|\beta| + |\delta| \leq |\alpha|$. In this case, $\alpha = \alpha\gamma\delta$, where γ is a subword of γ_X for all $X \in \mathscr{X}$. This means that $\gamma \in X$ for all $X \in \mathscr{X}$. Thus, $\gamma \in \bigcap \mathscr{X}$ whence $\alpha \in R\big(\bigcap \mathscr{X}\big)S$.

Therefore, the algebraic structure $\langle \mathscr{F}_\Sigma; \cap, \cdot, \cup, \Sigma^*, \{\lambda\}\rangle$ is a complete distributive l-semiring. □

Proposition 2. *The complete l-semiring* $\langle \mathscr{F}_\Sigma; \cap, \cdot, \cup, \Sigma^*, \{\lambda\}\rangle$ *is atomic with finitely many atoms for an arbitrary finite alphabet* Σ.

Proof. We put $A = \big\{\{a, \lambda\} \mid a \in \Sigma\big\}$. It is straightforward to see that A is a finite set of atoms in $\mathscr{F}_\Sigma$ and all conditions of Definition 5 are satisfied for the set A. □

From Propositions 1, 2, and Theorem 1, we obtain

Corollary 1. *Each nontrivial factorial language* $L \subseteq \Sigma^*$ *has a canonical decomposition for an arbitrary finite alphabet* Σ.

In the proof of Proposition 3 below, we use an argument by Avgustinovich and Frid (see [1, p. 157, Case 1]).

Proposition 3. *The sentence* (C_0) *holds in the algebraic structure* $\langle \mathscr{F}_\Sigma; \cap, \cdot, \cup, \Sigma^*, \{\lambda\}\rangle$ *for an arbitrary alphabet* Σ.

Proof. Suppose that $A_0A_1 = B_0B_1$ for some factorial languages $A_0, A_1, B_0, B_1 \in \mathscr{F}_\Sigma$; i.e., the premise of (C_0) holds. The following cases are possible:

Case 1: $A_0 \subseteq B_0$ and $B_1 \nsubseteq A_1$. In this case, there is a word $\alpha \in B_1 \backslash A_1$. Let β denote the longest suffix of α which belongs to A_1. In this case, $\alpha = \gamma\beta$ for some nonempty word $\gamma \in \Sigma^*$. As $\alpha \in B_1$, we conclude that $\delta\gamma\beta = \delta\alpha \in B_0B_1 = A_0A_1$ for each $\delta \in B_0$. As $\gamma \neq \lambda$, the word β is the longest suffix of $\delta\gamma\beta$ which belongs to A_1. This means that $\delta\gamma \in A_0$ whence $\delta \in A_0$, since A_0 is a factorial language. Consequently, we proved that $B_0 \subseteq A_0$. Therefore, $A_0 = B_0$.

CASE 2: $A_0 \subseteq B_0$ and $B_1 \subseteq A_1$. Hence, $B_0 \leq A_0$ and $A_1 \leq B_1$.

CASE 3: $A_0 \nsubseteq B_0$ and $B_1 \subseteq A_1$. In this case, there is a word $\alpha \in A_0 \backslash B_0$. Let β denote the longest prefix of α which belongs to B_0. In this case, $\alpha = \beta\gamma$ for some nonempty $\gamma \in \Sigma^*$. As $\alpha \in A_0$, we conclude that $\beta\gamma\delta = \alpha\delta \in A_0A_1 = B_0B_1$ for each $\delta \in A_1$. Since $\gamma \neq \lambda$, the word β is the longest prefix of $\beta\gamma\delta$ belonging to B_0. This implies that $\gamma\delta \in B_1$ whence $\delta \in B_1$ as B_1 is a factorial language. Hence, we proved that $A_1 \subseteq B_1$. Therefore, $A_1 = B_1$.

CASE 4: $A_0 \nsubseteq B_0$ and $B_1 \nsubseteq A_1$. In this case, there are words $\alpha \in A_0 \backslash B_0$ and $\beta \in A_1 \nsubseteq B_1$. Arguing as in Case 1, we prove that $B_0 \subseteq A_0$. Arguing as in Case 3, we show that $A_1 \subseteq B_1$. Thus, $A_0 \leq B_0$ and $B_1 \leq A_1$.

We established therefore that the conclusion of (C_0) also holds for A_0, A_1, B_0, and B_1. $\square$

Proposition 4. *The sentence* (C_1) *holds in the algebraic structure* $\langle \mathscr{F}_\Sigma; \cap, \cdot, \cup, \Sigma^*, \{\lambda\}\rangle$ *for an arbitrary alphabet* Σ.

