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Devoted to memory of Alexei Alexandrovich Starobinsky

De Sitter Local Thermodynamics in $f(R)$ Gravity

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We consider the local thermodynamics of the de Sitter state in the $f(\mathcal{R})$ gravity. The local temperature, which is the same for all points of the de Sitter space, is $T = H/\pi$, where H is the Hubble parameter. It is twice larger than the Gibbons–Hawking temperature of the cosmological horizon, $T_{\rm GH} = H/2\pi$. The local temperature is not related to the cosmological horizon. It determines the rate of the activation processes, which are possible in the de Sitter environment. The typical example is the process of the ionization of the atom in the de Sitter environment, which rate is determined by temperature $T = H/\pi$. The local temperature determines the local entropy of the de Sitter vacuum state, and this allows to calculate the total entropy inside the cosmological horizon. The result reproduces the Gibbons–Hawking area law, which corresponds to the Wald entropy, $S_{\rm hor} = 4\pi KA$. Here K is the effective gravitational coupling, $K = df/d\mathcal{R}$. In the local thermodynamic approach, K is the thermodynamic variable, which is conjugate to the Ricci scalar curvature $\mathcal{R}$. The holographic connection between the bulk entropy of the Hubble volume and the surface entropy of the cosmological horizon supports the suggestion that the de Sitter quantum vacuum is characterized by the local thermodynamics with the local temperature $T = H/\pi$. The local temperature $T = H/\pi$ of the de Sitter vacuum suggests that the de Sitter vacuum is locally unstable towards the creation of matter and its further heating. The decay of the de Sitter vacuum due to such processes determines the quantum breaking time of the space-times with positive cosmological constant.

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I. Introduction. The $f(\mathcal{R})$ gravity in terms of the Ricci scalar $\mathcal{R}$ is one of the simplest geometrical models, which describes the dark energy and de Sitter expansion of the Universe [1–7]. It was used to construct an inflationary model of the early Universe – the Starobinsky inflation, which is controlled by the $\mathcal{R}^2$ contribution to the effective action. This class of models, $f(\mathcal{R}) \propto \mathcal{R} - \mathcal{R}^2/M^2$, was also reproduced in the so-called q-theory [8], where q is the 4-form field introduced by Hawking [9] for the phenomenological description of the physics of the deep (ultraviolet) vacuum (here the sign convention for $\mathcal{R}$ is opposite to that in [2]). The Starobinsky model is in good agreement with the observations. However, despite the observational success, the theory of Starobinsky inflation is still phenomenological. Due to a rather small mass scale M compared with the Planck scale it is difficult to embed the model into a ultraviolet (UV) complete theory [10–12].

In this paper we do not discuss the problem of the UV-completion. We consider the de Sitter stage of the expansion of the Universe, and use the $f(\mathcal{R})$ gravity for the general consideration of the local thermodynamics of the de Sitter state. The term "local" means that we consider the de Sitter vacuum as the thermal state, which is characterized by the local temperature. This consideration is based on observation, that matter immersed in the de Sitter vacuum feels this vacuum as the heat bath with the local temperature $T = H/\pi$, where H is the Hubble parameter. This temperature is twice larger than the Gibbons–Hawking one, and it has no relation to the cosmological horizon. The existence of the local temperature suggests the existence of the other local thermodynamic quantities, which participate in the local thermodynamics of the de Sitter state. In addition to the local entropy density s and local vacuum energy density ϵ, there are also the local thermodynamic variables related to the gravitational degrees of freedom.

The $f(\mathcal{R})$ theory demonstrates that the effective gravitational coupling K (it is the inverse Newton constant, $K = 1/16\pi G$) and the scalar curvature $\mathcal{R}$ are connected by equation $K = df/d\mathcal{R}$. This suggests that K and $\mathcal{R}$ are the thermodynamically conjugate variables

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[13, 14]. This pair of the gravitational variables is similar to the pair of the electrodynamic variables, electric field **E** and electric induction **D**, which participate in the thermodynamics of dielectrics.

