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CONDENSED MATTER

Pairing and Collective Excitations in Ising Superconductors

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Two-dimensional Ising superconductivity formed in $NbSe_2$, MoS_2, WS_2, etc. transition-metal dichalcogenides is considered. For the superconducting state, the effective low-energy action for phases of the order parameters has been obtained and collective modes in the system have been studied. It has been shown that the system contains not only the Goldstone mode but also the Leggett mode with a mass related to the difference between the singlet and triplet pairing constants. The effect of a low magnetic field parallel to the plane of the system has also been discussed.

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1. INTRODUCTION

Two-dimensional transition-metal dichalcogenides have recently attracted attention due to their extraordinary properties and to their hypothesized potential application in nanoelectronics (see, e.g., [1] and reference therein). These materials have the general formula MX_2, where M is a transition metal (niobium Nb, molybdenum Mo, and tungsten W) and X is a selenium Se, sulfur S, or tellurium Te atom. There are several isomers with such a chemical formula, but isomers with the unit cell consisting of the M atom between two X atoms are considered in this work. As a result, a lattice with the D_{3h} symmetry, where the center of inversion is absent, is formed [2–4].

In this work, the main interest is focused on superconductivity observed in such systems, in particular, in $NbSe_2$ [5], MoS_2 [6, 7], and WS_2 monolayers [8]. Superconductivity in such systems has a number of interesting properties due to the strong internal spin–orbit coupling. In particular, it leads both to the orientation of electron spins perpendicular to the system plane and to the stability of the superconducting order with respect to the in-plane magnetic field. For this reason, such systems are called Ising superconductors. Unconventional superconductivity with triplet pairing between electrons can also be observed in such systems [3, 4].

It is noteworthy that systems considered in this work are purely two-dimensional materials, where various fluctuations play a significant role, and the formation of a superconducting state should occur through the Berezinskii–Kosterlitz–Thouless transition, where quantum vortices play a key role [9].

The aim of this work is to discuss a possible superconducting state in such systems and to derive an effective low-energy action describing fluctuations in Ising superconductors. The spectrum of low-energy long-wavelength collective excitations with frequencies smaller than the superconducting gap is also determined and analyzed. Possible consequences from the presented physical picture of the superconducting state are finally discussed.

2. ISING SUPERCONDUCTOR

In this work, $NbSe_2$, MoS_2, WS_2, etc. transition-metal dichalcogenide monolayers are considered. The center of inversion is absent in these materials, but time reversal invariance nevertheless holds. The band Hamiltonian of this system has the general form

$$\hat{H}_{\rm TMD} = \sum_{\substack{\alpha,\beta=\uparrow,\downarrow \\ i=x,y,z}} \int_{BZ} \frac{d^2\mathbf{p}}{(2\pi)^2}(\varepsilon_{\mathbf{p}}\delta_{\alpha\beta} + \gamma^i_{\mathbf{p}}\sigma^i_{\alpha\beta})\hat{c}^{\dagger}_{\mathbf{p},\alpha}\hat{c}_{\mathbf{p},\beta}. \quad (1)$$

Here, the internal Zeeman spin–orbit coupling with $\gamma^i_{\mathbf{p}} = -\gamma^i_{-\mathbf{p}}$ is present in addition to the standard term with $\varepsilon_{\mathbf{p}} = \varepsilon_{-\mathbf{p}}$. Due to symmetry, $\gamma_{\mathbf{p}}$ is directed along the z axis perpendicular to the sample plane. Figure 1 shows the Brillouin zone of this system and Fermi contours for different directions of the (pseudo)spin of an electron excitation. The sign of the spin–orbit splitting varies from point to point inside the Brillouin zone in such a way that the time reversal invariance is not broken and the Kramers degeneracy of the spectrum exists. Each state with the quasimomentum $\mathbf{p}$ corresponds to the state with the quasimomentum $-\mathbf{p}$, opposite spin, and the same energy. The valence band

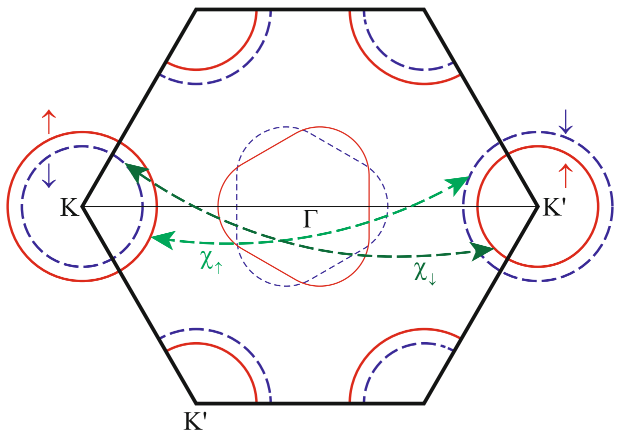

Fig. 1. (Color online) Brillouin zone of the system under consideration. Solid (dashed) lines are the Fermi contours for spin-up (spin-down) states. Two types of pairing are also illustrated.