PROOF. Suppose that $AA_0A_1 = AB_0B_1$, $AA_0 \leq AB_0$, and $B_1 \leq A_1$ for some factorial languages A, A_0, A_1, B_0, $B_1 \in \mathscr{F}_\Sigma$; i.e., the premise of ($C_1$) holds. The last two inequalities mean that $AB_0 \subseteq AA_0$ and $A_1 \subseteq B_1$. We choose an arbitrary word $\mu \in AA_0A_1 = AB_0B_1$. Let $\delta(\mu)$ denote the longest prefix of μ belonging to AB_0 and let $\alpha_1(\mu)$ denote the longest suffix of μ belonging to A_1.

The following cases are possible:

CASE 1: $|\delta(\mu)| + |\alpha_1(\mu)| \geq |\mu|$. We put $\gamma(\mu) = \lambda$ in this case.

CASE 2: $|\delta(\mu)| + |\alpha_1(\mu)| < |\mu|$. In this case, $\mu = \delta(\mu)\gamma(\mu)\alpha_1(\mu)$ for some nonempty word $\gamma(\mu)$.
We put $B = F(B_0 \backslash P(A)B_0)$ and $C = F(\gamma(\mu) \mid \mu \in AA_0A_1)$. It is obvious that $B, C \in \mathscr{F}_\Sigma$.

Claim 1. *$B \subseteq B_0$ and $AB_0 = AB$.*

PROOF. The first inclusion is obvious. We have therefore that $AB \subseteq AB_0$. In order to prove the reverse inclusion, we choose arbitrary words $\alpha \in A$ and $\beta \in B_0$ and show that $\alpha\beta \in AB$. Let α' denote the longest prefix of $\alpha\beta$ which belongs to A. As $\alpha \in A$, we conclude that $\alpha\beta = \alpha'\gamma$ where γ is a suffix of β. Thus, $\gamma \in B_0$.

Suppose that $\gamma = \delta\gamma'$ for some word $\delta \in P(A)$. In this case, $\gamma' \in B_0$ and $\alpha\beta = \alpha'\gamma = \alpha'\delta\gamma'$. Moreover, α' is a proper prefix of $\alpha'\delta \in A$. In view of the maximality of α', this is impossible, whence $\gamma \notin P(A)B_0$ and $\gamma \in B_0 \backslash P(A)B_0 \subseteq B$. This means that $\alpha\beta = \alpha'\gamma \in AB$. $\square$

Claim 2. *$AA_0A_1 = ABCA_1$.*

PROOF. In view of Claim 1, it suffices to show that $AA_0A_1 = AB_0CA_1$. By definition, for each word $\mu \in AA_0A_1$, we have $\mu = \delta(\mu)\gamma(\mu)\alpha'(\mu)$ where $\alpha'(\mu)$ is a suffix of $\alpha_1(\mu)$. As $\alpha_1(\mu) \in A_1$ and A_1 is a factorial language; therefore, $\alpha'(\mu) \in A_1$. Moreover, by construction and Claim 1, $\delta(\mu) \in AB_0$ and $\gamma(\mu) \in C$. Thus, $\mu \in AB_0CA_1$ whence $AA_0A_1 \subseteq AB_0CA_1$. In order to prove the reverse inclusion, we consider arbitrary words $\alpha \in A$, $\beta \in B_0$, $\gamma \in C$, and $\xi \in A_1$. We have to show that $\alpha\beta\gamma\xi \in AA_0A_1$. The two cases are possible:

CASE 1: $\gamma = \lambda$. In this case, we have by assumption that

$$\alpha\beta\gamma\xi = \alpha\beta\xi \in AB_0A_1 \subseteq AA_0A_1.$$

CASE 2: $\gamma \neq \lambda$. In this case, there are words $\mu \in AA_0A_1$ and $\gamma_0, \gamma_1 \in \Sigma^*$ such that $\gamma_0\gamma\gamma_1 = \gamma(\mu)$. Then $\delta(\mu)\gamma_0\gamma\gamma_1\alpha_1(\mu) = \mu \in AA_0A_1$. As $\alpha_1(\mu)$ is the longest suffix of μ belonging to A_1; therefore, $\delta(\mu)\gamma_0\gamma\gamma_1 \in AA_0$. But then $\delta(\mu)\gamma_0\gamma \in AA_0$ as AA_0 is a factorial language. Thus, $\eta = \delta(\mu)\gamma_0\gamma\xi \in AA_0A_1 = AB_0B_1$. As $\delta(\mu)$ is the longest prefix of η belonging to AB_0, we conclude that $\gamma_0\gamma\xi \in B_1$ and $\gamma\xi \in B_1$, since B_1 is a factorial language. Hence, $\alpha\beta\gamma\xi \in AB_0B_1 = AA_0A_1$. $\square$

Claim 3. $BC \subseteq A_0$.