Example of the influence of the de Sitter vacuum to the matter immersed into this vacuum is provided by an atom in the de Sitter environment. As distinct from the atom in the flat space, the atom in the de Sitter vacuum has a certain probability of ionization. The rate of ionization is similar to the rate of ionization in the presence of the thermal bath with temperature $T = H/\pi$ [15–17]. The same temperature determines the other activation processes, which are energetically forbidden in the Minkowski spacetime, but are allowed in the de Sitter background, see also [18, 19]. That is why it is natural to consider the temperature $T = H/\pi$ as the local temperature of the de Sitter vacuum. Although the local temperature is twice larger than the Gibbons–Hawking temperature assigned to the horizon, $T_{\rm GH} = H/2\pi$, there is the certain connection between the local thermodynamics and the thermodynamics of the event horizon. It appears that the total entropy of the volume V_H bounded by the cosmological horizon coincides with the Gibbons–Hawking entropy, $S_{\rm bulk} = sV_H = A/4G = S_{\rm hor}$. This demonstrates that the local thermodynamics in the $(3+1)$ de Sitter is consistent with the global thermodynamics assigned to the cosmological horizon, although the origin of such bulk-surface correspondence is not very clear.

Here we extended the thermodynamic consideration to the $f(\mathcal{R})$ gravity. Using the local thermodynamics with $T = H/\pi$, we obtained the general result for the total entropy inside the horizon, $S_{\rm bulk} = sV_H = 4\pi KA = S_{\rm hor}$, where $K = df/d\mathcal{R}$ is the effective gravitational coupling. This is in agreement with the global thermodynamics of de Sitter cosmological horizon, which provides the further support for the local thermodynamics with the local temperature $T = H/\pi$ in the de Sitter vacuum in the $(3+1)$-dimensional spacetime.

II. Thermodynamics of the de Sitter state.

A. Local de Sitter temperature. We consider the de Sitter thermodynamics using the Painlevé–Gullstrand (PG) form [20, 21], where the metric in the de Sitter expansion is

$$ds^2 = -dt^2 + (dr - v(r)dt)^2 + r^2 d\Omega^2 \,. \qquad (1)$$

Here the shift velocity is $v(r) = Hr$. This metric is stationary, i.e. does not depend on time, and it does not have the unphysical singularity at the cosmological horizon. That is why it is appropriate for consideration of the local thermodynamics both inside and outside the horizon.

Now let us consider an atom at the origin, $r = 0$. The atom is the external object in the de Sitter spacetime, which is playing the role of the detector (or the role of the static observer) in this spacetime. The electron bounded to an atom may absorb the energy from the gravitational field of the de Sitter background and escape from the electric potential barrier. If the ionization potential is much smaller than the electron mass but is much larger than the Hubble parameter, $H \ll \epsilon_0 \ll m$, one can use the nonrelativistic quantum mechanics to estimate the tunneling rate through the barrier. The corresponding radial trajectory $p_r(r)$ is obtained from the classical equation $p_r^2/2m + p_r v(r) = -\epsilon_0$, where $p_r(r)v(r)$ is the Doppler shift:

$$p_r(r) = -mv(r) + \sqrt{m^2v^2(r) - 2m\epsilon_0}. \qquad (2)$$

The integral of $p_r(r)$ over the classically forbidden region, $0 < r < r_0 = \sqrt{2\epsilon_0/mH^2}$, gives the ionization rate

$$w \sim \exp\left(-2\,{\rm Im}\, S\right) = \exp\left(-\frac{\pi\epsilon_0}{H}\right). \qquad (3)$$

This is equivalent to the thermal radiation with temperature $T = H/\pi$, see also [17].

The same local temperature describes the process of the splitting of the composite particle with mass m into two components with $m_1 + m_2 > m$, which is also not allowed in the Minkowski vacuum [15, 22–24]. In the limit $m \gg H$, the rate of such decay of the composite particle is $w \sim \exp\left(-\frac{\pi(m_1+m_2-m)}{H}\right)$. The similar processes take place in the so-called Cosmological Collider [18, 19], where the new particle created by the Hawking radiation plays the role of the external object which produces the heavy particles. Here there are two different physical processes, which are described by different temperatures. The Hawking radiation from the de Sitter vacuum is determined by the Hawking temperature $T_{\rm GH}$ of the cosmological horizon, while the further process – the splitting of the created particles – is determined by the local temperature $T = 2T_{\rm GH}$.