crosses the Fermi level several times, forming pockets near the K, K', and Γ points. The authors of [2–4, 10] discussed that electrons near the K and K' points form one subsystem, whereas electrons near the Γ point constitute the other subsystem, and these two systems are weakly coupled to each other. Furthermore, the superconducting gap in the Γ subsystem can be much smaller [10]. For this reason, only states near the Fermi surface in the vicinities of the K and K' points and superconducting states formed at the interaction between them. The effective Hamiltonian of this subsystem has the form

$$\hat{H}_0 = \sum_{\alpha=\uparrow,\downarrow}\int\frac{d^2\mathbf{p}}{(2\pi)^2}\left(\frac{(\mathbf{p}+\mathbf{K})^2}{2m_\alpha} - \varepsilon_\alpha^F\right)\hat{a}^\dagger_{\mathbf{p},\alpha}\hat{a}_{\mathbf{p},\alpha} + \sum_{\alpha=\uparrow,\downarrow}\int\frac{d^2\mathbf{p}}{(2\pi)^2}\left(\frac{(\mathbf{p}-\mathbf{K})^2}{2m_{-\alpha}} - \varepsilon_{-\alpha}^F\right)\hat{b}^\dagger_{\mathbf{p},\alpha}\hat{b}_{\mathbf{p},\alpha}. \quad (2)$$

Here, $\hat{a}^\dagger_{\mathbf{p},\alpha}, \hat{a}_{\mathbf{p},\alpha}(\hat{b}^\dagger_{\mathbf{p},\alpha}, \hat{b}_{\mathbf{p},\alpha})$ are the operators describing states near the K (K') point, m_α and $m_{-\alpha}$ are the effective masses, and $\varepsilon_\uparrow^F = \varepsilon^F - \eta_{SO}$ and $\varepsilon_\downarrow^F = \varepsilon^F + \eta_{SO}$ ($\varepsilon_\downarrow^F = \varepsilon^F + \eta_{SO}$ and $\varepsilon_\uparrow^F = \varepsilon^F - \eta_{SO}$), where η_{SO} specifies the magnitude of the internal spin–orbit coupling, are the distances from the energies of electron states with the (pseudo)spin-up and -down near the K (K') point to the bottom of the band, respectively. It is assumed that the effective masses $m_\uparrow$ and $m_\downarrow$ are slightly different; therefore, the densities of states $\nu_\uparrow = m_\uparrow/(2\pi)$ and $\nu_\downarrow = m_\downarrow/(2\pi)$ near the corresponding Fermi surfaces are also slightly different. For further consideration, field operators are introduced in the form

$$\hat{\psi}_{a,\alpha}(\mathbf{x}) = \int\frac{d^2\mathbf{p}}{2\pi}e^{i(\mathbf{p}+\mathbf{K})\mathbf{x}}\hat{a}_{\mathbf{p},\alpha},$$

$$\hat{\psi}_{b,\alpha}(\mathbf{x}) = \int\frac{d^2\mathbf{p}}{2\pi}e^{i(\mathbf{p}-\mathbf{K})\mathbf{x}}\hat{b}_{\mathbf{p},\alpha}.$$

The Coulomb interaction is described by fluctuations of a scalar potential induced by electrons of the system. The interaction term in the Hamiltonian at spatial scales exceeding the lattice constant has the form

$$\hat{H}_\phi = e\int d^2\mathbf{x}\hat{\phi}(\mathbf{x})\left(\hat{\rho}_\uparrow(\mathbf{x}) + \hat{\rho}_\downarrow(\mathbf{x}) - \rho_0\right), \quad (3)$$

where $\hat{\rho}_\alpha(\mathbf{x}) = \hat{\psi}^\dagger_{a,\alpha}(\mathbf{x})\hat{\psi}_{a,\alpha}(\mathbf{x}) + \hat{\psi}^\dagger_{b,\alpha}(\mathbf{x})\hat{\psi}_{b,\alpha}(\mathbf{x})$ is the electron density with the spin α and ρ_0 is the compensating background of ions. The interaction with the external vector potential $\mathbf{A}(\mathbf{x})$ is introduced by the standard long derivative $\nabla \to \nabla - ie\mathbf{A}(\mathbf{x})$.

To examine superconducting properties of the considered system, it is necessary to take into account the pairing interaction between electrons. Strictly speaking, its form and even mechanisms are incompletely known. In particular, ab initio calculations indicate that not only the electron–phonon coupling but also exchange by spin degrees of freedom can play an important role in pairing [10]. The below analysis is based on symmetry reasons. For these reasons, time reversed states can be paired. This means that a spin-up electron from the K valley is paired with a spin-down electron from the K' valley and vice versa. Thus, the system has two types of Cooper pairs with the creation operators (see Fig. 1)

$$\hat{\chi}^\dagger_\uparrow(x) = \hat{\psi}^\dagger_{a,\uparrow}(\mathbf{x})\hat{\psi}^\dagger_{b,\downarrow}(\mathbf{x}), \quad \hat{\chi}^\dagger_\downarrow(\mathbf{x}) = \hat{\psi}^\dagger_{a,\downarrow}(\mathbf{x})\hat{\psi}^\dagger_{b,\uparrow}(\mathbf{x}). \quad (4)$$