Proof. By Lemma 6, the language BC coincides with $F\big(B_0C\backslash P(A)B_0C\big)$. As A_0 is a factorial language, it suffices to verify that

$$B_0C\backslash P(A)B_0C \subseteq A_0.$$

Indeed, we choose arbitrary words $\beta \in B_0$ and $\gamma \in C$ such that $\beta\gamma \notin P(A)B_0C$. We have to show that $\beta\gamma \in A_0$. The following cases are possible:

Case 1: $\gamma = \lambda$. In this case, $\beta\gamma = \beta \notin P(A)B_0$ whence $\beta \in B$. Given an arbitrary word $\alpha \in A$, we have $\alpha\beta \in AB \subseteq AB_0 \subseteq AA_0$. Let α' denote the longest prefix of the word $\alpha\beta$ which belongs to A. Since $\alpha \in A$, we conclude that $\alpha' = \alpha\delta(\alpha)$ where $\delta(\alpha)$ is a prefix of β; i.e., $\beta = \delta(\alpha)\beta'(\alpha)$ and $\beta'(\alpha) \in B_0$. We choose α_0 so that the word $\delta(\alpha_0)$ is of least length among the words in $X = \{\delta(\alpha) \mid \alpha \in A\}$.

Suppose that $\delta(\alpha_0) \neq \lambda$ and prove that $\delta(\alpha_0) \in P(A)$ in this case. Indeed, consider an arbitrary word $\alpha \in A$. Then $\alpha\beta = \alpha\delta(\alpha)\beta'(\alpha)$. As $\delta(\alpha_0)$ is of least length in X, we conclude that $\delta(\alpha_0)$ is a prefix $\delta(\alpha)$ whence $\alpha\delta(\alpha_0)$ is a prefix of $\alpha\delta(\alpha) \in A$. Therefore, $\alpha\delta(\alpha_0) \in A$ and $\delta(\alpha_0) \in P(A)$. Furthermore, as $\beta = \delta(\alpha_0)\beta'(\alpha_0) \in P(A)B_0$, we obtain a contradiction. This contradiction shows that $\delta(\alpha_0) = \lambda$; i.e., α_0 is the longest prefix of $\alpha_0\beta \in AA_0$ which belongs to A. Hence, $\beta\gamma = \beta \in A_0$.

Case 2: $\gamma \neq \lambda$. In this case, there are words $\mu \in AA_0A_1$ and $\gamma_0, \gamma_1 \in \Sigma^*$ such that $\gamma_0\gamma\gamma_1 = \gamma(\mu) \in C$. Then by Claims 1 and 2, for each word $\alpha \in A$, we have

$$\eta = \alpha\beta\gamma\gamma_1\alpha_1(\mu) \in AB_0CA_1 = ABCA_1 = AA_0A_1.$$

Since $\alpha_1(\mu)$ is the longest suffix of η belonging to A_1, we conclude that $\alpha\beta\gamma\gamma_1 \in AA_0$ whence $\alpha\beta\gamma \in AA_0$ for all $\alpha \in A$. Using the same argument as in Case 1, we can show that there is $\alpha_0 \in A$ such that α_0 is the longest prefix of $\alpha_0\beta\gamma$ which belongs to A. As $\alpha_0\beta\gamma \in AA_0$, this yields $\beta\gamma \in A_0$. □

Claim 4. $CA_1 \subseteq B_1$.

Proof. Let $\gamma \in C$ and $\alpha \in A_1$. We have to show that $\gamma\alpha \in B_1$. The following cases are possible:

Case 1: $\gamma = \lambda$. In this case, $\gamma\alpha = \alpha \in A_1 \subseteq B_1$ by assumption.

Case 2: $\gamma \neq \lambda$. In this case, there are words $\mu \in AA_0A_1$ and $\gamma_0, \gamma_1 \in \Sigma^*$ such that $\gamma_0\gamma\gamma_1 = \gamma(\mu) \in C$. Then

$$\eta = \delta(\mu)\gamma_0\gamma\alpha \in AB_0CA_1 = ABCA_1 = AA_0A_1 = AB_0B_1$$

by Claims 1 and 2. Since $\gamma_0\gamma \neq \lambda$, the word $\delta(\mu)$ is the longest prefix of η which belongs to AB_0. Therefore, $\gamma_0\gamma\alpha \in B_1$ whence $\gamma\alpha \in B_1$ as B_1 is a factorial language. □

It follows from Claims 1–4 that the conclusion of (C_1) also holds for A, A_0, A_1, B_0, and B_1. □

From Propositions 3, 4, and Corollary 1 we obtain the following

Corollary 2 [1, Theorem 1]. *Each nontrivial factorial language $L \subseteq \Sigma^*$ has the unique canonical decomposition for an arbitrary finite alphabet Σ.*

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