Moreover, the local temperature $T = H/\pi$ also determines the process of the Hawking radiation from the cosmological horizon and the Gibbons–Hawking temperature $T_{\rm GH} = H/2\pi$. The reason is that in the Hawking process, two particles are coherently created: one particle is created inside the horizon, while its partner is simultaneously created outside the horizon. The rate of the coherent radiation of two particles, each with energy E, is $w \propto \exp(-\frac{2E}{T})$. However, the observer can detect only the particle created inside the horizon. For this observer the creation rate $w \propto \exp(-\frac{2E}{T})$ is perceived as $w \propto \exp(-\frac{E}{T/2}) = \exp(-\frac{E}{T_{\rm GH}})$ with the Gibbons–Hawking temperature $T_{\rm GH} = T/2 = H/2\pi$.

On the contrary, in the local process of the decay of the atom, which is not related to the cosmological horizon, only single particle (electron) is radiated from the atom. This process is fully determined by the local temperature, $w \propto \exp(-\frac{\epsilon_0}{T})$.

B. From local temperature to local entropy. Since the de Sitter state serves as the thermal bath for matter, it is not excluded that the de Sitter quantum vacuum may have its own temperature and entropy [25]. If so, then the quasi-equilibrium states of the expanding Universe is described by two different temperatures: the temperature of the gravitational vacuum and the temperature of the matter degrees of freedom [26]. In this section we discuss the pure de Sitter vacuum without the excited matter ignoring for the moment the thermally activated creation of matter from the vacuum. The excitation and thermalization of matter by the de Sitter thermal bath will be discussed in Section IV.

If the vacuum thermodynamics is determined by the local activation temperature $T = H/\pi$, then in the Einstein gravity with cosmological constant the vacuum energy density is quadratic in temperature:

$$\epsilon_{\rm vac} = \frac{3}{8\pi G} H^2 = \frac{3\pi}{8G} T^2. \tag{4}$$

This leads to the free energy density of the de Sitter vacuum, $F = \epsilon_{\rm vac} - T d\epsilon_{\rm vac}/dT$, which is also quadratic in T, and thus the entropy density $s_{\rm vac}$ in the de Sitter vacuum is linear in T:

$$s_{\rm vac} = -\frac{\partial F}{\partial T} = \frac{3\pi}{4G} T = 12\pi^2 K T. \tag{5}$$

The temperature T and the entropy density $s_{\rm vac}$ are the local quantities which can be measured by the local static observer.

C. Gibbs–Duhem relation. The T^2 dependence of vacuum energy on temperature suggests the modification of the thermodynamic Gibbs–Duhem relation for quantum vacuum and to the reformulation of the vacuum pressure. The conventional vacuum pressure $P_{\rm vac}$ obeys the equation of state $w = -1$ and enters the energy momentum tensor of the vacuum medium in the form:

$$T^{\mu\nu} = \Lambda g^{\mu\nu} = {\rm diag}(\epsilon_{\rm vac}, P_{\rm vac}, P_{\rm vac}, P_{\rm vac}),\ \ P_{\rm vac} = -\epsilon_{\rm vac}. \tag{6}$$

In the de Sitter state the vacuum pressure is negative, $P_{\rm vac} = -\epsilon_{\rm vac} < 0$.

This pressure $P_{\rm vac}$ does not satisfy the standard thermodynamic Gibbs–Duhem relation, $T s_{\rm vac} = \epsilon_{\rm vac} + P_{\rm vac}$, because the right hand side of this equation is zero. The reason for that is that in this equation we did not take into account the gravitational degrees of freedom of quantum vacuum. Earlier it was shown, that gravity contributes with the pair of the thermodynamically conjugate variables: the gravitational coupling $K = \frac{1}{16\pi G}$ and the scalar Riemann curvature $\mathcal{R}$, see [8, 27, 28]. The contribution of the term $K\mathcal{R}$ to thermodynamics is similar to the work density [29–32].