It should be emphasized that these types of pairing are independent because electrons from different valleys are paired. These Cooper pairs can form a singlet $\hat{\chi}_s = (\hat{\chi}_\uparrow - \hat{\chi}_\downarrow)/\sqrt{2}$ and a triplet $\hat{\chi}_t = (\hat{\chi}_\uparrow + \hat{\chi}_\downarrow)/\sqrt{2}$ from the trivial representation. In the general case, pairing can occur in all channels allowed by the symmetry of the crystal lattice. In this work, it is assumed that the pairing interaction exists only in the singlet and triplet channels indicated above:

$$\hat{H}_p = g_s\int d^2\mathbf{x}\hat{\chi}^\dagger_s(\mathbf{x})\hat{\chi}_s(\mathbf{x}) + g_t\int d^2\mathbf{x}\hat{\chi}^\dagger_t(\mathbf{x})\hat{\chi}_t(\mathbf{x}). \quad (5)$$

If this part of the Hamiltonian is expressed in terms of electron operators, then it can be seen that the contributions proportional to $g_s + g_t$ enter in the form of the combinat $(g_s + g_t)\hat{\psi}^\dagger_{a,\uparrow}\hat{\psi}^\dagger_{b,\downarrow}\hat{\psi}_{a,\uparrow}\hat{\psi}_{b,\downarrow}$. They can be interpreted as a density–density interaction caused by the exchange by phonons. The a $(g_s - g_t)\hat{\psi}^\dagger_{a,\uparrow}\hat{\psi}^\dagger_{b,\downarrow}\hat{\psi}_{a,\downarrow}\hat{\psi}_{b,\uparrow}$ combinations, which can be attributed to the exchange by spin fluctuations, are also present. When the contribution of such processes is small, the singlet and triplet coupling constants will be close to each other; i.e., $g_s - g_t \ll g_s$. The case where these two coupling constants coincide can be considered as the first approximation, and the case where this difference is small but nonzero can be then analyzed.

3. CASE OF IDENTICAL COUPLING CONSTANTS

The case $g_t = g_s$ is considered first. The subject of this work is the low-energy long-wavelength behavior of the system under consideration, i.e., the behavior at frequencies below the superconducting gap and at spatial scales exceeding the coherence length. The effective action for this case can be derived as follows. First, the partition function is represented in the form of a Grassmann functional integral over coherent states [11]:

$$\mathcal{Z} = \int \mathcal{D}\phi \int \mathcal{D}\bar{\psi}_{a,\alpha}\mathcal{D}\psi_{a,\alpha}\mathcal{D}\bar{\psi}_{b,\alpha}\mathcal{D}\psi_{b,\alpha} e^{-S[\bar{\psi},\psi,\phi]}. \quad (6)$$

Here, the electron degrees of freedom and fluctuations of the electric potential that lead to the Coulomb interaction are taken into account. Fluctuations of the vector potential are neglected, assuming that it is fixed by an external source. The action has the form

$$\begin{aligned} S = {} & S_{\mathrm{EM}}[\phi] - e\rho_0 \int_0^\beta d\tau \int d^2\mathbf{x}\phi \\ & + \sum_\alpha \int_0^\beta d\tau \int d^2\mathbf{x} \bar{\psi}_{a,\alpha}\left(\partial_\tau - \frac{(\nabla - ie\mathbf{A})^2}{2m_\alpha} + ie\phi - \varepsilon_\alpha^F\right)\psi_{a,\alpha} \\ & + \sum_\alpha \int_0^\beta d\tau \int d^2\mathbf{x} \bar{\psi}_{b,\alpha}\left(\partial_\tau - \frac{(\nabla - ie\mathbf{A})^2}{2m_{-\alpha}} + ie\phi - \varepsilon_{-\alpha}^F\right)\psi_{b,\alpha} \\ & - g_s \sum_\alpha \int_0^\beta d\tau \int d^2x \bar{\psi}_{a,\alpha}\bar{\psi}_{b,-\alpha}\psi_{b,-\alpha}\psi_{a,\alpha}. \end{aligned} \quad (7)$$

Here, S_{EM} describes fluctuations of the electric field, i.e., the Coulomb interaction. After the decoupling of each of the pairing interactions by the Hubbard–Stratonovich transformation, the partition function takes the form

$$\begin{aligned} \mathcal{Z} = {} & \int \mathcal{D}\bar{\Delta}_\alpha \mathcal{D}\Delta_\alpha \int \mathcal{D}\bar{\Psi}_\alpha \mathcal{D}\Psi_\alpha e^{-S_{\mathrm{EM}}[\phi]} \\ & \times e^{e\rho_o \int_0^\beta d\tau \int d^2x\phi - \sum_\alpha \int_0^\beta d\tau \int d^2x \left(\frac{|\Delta_\alpha|^2}{g_s} - \bar{\Psi}_\alpha \check{M}_\alpha \Psi_\alpha\right)}. \end{aligned} \quad (8)$$

Here, $\Delta_\alpha = \Delta_\alpha(x, \tau)$, $\alpha = \uparrow, \downarrow$, are complex fields,

$$\bar{\Psi}_\alpha = (\bar{\psi}_{a,\alpha} \;\; \psi_{b,-\alpha}), \quad \Psi_\alpha = \begin{pmatrix} \psi_{a,\alpha} \\ \bar{\psi}_{b,-\alpha} \end{pmatrix} \quad (9)$$

are spinors in the Nambu space, and

$$\begin{aligned} \check{M}_\alpha = {} & -\partial_\tau + \frac{(\nabla - ie\check{\tau}^z\mathbf{A})^2}{2m_\alpha} + \varepsilon_\alpha^F \\ & - ie\phi\check{\tau}^z + \Delta_\alpha\check{\tau}^+ + \bar{\Delta}_\alpha\check{\tau}^-, \end{aligned} \quad (10)$$

where $\check{\tau}^i$ are the standard Pauli matrices. The functional integral, where the integrand is a Gaussian function of the electron degrees of freedom, can be calculated analytically; the resulting effective action for collective degrees of freedom is obtained in the form

$$\begin{aligned} S_{\mathrm{eff}}[\bar{\Delta}, \bar{\Delta}, \phi] = {} & S_{\mathrm{EM}}[\phi] - e\rho_0 \int_0^\beta d\tau \int d^2x\phi \\ & + \sum_\alpha \int_0^\beta d\tau \int d^2\mathbf{x} \frac{|\Delta_\alpha|^2}{g_s} - \sum_\alpha \mathrm{Tr}\log[\check{M}_\alpha]. \end{aligned} \quad (11)$$

This action is similar to the action for a conventional superconductor but this system consists of two subsystems (specified by the subscript α), which interact with each other only through the scalar potential ϕ. Consequently, the behavior of the considered Ising superconductor can be analyzed in terms of the physics of the conventional superconductor. If fluctuations of ϕ are disregarded, each subsystem has its own transition temperature T_α^c below which the superconducting order appears with a nonzero average value $|\Delta_\alpha|$. In the mean field approximation, the magnitudes of the order parameters $\tilde{\Delta}_\alpha \equiv |\Delta_\alpha|$ are determined from the self-consistency equation

$$\frac{1}{\nu_\alpha g_s} = \int_0^{\omega_c} d\xi \frac{\tanh\left(\frac{\sqrt{\xi^2 + \tilde{\Delta}_\alpha^2}}{2T}\right)}{\sqrt{\xi^2 + \tilde{\Delta}_\alpha^2}}, \quad (12)$$

where ω_c is the width of a band near the Fermi surface, where the pairing interaction acts, whereas the phases of the order parameters are indefinite. Unlike the conventional superconductor, the relative phase of the order parameters significantly affects the dominant pairing type because the singlet and triplet order parameters are composed from two independent fields:

$$\Delta_s = \frac{\Delta_\uparrow - \Delta_\downarrow}{\sqrt{2}}, \quad \Delta_t = \frac{\Delta_\uparrow + \Delta_\downarrow}{\sqrt{2}}.$$

In particular, if the relative phase of the order parameters is π, singlet pairing is dominant; otherwise, triplet pairing is dominant. It is also worth noting that both the singlet and the triplet components are always present in a system where the densities of states $\nu_\uparrow$ and $\nu_\downarrow$ on the Fermi surfaces are different. As known, fluctuations of the phase of the order parameter in low-dimensional systems strongly affect the behavior of these systems and even determine the type of the superconducting phase transition [9, 12, 13]. For this reason, the phases Δ_α will be the main low-energy degrees of freedom in this system, whereas fluctuations of the magnitude of the order parameter can be neglected. In this case, small fluctuations of the magnitudes of the order parameters are represented by two Higgs modes with energies starting from $2\tilde{\Delta}_\alpha$. The effective action for the phases of the order parameters can be obtained in the standard way [11, 12, 14]. Only the main steps are described below. First, the field Δ_α is represented in the form

$$\Delta_\alpha(\mathbf{x},\tau) = \tilde{\Delta}_\alpha e^{i\varphi_\alpha(\mathbf{x},\tau)}$$

and the invariance of the trace with respect to unitary transformations is used. After the transformation

$$\check{M}_\alpha \to e^{-i\frac{\varphi_\alpha}{2}\check{\tau}^z}\check{M}_\alpha e^{i\frac{\varphi_\alpha}{2}\check{\tau}^z} = -\partial_\tau - ie\phi\check{\tau}^z - \frac{i\partial_\tau\varphi_\alpha\check{\tau}^z}{2} + \frac{(\nabla - ie\check{\tau}^z\mathbf{A} + i\nabla\varphi_\alpha\check{\tau}^z/2)^2}{2m_\alpha} + \varepsilon^F_\alpha + \tilde{\Delta}_\alpha\check{\tau}^x, \quad (13)$$

the dependence on the phases of the order parameters enters only through gauge-invariant combinations

$$\tilde{\mathbf{A}}_\alpha = \mathbf{A} - \frac{1}{2e}\nabla\varphi_\alpha, \quad \tilde{\phi}_\alpha = \phi + \frac{1}{2e}\partial_\tau\varphi_\alpha, \quad (14)$$

which are small in the limit of interest. To expand in these quantities, it is necessary to introduce the Green's function $\check{G}_\alpha(\mathbf{x},\tau)$ as a solution of the equation

$$\left(-\partial_\tau + \frac{\nabla^2}{2m_\alpha} + \varepsilon^F_\alpha + \tilde{\Delta}_\alpha\check{\tau}^z\right)\check{G}_\alpha(\mathbf{x},\tau) = \delta(\mathbf{x})\delta(\tau), \quad (15)$$

and perturbation operators

$$\check{V}_1 = -ie\phi_\alpha\check{\tau}^z - \frac{ie}{2m_\alpha}\{\nabla,\tilde{\mathbf{A}}_\alpha\}, \quad \check{V}_2 = -\frac{e^2\tilde{\mathbf{A}}^2_\alpha}{2m_\alpha}\check{\tau}^z. \quad (16)$$