The quantities K and $\mathcal{R}$ can be considered as the local thermodynamic variables, which are similar to temperature, pressure, chemical potential, number density, spin density, etc., in condensed matter physics. Indeed, since the de Sitter spacetime is maximally symmetric, its local structure is characterized by the scalar curvature alone, while all the other components of the Riemann curvature tensor are expressed via $\mathcal{R}$:

$$R_{\mu\nu\alpha\beta} = \frac{1}{12}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right)\mathcal{R}. \tag{7}$$

That is why the scalar Riemann curvature as the covariant quantity naturally serves as one of the thermodynamical characteristics of the macroscopic matter [33, 34]. Another argument is related to the so-called Larkin–Pikin effect [35]. This is the jump in the number of degrees of freedom, when the fully homogeneous state is considered. One has the extra parameters, which are space independent, but participate in thermodynamics [36–38]. The same concerns the constant electric and magnetic fields *in vacuo*, which add three more degrees of freedom. These constant fields are mutually independent, in contrast to the spacetime-dependent fields connected by the Maxwell equations [37]. The scalar curvature $\mathcal{R}$ in the de Sitter vacuum, which is constant in space-time, also serves as such thermodynamic parameter. Then the gravitational coupling $K = df/d\mathcal{R}$ serves as the analog of the chemical potential, which is constant in the full equilibrium.

The new thermodynamic variables, which come from the gravity, and Eq. (5) for the entropy density allow us to introduce the corresponding Gibbs–Duhem relation for de Sitter vacuum, which has the conventional form:

$$T s_{\rm vac} = \epsilon_{\rm vac} + P_{\rm vac} - K\mathcal{R}. \tag{8}$$

This equation is obeyed, since $\epsilon_{\rm vac} + P_{\rm vac} = 0$; $\mathcal{R} = -12H^2$; and $T s_{\rm vac} = 12\pi^2 K T^2 = 12 K H^2$, which supports the earlier proposal that K and $\mathcal{R}$ can be considered as the thermodynamically conjugate variables [27, 28].

The Equation (8) can be also written using the effective vacuum pressure, which absorbs the gravitational degrees of freedom:

$$P = P_{\rm vac} - K\mathcal{R}. \tag{9}$$

Then the conventional Gibbs–Duhem relation is satisfied:

$$Ts_{\rm vac} = \epsilon_{\rm vac} + P. \tag{10}$$

The equation (10) is just another form of writing the Gibbs–Duhem relation (8). But it allows to make different interpretation of the de Sitter vacuum state. The introduced effective de Sitter pressure P is positive, $P = \epsilon_{\rm vac} > 0$, and satisfies equation of state $w = 1$, which is similar to matter with the same equation of state. As a result, due to the gravitational degrees of freedom, the de Sitter state has many common properties with the non-relativistic Fermi liquid, where the thermal energy is proportional to T^2, and also with the relativistic stiff matter with $w = 1$ introduced by Zel'dovich [39].

D. Hubble volume entropy vs entropy of the cosmological horizon. Using the entropy density in Eq. (5), one may find the total entropy of the Hubble volume V_H – the volume surrounded by the cosmological horizon with radius $R = 1/H$:

$$S_{\rm bulk} = s_{\rm vac} V_H = \frac{4\pi R^3}{3} s_{\rm vac} = \frac{\pi}{GH^2} = 4\pi K A = S_{\rm hor}, \tag{11}$$

where A is the horizon area. This Hubble-volume entropy coincides with the Gibbons–Hawking entropy of the cosmological horizon. However, here it is the thermodynamic entropy coming from the local entropy of the de Sitter quantum vacuum, rather than the entropy of the horizon degrees of freedom.

Anyway, the relation between the bulk and surface entropies in the local vacuum thermodynamics suggests some holographic origin. Such bulk-surface correspondence is valid only in the $(3 + 1)$-dimension[2)]. In the general $d+1$ dimension of spacetime, the same approach gives the factor $(d-1)/2$ in the relation between the entropy of the Hubble volume and the Gibbons–Hawking entropy of the cosmological horizon, $S_{\rm bulk} = \frac{d-1}{2} S_{\rm hor}$. This may add to the peculiarities of the $d = 3$ space dimension [40], where in particular the mass dimension of the gravitational coupling, $[K] = d - 1$, coincides with the mass dimension of curvature, $[\mathcal{R}] = 2$.