The terms of the expansion up to the second order in $\tilde{\phi}_\alpha, \tilde{\mathbf{A}}_\alpha$ have the form

$$\mathrm{Tr}\log[\check{M}_\alpha] \approx -\mathrm{Tr}\log[\check{G}_\alpha] + \mathrm{Tr}[\check{V}_1\check{G}_\alpha] + \mathrm{Tr}[\check{V}_2\check{G}_\alpha] - \frac{1}{2}\mathrm{Tr}[\check{V}_1\check{G}_\alpha\check{V}_1\check{G}_\alpha] + \ldots. \quad (17)$$

The first two terms are local and contains $\rho^\Delta_\alpha = \mathrm{tr}[\check{\tau}^z\check{G}_\alpha(\mathbf{x},\tau)]$, which can be represented as

$$\rho^\Delta_\alpha = 2T\sum_n\int\frac{d^2\mathbf{p}}{(2\pi)^2}\frac{-iz_n - \mathbf{p}^2/(2m_\alpha) + \varepsilon^F_\alpha}{z_n^2 + \omega^2_{\mathbf{p},\alpha}} \quad (18)$$

and tends in the limit $\tilde{\Delta}_\alpha \to 0$ to the doubled number of electrons with the spin α near the K point:

$$\rho_\alpha = 2\int\frac{d^2\mathbf{p}}{(2\pi)^2}N_F(\mathbf{p}^2/(2m_\alpha) - \varepsilon^F_\alpha). \quad (19)$$

Here, $N_F(x) = (e^{x/T}+1)^{-1}$ is the Fermi function. In Eq. (18), $z_n = \pi(2n+1)T$ are the fermion Matsubara frequencies and $\omega_{\mathbf{p},\alpha} = \sqrt{(\mathbf{p}^2/(2m_\alpha) - \varepsilon^F_\alpha)^2 + \tilde{\Delta}^2_\alpha}$ is the energy of quasiparticle excitations. As in the conventional superconductor, $\rho^\Delta_\alpha - \rho_\alpha$ is small in the parameter $(\tilde{\Delta}_\alpha/\varepsilon^F_\alpha)^2$. Due to electroneutrality, the first order correction in $\tilde{\phi}_\alpha$ is cancelled with the compensating contribution from ions $\rho_0 = \rho_\uparrow + \rho_\downarrow$, whereas

$$\mathrm{Tr}[\check{V}_2\check{G}_\alpha] = \frac{e^2\rho^\Delta_\alpha}{2m_\alpha}\int_0^\beta d\tau\int d^2\mathbf{x}\tilde{\mathbf{A}}^2_\alpha(\mathbf{x},\tau). \quad (20)$$

The last term in Eq. (17) is nonlocal and includes various polarization bubbles of the form $\sim\check{G}_\alpha(\mathbf{x}-\mathbf{x}',\tau-\tau')\check{G}_\alpha(\mathbf{x}'-\mathbf{x},\tau'-\tau)$. Since slow long-wavelength fluctuations of the phase are of interest, bubbles can be calculated in the local approximation at zero frequency and momentum. This approximation is valid when the characteristic scale of phase variation is larger than the coherence length ξ_c. The standard calculations [11, 14] give

$$\frac{1}{2}\mathrm{Tr}[\check{V}_1\check{G}_\alpha\check{V}_1\check{G}_\alpha] \approx \int_0^\beta d\tau\int d^2\mathbf{x}(\zeta^\phi_\alpha\tilde{\phi}^2_\alpha + \zeta^\mathbf{A}_\alpha\tilde{\mathbf{A}}^2_\alpha),$$

where

$$\zeta_\alpha = e^2\int\frac{d^2\mathbf{p}}{(2\pi)^2}\left(\frac{\tilde{\Delta}^2_\alpha(1-2N_F(\omega_{\mathbf{p},\alpha}))}{2\omega^3_{\mathbf{p},\alpha}} - \frac{(\omega^2_{\mathbf{p},\alpha} - \tilde{\Delta}^2_\alpha)N'_F(\omega_{\mathbf{p},\alpha})}{\omega^2_{\mathbf{p},\alpha}}\right) \approx \frac{m_\alpha e^2}{2\pi} = e^2\nu_\alpha, \quad (21)$$

$$\zeta^\mathbf{A}_\alpha = \frac{e^2}{2m^2_\alpha}\int\frac{d^2\mathbf{p}}{(2\pi)^2}\mathbf{p}^2N'_F(\omega_{\mathbf{p},\alpha}). \quad (22)$$

The approximate value for ζ_α is valid in the same limit as before when the distances to the bottom of the band ε^F_α are the highest energy parameters in the problem. The sum of all terms gives the expression for the effective action for the phases of the order parameters:

$$S_{\mathrm{eff}}[\varphi_\alpha,\phi] = \sum_\alpha\int_0^\beta d\tau\int d^2\mathbf{x}\left(\frac{\nu_\alpha}{4}(\partial_\tau\varphi_\alpha + 2e\phi)^2 + \frac{\rho^s_\alpha}{8m_\alpha}(\nabla\varphi_\alpha - 2e\mathbf{A})^2\right) + S_{\mathrm{EM}}[\phi]. \quad (23)$$

Here, $\rho^s_\alpha = \rho^\Delta_\alpha + 2m_\alpha\zeta^\mathbf{A}_\alpha/e^2$ has the meaning of the superfluid density of the corresponding electron subsystem. To obtain the resulting action for the phases of the order parameters, it remains to take into account the Coulomb interaction and to average over fluctuations of ϕ.

4. COULOMB INTERACTION

Since the system under consideration is two-dimensional, whereas the scalar potential acts in three-dimensional space, its fluctuations directly depend on the environment of the considered sample. In the simplest case where retardation effects can be neglected and the behavior is determined by electrostatics,

$$S_{\rm EM}[\phi] = \frac{1}{2}\int_0^\beta d\tau \int d^2\mathbf{x} \int d^2\mathbf{y}\,\phi(\mathbf{x},\tau) I(\mathbf{x}-\mathbf{y})\phi(\mathbf{y},\tau), \quad (24)$$

where the form of the kernel $I(\mathbf{x}-\mathbf{y})$ depends on the screening of the considered system. If the screening is strong, $I(\mathbf{x}-\mathbf{y}) = C\delta(\mathbf{x}-\mathbf{y})$, where C is the capacitance per unit area. In the opposite limit, this kernel describes the long-range Coulomb interaction and its Fourier transform is

$$I_{\mathbf{p}} = \int d^2\mathbf{x} e^{i\mathbf{p}\mathbf{x}} I(\mathbf{x}) = \frac{|\mathbf{p}|}{2\pi}. \quad (25)$$

The integration over $\phi(\mathbf{x}, \tau)$ gives the effective action for the phases of the order parameters:

$$\begin{aligned} S_\varphi[\varphi_\alpha] &= \sum_\alpha \int_0^\beta d\tau \int d^2\mathbf{x} \frac{\rho_\alpha^s}{8m_\alpha}(\nabla\varphi_\alpha - 2e\mathbf{A})^2 \\ &+ \sum_{\alpha,\beta}\int_0^\beta d\tau \int d^2\mathbf{x} \int d^2\mathbf{y}\,\partial_\tau\varphi_\alpha K^{\alpha\beta}(\mathbf{x}-\mathbf{y})\partial_\tau\varphi_\beta, \end{aligned} \quad (26)$$

where

$$\int d^2\mathbf{x} e^{i\mathbf{p}\mathbf{x}} K^{\alpha\beta}(\mathbf{x}) = \frac{\nu_\alpha\delta_{\alpha\beta}}{4} - \frac{e^2\nu_\alpha\nu_\beta}{2(I_{\mathbf{p}} + 2e^2(\nu_\uparrow + \nu_\downarrow))}. \quad (27)$$

Since $I_{\mathbf{p}} \ll e^2\nu_\alpha$ in the physically interesting cases, only this limit is considered below. The derived quadratic action describes two coupled collective modes with the dispersion relations

$$\begin{aligned} \omega_{\rm G} &\approx e|\mathbf{p}|\sqrt{\frac{\rho_\uparrow^s m_\downarrow + \rho_\downarrow^s m_\uparrow}{I_p m_\uparrow m_\downarrow}} \\ &\times\left(1 + \frac{(\rho_\uparrow^{s\,2} m_\downarrow^2 \nu_\downarrow + \rho_\downarrow^{s\,2} m_\uparrow^2 \nu_\uparrow) I_{\mathbf{p}}}{4e^2\nu_\uparrow\nu_\downarrow(\rho_\uparrow^s m_\downarrow + \rho_\downarrow^s m_\uparrow)^2} + \ldots\right), \end{aligned} \quad (28)$$

$$\begin{aligned} \omega_{\rm L} &\approx |\mathbf{p}|\sqrt{\frac{(\nu_\uparrow + \nu_\downarrow)\rho_\uparrow^s\rho_\downarrow^s}{2\nu_\uparrow\nu_\downarrow(\rho_\uparrow^s m_\downarrow + \rho_\downarrow^s m_\uparrow)}} \\ &\times\left(1 - \frac{(\rho_\uparrow^s m_\downarrow\nu_\downarrow - \rho_\downarrow^s m_\uparrow\nu_\uparrow)^2 I_{\mathbf{p}}}{4e^2\nu_\uparrow\nu_\downarrow(\nu_\uparrow + \nu_\downarrow)(\rho_\uparrow^s m_\downarrow + \rho_\downarrow^s m_\uparrow)^2} + \ldots\right). \end{aligned} \quad (29)$$

The first collective mode is similar to the Goldstone mode [11], where the phases of the both order parameters oscillate together. It has an acoustic dispersion relation $\omega_{\rm G} \sim |\mathbf{p}|$ if the Coulomb interaction is screened and the square root dispersion relation $\omega_{\rm G} \sim \sqrt{|\mathbf{p}|}$ in the opposite case. The second collective mode is similar to the Leggett mode [15], where the phases of the both order parameters oscillate out-of-phase and the Coulomb interaction hardly affects this mode.