E. Hubble volume vs the volume of Universe, and thermal fluctuations of de Sitter state. It is not excluded that our Universe is finite. Its volume V might be comparatively small, not much larger than the currently observed Hubble volume V_H [41].

If the Universe is finite and if the de Sitter state represents the excited thermal state of the quantum vacuum, the thermal fluctuations of the deep quantum vacuum may become important. According to Landau–Lifshitz [42], the thermal fluctuations are determined by the compressibility of the system and by its volume. In case of the Universe with the volume V, the fluctuations of the vacuum energy density are given by [43]:

$$\left\langle (\Delta\epsilon_{\rm vac})^2 \right\rangle = \left\langle (\Delta P_{\rm vac})^2 \right\rangle = \frac{T}{V\chi_{\rm vac}}. \tag{12}$$

Here $\chi_{\rm vac}$ is the vacuum compressibility – the compressibility of the fully equilibrium Minkowski vacuum with $\epsilon_{\rm vac} = -P_{\rm vac} = 0$. As for the quantum fluctuations, their contribution to the vacuum energy density is typically on the order of $M_{\rm Pl}^4$, where $M_{\rm Pl}$ is the Planck mass. But in the equilibrium vacuum this contribution is cancelled by the trans-Planckian degrees of freedom due to thermodynamic Gibbs–Duhem relation [37, 43]. On the other hand, the vacuum compressibility is determined by the Planck energy scale, $\chi_{\rm vac} \sim 1/M_{\rm Pl}^4$.

In the excited vacuum – the de Sitter state with the temperature $T = H/\pi$ and the energy density $<\epsilon_{\rm vac}> \sim M_{\rm Pl}^2 H^2$ – the relative magnitude of thermal fluctuations is determined by the ratio of the Hubble volume to the volume of the Universe:

$$\frac{\left\langle (\Delta\epsilon_{\rm vac})^2 \right\rangle}{\left\langle \epsilon_{\rm vac} \right\rangle^2} \sim \frac{V_H}{V}. \tag{13}$$

The volume of the present Universe exceeds the Hubble volume, $V > V_H$, and thus the thermal fluctuations of the vacuum energy density are still relatively small.

III. Thermodynamics of de Sitter state in $f(R)$ gravity.

A. Gibbs–Duhem relation in $f(R)$ gravity. Let us show that equation $S_{\rm hor} = 4\pi K A$ remains valid also in the $f(\mathcal{R})$ gravity, but with the gravitational coupling determined as the thermodynamic conjugate to the curvature. In the $f(\mathcal{R})$ gravity the action is:

$$S = -\int d^4x \sqrt{-g} f(\mathcal{R})\,. \tag{14}$$

The generalization of the modified Gibbs–Duhem relation for the de Sitter states (i.e. for the states with constant four-dimensional curvature) in the $f(\mathcal{R})$ gravity is:

$$Ts_{\rm vac} = \epsilon_{\rm vac} + P_{\rm vac} - K\mathcal{R} = -K\mathcal{R}, \tag{15}$$

$$\epsilon_{\rm vac} = f(\mathcal{R}) - K\mathcal{R},\ K = \tfrac{df}{d\mathcal{R}}. \tag{16}$$

Here K is the natural definition of the variable, which is thermodynamically conjugate to the curvature $\mathcal{R}$, while $\epsilon_{\rm vac}$ serves as the corresponding thermodynamic potential. In the equilibrium de Sitter state the curvature is determined by equation:

$$2f(\mathcal{R}) = \mathcal{R}\frac{df}{d\mathcal{R}}\,. \tag{17}$$

[2)] I thank the referee for this comment.