5. CASE OF SLIGHTLY DIFFERENT COUPLING CONSTANTS

Both collective modes obtained in the preceding section are gapless. It means that the phases can freely and independently rotate at $g_{\rm s} = g_{\rm t}$, and the established state can be any superposition of singlet and triplet pairing. The situation is fundamentally different when the coupling constants are not equal. The case $g_{\rm s} \neq g_{\rm t}$ but $|g_{\rm s} - g_{\rm t}|$ is small so that self-consistency equations change slightly is considered below. In this case, the decoupling of the pairing interaction by the Hubbard–Stratonovich transformation should be performed separately in each of the channels by introducing the fields $\Delta_{\rm s}(\mathbf{x}, \tau)$ and $\Delta_{\rm t}(\mathbf{x}, \tau)$ with the action

$$S_\Delta = \int_0^\beta d\tau \int d^2\mathbf{x}\left(\frac{|\Delta_{\rm s}|^2}{g_{\rm s}} + \frac{|\Delta_{\rm t}|^2}{g_{\rm t}}\right). \quad (30)$$

The orthogonal transformation

$$\Delta_{\rm s}(\mathbf{x},\tau) = \frac{\Delta_\uparrow(\mathbf{x},\tau) - \Delta_\downarrow(\mathbf{x},\tau)}{\sqrt{2}}, \quad (31)$$

$$\Delta_{\rm t}(\mathbf{x},\tau) = \frac{\Delta_\uparrow(\mathbf{x},\tau) + \Delta_\downarrow(\mathbf{x},\tau)}{\sqrt{2}} \quad (32)$$

gives the previously used fields $\Delta_\alpha(\mathbf{x}, \tau)$. The electronic part of the action is the same as above and

$$\begin{aligned} S_\Delta &= \frac{g_{\rm s} + g_{\rm t}}{2g_{\rm s}g_{\rm t}}\sum_\alpha \int_0^\beta d\tau \int d^2\mathbf{x}|\Delta_\alpha|^2 \\ &+ \frac{g_{\rm s} - g_{\rm t}}{2g_{\rm s}g_{\rm t}}\int_0^\beta d\tau \int d^2\mathbf{x}\left(\bar{\Delta}_\uparrow\Delta_\downarrow + \bar{\Delta}_\downarrow\Delta_\uparrow\right). \end{aligned} \quad (33)$$

The first term of the action is similar to the previous one but with the coupling constant $g_{\rm p} = 2g_{\rm s}g_{\rm t}/(g_{\rm s} + g_{\rm t})$. The second term is small at $|g_{\rm s} - g_{\rm t}| \ll g_{\rm s}$ and can be considered as perturbation. The same calculations neglecting the contribution of this term to $\tilde{\Delta}_\alpha$ yield an additional contribution to the effective action for the phases of the order parameters in the form

$$\begin{aligned} S_\varphi^{g_{\rm s}\neq g_{\rm t}}[\varphi_\alpha] &= S_\varphi[\varphi_\alpha] \\ &+ \frac{(g_{\rm s} - g_{\rm t})\tilde{\Delta}_\uparrow\tilde{\Delta}_\downarrow}{g_{\rm s}g_{\rm t}}\int_0^\beta d\tau \int d^2\mathbf{x}\cos\left(\varphi_\uparrow - \varphi_\downarrow\right). \end{aligned} \quad (34)$$

According to this expression, the equilibrium phases of the order parameters are no longer independent. In the case $g_{\rm s} > g_{\rm t}$, the minimum of the action is reached at $\varphi_\uparrow = \varphi_\downarrow + \pi$ and the established state is predominantly singlet with a small addition of the triplet component. This additional contribution also affects the spectrum of collective excitations. This contribution slightly affects the Goldstone mode but strongly modifies the Leggett mode:

$$\omega_L \approx \sqrt{\omega_\Delta^2 + \mathbf{p}^2 \frac{(\nu_\uparrow + \nu_\downarrow)\rho_\uparrow^s\rho_\downarrow^s}{2\nu_\uparrow\nu_\downarrow(\rho_\uparrow^s m_\downarrow + \rho_\downarrow^s m_\uparrow)}} \times \left(1 - \frac{(\rho_\uparrow^s m_\downarrow \nu_\downarrow - \rho_\downarrow^s m_\uparrow \nu_\uparrow)^2 I_\mathbf{p}}{4e^2\nu_\uparrow\nu_\downarrow(\nu_\uparrow+\nu_\downarrow)(\rho_\uparrow^s m_\downarrow + \rho_\downarrow^s m_\uparrow)^2} + \ldots\right), \quad (35)$$

$$\omega_\Delta^2 = \frac{2(\nu_\uparrow + \nu_\downarrow)|g_s - g_t|\tilde{\Delta}_\uparrow\tilde{\Delta}_\downarrow}{\nu_\uparrow\nu_\downarrow g_s g_t}. \quad (36)$$

It is seen that the Leggett mode has a gap and the gap width directly depends on the difference of the coupling constants. For the described approximation to be valid, this mode should be low-energy; i.e., $\omega_\Delta \lesssim \tilde{\Delta}_\alpha$ and, thereby, $|g_s - g_t| \lesssim \nu_\alpha g_s^2$. The results are discussed in the next section.