B. Entropy of cosmological horizon in terms of effective gravitational coupling. The value of the local entropy of the de Sitter state $s_{\rm vac}$ follows from Eq. (15), assuming that the local temperature of the equilibrium dS states is $T = H/\pi$. Then the total entropy of the Hubble volume V_H is given by the same Eq. (11):

$$S_{\rm bulk} = s_{\rm vac} V_H = 4\pi K A = S_{\rm hor}. \quad (18)$$

But now K is the effective gravitational coupling in Eq. (16). This generalization of the Gibbons–Hawking entropy was discussed in [8, 44–46]. But here it was obtained using the local thermodynamics of the de Sitter vacuum. This demonstrates that the local thermodynamics of the de Sitter vacuum is valid also for the $f(\mathcal{R})$ gravity. The effective gravitational coupling K serves as one of the thermodynamic variable of the local thermodynamics. This quantity plays the role of the chemical potential, which is thermodynamically conjugate to the curvature $\mathcal{R}$, and it is constant in the thermodynamic equilibrium state of de Sitter spacetime.

For illustration, we consider an example of the modification of the gravitational coupling K in the de Sitter environment. In the conventional Einstein gravity, where $f(\mathcal{R}) = K_0\mathcal{R} + \Lambda$, the de Sitter state has the equilibrium value of the curvature, $\mathcal{R}_0 = -2\Lambda/K_0 = -12H^2$. Let us add the quadratic term to the Einstein action [44, 8]:

$$f(\mathcal{R}) = K_0\mathcal{R} - p\mathcal{R}^2 + \Lambda. \quad (19)$$

Then one obtains the following equations for the equilibrium value of the curvature $\mathcal{R}_0$, the entropy of the Hubble volume $S_{\rm hor}$ and the equilibrium value of the effective coupling K:

$$\mathcal{R}_0 = -2\frac{\Lambda}{K_0} = -12H^2, \quad (20)$$

$$S_{\rm hor} = s_{\rm vac} V_H = 4\pi K A, \quad (21)$$

$$K = \frac{df}{d\mathcal{R}}\Big|_{\mathcal{R}=\mathcal{R}_0} = K_0 + p\frac{\Lambda}{K_0}. \quad (22)$$

The equilibrium curvature in the de Sitter space $\mathcal{R}_0$ is obtained from Eq. (17). It is the same as in Einstein gravity, because the quadratic terms in Eq. (17) are cancelled. The local entropy $s_{\rm vac}$, which follows from Eq. (15), is determined by the modified gravitational coupling K. As a result, the entropy of the Hubble volume in Eq. (21), which we identify with the entropy of the horizon $S_{\rm hor}$, is also determined by the modified coupling K. The latter is given by Eq. (22).

The local entropy $s_{\rm vac}$ changes sign for $K < 0$, while the cosmological expansion is still described by the de Sitter metric. However, the negative K requires the negative parameter $p < 0$, which marks the instability of such de Sitter vacuum [44].

IV. Local temperature and de Sitter decay. The extension of the thermodynamics to the $f(\mathcal{R})$ gravity supports the idea that the de Sitter vacuum is the thermal state with the local temperature $T = H/\pi$. On the other hand the nonzero local temperature of the vacuum suggests that the de Sitter vacuum is locally unstable towards the creation of thermal matter from the vacuum by thermal activation. This is distinct from the mechanism of creation of the pairs of particles by Hawking radiation from the cosmological horizon, which may or may not lead to the decay of the vacuum energy. There are still controversies concerning the stability of the de Sitter vacuum caused by Hawking radiation, see, e.g., [38, 47, 48] and references therein.

To describe the decay of the vacuum due to activation and thermalization of matter, the extension of the Starobinsky analysis of the vacuum decay [49–52] is needed. The thermal exchange between the vacuum and the excited matter generates the thermal relativistic gas. The temperature of relativistic gas tends to approach the temperature $T = H/\pi$ of the de Sitter heat bath. Then the matter energy density ϵ_M tends to approach the value $\epsilon_M \sim T^4$, i.e. due to the heating in the de Sitter thermal bath the matter energy density tends to approach the local thermal equilibrium, $\epsilon_M \to bH^4$, where the dimensionless parameter b depends on the number of the massless relativistic fields. The energy exchange between the vacuum heat bath and matter can be described by the following dynamical modification of the Friedmann equations [53], where the dissipative Hubble friction equation $\partial_t \epsilon_M = -4H\epsilon_M$ is extended to

$$\partial_t \epsilon_M = -4H(\epsilon_M - bH^4). \quad (23)$$

This equation describes the tendency of matter to approach the local temperature of the vacuum, $T = H/\pi$. The extra gain of the matter energy, $4bH^5$, must be compensated by the corresponding loss of the vacuum energy:

$$\partial_t \epsilon_{\rm vac} = -4bH^5. \quad (24)$$

Here we use for simplicity the conventional general relativity with $\epsilon_{\rm vac} = \Lambda = -\frac{1}{2}K_0\mathcal{R}$. This phenomenological description of the energy exchange between vacuum and matter does not depend on the details of the microscopic (UV) theory, and requires only the condition for slow variation of the Hubble parameter, $\dot{H} \ll H^2$.

Since the vacuum energy density is $\epsilon_{\rm vac} \propto KH^2$, one obtains from Eq. (24) the following time dependence of the Hubble parameter and of energy densities:

$$H \sim M_{\rm Pl} \left(\frac{t_{\rm Pl}}{t+t_0}\right)^{1/3}, \quad (25)$$

$$\epsilon_{\rm vac} \sim M_{\rm Pl}^4 \left(\frac{t_{\rm Pl}}{t+t_0}\right)^{2/3}, \quad (26)$$

$$\epsilon_M = bH^4 \sim M_{\rm Pl}^4 \left(\frac{t_{\rm Pl}}{t+t_0}\right)^{4/3}. \quad (27)$$

Here $M_{\rm Pl}$ is the Planck mass, $M_{\rm Pl}^2 = K$, and $t_{\rm Pl} = 1/M_{\rm Pl}$ is Planck time. We assume that $t_0 \gg t_{\rm Pl}$, and thus $\dot{H} \ll H^2$.

Thus the thermal character of the de Sitter state determines the process of its decay. The obtained power law decay of H in Eq. (25) agrees with that found in [54–58]. In [55–57] the parameter t_0 is related to the initial value of the Hubble parameter at the beginning of inflation at $t = 0$:

$$H(t=0) \sim M_{\rm Pl} \left(\frac{t_{\rm Pl}}{t_0}\right)^{1/3} \ll M_{\rm Pl}. \quad (28)$$

This $H(t = 0)$ corresponds to the scaleron mass M in Starobinsky inflation. The time $t_0 \sim E_{\rm Pl}^2/H_{t=0}^3$ is called the quantum breaking time of space-times with positive cosmological constant [59, 60].

V. Conclusion. The local thermodynamics of the de Sitter state in the Einstein gravity gives rise to the Gibbons–Hawking area law for the total entropy inside the cosmological horizon. Here we extended the consideration of the local thermodynamics to the $f(\mathcal{R})$ gravity. We obtained the same area law, but with the modified gravitational coupling $K = df/d\mathcal{R}$, which is in agreement with the global thermodynamics. This supports the suggestion that the de Sitter vacuum is the thermal state with the local temperature $T = H/\pi$, and that the local thermodynamics is based on the thermodynamically conjugate gravitational variables K and $\mathcal{R}$. The variable K plays the role of the chemical potential, which is constant in the thermal equilibrium.

The local temperature $T = H/\pi$ has the definite physical meaning. It is temperature, which is experienced by the external object in the de Sitter environment. In particular, this temperature determines the local activation processes, such as the process of ionization of an atom in the de Sitter environment. The nonzero local temperature of the de Sitter state suggests the thermal instability of this state due to the thermalization of matter. The process of thermalization of matter with the corresponding decay of the vacuum energy density determines the quantum breaking time of the space-times with positive cosmological constant.

The connection between the bulk entropy of the Hubble volume, and the surface entropy of the cosmological horizon suggests a kind of the bulk-surface correspondence, which may have the holographic origin [61–63]. It would be interesting to check this correspondence using the more general extensions of the Einstein gravity and also different types of the generalized entropy [32, 64–66].

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