6. DISCUSSION

In this work, the effective low-energy action for the Ising superconductor has been derived under the assumption that the singlet and triplet pairing constants are close to each other. The results obtained allow one to draw the following general picture of the superconducting state in this system. This state has two order parameters corresponding to two subsystems. The magnitudes of these order parameters fluctuate slightly, whereas the phases of the order parameters fluctuate strongly but are coupled to each other. Due to the difference between the singlet and triplet coupling constants, the mode associated with the difference between the phases of the order parameters in the system is gapped. This suppresses mutual fluctuations of the phases φ_α and promotes the establishment of the predominant singlet state (at $g_s > g_t$) with a small addition of the triplet component. The latter component appears only because the magnitudes of the order parameters are slightly different.

Some mechanisms can lead to the aligning of the phases of order parameters. The calculations is this work show that the corresponding contributions to the Hamiltonian of the system should mix electron states from different subsystems. In particular, scattering on nonmagnetic impurities inside a single valley does not change the behavior of the system qualitatively but only leads to the renormalization of the superfluid densities ρ_α^s. On the contrary, intervalley scattering (contributions $\sim\hat{\psi}_{a,\alpha}^\dagger\hat{\psi}_{b,\alpha}$) can affect fluctuations of the difference between the phases of the order parameters. This problem will be studied in detail in future, but it can be expected that this mechanism is suppressed because it corresponds to processes of scattering with a high (quasi)momentum transfer. Magnetic impurities, which can ensure scattering even inside a single valley, can also provide a significant effect [16]. Another mechanism is due to a magnetic field parallel to the system plane (e.g., along the x axis). The corresponding Zeeman contribution to the Hamiltonian is given by the expression (the spin magnetic moment is taken into account in the definition of the field)

$$\hat{H}_B = -B_x \sum_\alpha \int d^2\mathbf{x} \left(\hat{\psi}_{a,\alpha}^\dagger \hat{\psi}_{a,-\alpha} + \hat{\psi}_{b,\alpha}^\dagger \hat{\psi}_{b,-\alpha}\right).$$

As above, B_x is considered to be small, so this contribution can be taken into account by perturbation theory. The resulting addition to the effective action has the form

$$\delta S_B = B_x^2 \mathrm{Tr}[\check{G}_\uparrow \check{\tau}^z e^{i\frac{\varphi_\downarrow - \varphi_\uparrow}{2}\check{\tau}^z} \check{G}_\downarrow \check{\tau}^z e^{i\frac{\varphi_\uparrow - \varphi_\downarrow}{2}\check{\tau}^z}].$$

In the long-wavelength limit, this term leads both to insignificant renormalization of the quadratic part of the effective action and to the additional contribution in the form

$$S_\varphi^B[\varphi_\alpha] = -B_x^2 \kappa \int_0^\beta d\tau \int d^2\mathbf{x} \cos(\varphi_\uparrow - \varphi_\downarrow), \quad (37)$$

where

$$\kappa = \int \frac{d^2\mathbf{p}}{(2\pi)^2} \frac{\tilde{\Delta}_\uparrow\tilde{\Delta}_\downarrow\left(\omega_{\mathbf{p},\uparrow}\tanh\left(\frac{\omega_{\mathbf{p},\downarrow}}{2T}\right) - \omega_{\mathbf{p},\downarrow}\tanh\left(\frac{\omega_{\mathbf{p},\uparrow}}{2T}\right)\right)}{(\omega_{\mathbf{p},\uparrow}^2 - \omega_{\mathbf{p},\downarrow}^2)\omega_{\mathbf{p},\uparrow}\omega_{\mathbf{p},\downarrow}}. \quad (38)$$

Since $\kappa > 0$, the in-plane magnetic field reduces the term with cosine in the effective action and softens the Leggett mode. It can even change the sign of this contribution, which leads to the conversion of the dominant type of pairing from singlet to triplet. The authors of [17] observed an additional peak in the tunnel differential conductivity, which was attributed to the Leggett mode similar to that considered in this work. The above reasoning can be used to verify this fact. If the position of the peak changes in the in-plane magnetic field, it is most likely that the origin of this peak is due to mutual oscillations of the phases of the order parameters. The detailed quantitative analysis of this problem in the general case, as well as the inclusion of the effect of soft modes on the single-electron density of states [13], is a goal for future works.

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CONFLICT OF INTEREST

The author of this work declare that he has no conflicts of interest.

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