ISSN 0021-3640, JETP Letters, 2023, Vol. 117, No. 8, pp. 630–634. © The Author(s), 2023. This article is an open access publication.
ISSN 0021-3640, JETP Letters, 2023. © The Author(s), 2023. This article is an open access publication.

METHODS
OF THEORETICAL PHYSICS

Gauge Equivalence Between 1 + 1 Rational Calogero–Moser Field Theory and Higher Rank Landau–Lifshitz Equation

K. Atalikov[a,b,]* and A. Zotov[a,b,c,]**

[a] *Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, 119991 Russia*
[b] *National Research Center Kurchatov Institute, Moscow, 123182 Russia*
[c] *National Research University Higher School of Economics, Moscow, 119048 Russia*
**e-mail: kantemir.atalikov@yandex.ru*
***e-mail: zotov@mi-ras.ru*

Received March 14, 2023; revised March 14, 2023; accepted March 18, 2023

In this paper we study 1 + 1 field generalization of the rational N-body Calogero–Moser model. We show that this model is gauge equivalent to some special higher rank matrix Landau–Lifshitz equation. The latter equation is described in terms of GL_N rational R-matrix, which turns into the 11-vertex R-matrix in the $N = 2$ case. The rational R-matrix satisfies the associative Yang–Baxter equation, which underlies construction of the Lax pair for the Zakharov–Shabat equation. The field analogue of the IRF-Vertex transformation is proposed. It allows to compute explicit change of variables between the field Calogero–Moser model and the Landau–Lifshitz equation.

DOI: 10.1134/S0021364023600714

1. CALOGERO–MOSER FIELD THEORY

1.1. 1 + 1 Field Generalization[1] of the Calogero–Moser Model

This model was proposed in [1, 2] (see also [3]). The Hamiltonian is given by the expression[2]

$$\begin{aligned}\mathcal{H}^{\mathrm{2dCM}} &= \oint dx\, H^{\mathrm{2dCM}}(x),\\ H^{\mathrm{2dCM}}(x) &= \sum_{i=1}^{N} p_i^2 (c - kq_{ix}) - \frac{1}{Nc}\left(\sum_{i=1}^{N} p_i (c - kq_{ix})\right)^2 - \sum_{i=1}^{N} \frac{k^4 q_{ixx}^2}{4(c - kq_{ix})}\\ &+ \frac{k^3}{2}\sum_{i\neq j}^{N} \frac{q_{ix}q_{jxx} - q_{jx}q_{ixx}}{q_i - q_j} - \frac{1}{2}\sum_{i\neq j}^{N} \frac{1}{(q_i - q_j)^2}[(c - kq_{ix})^2 (c - kq_{jx})\\ &+ (c - kq_{ix})(c - kq_{jx})^2 - ck^2 (q_{ix} - q_{jx})^2],\end{aligned} \tag{1.1}$$

where x is the (space) field variable. It is a coordinate on a unit circle. Dynamical variables are the ($\mathbb{C}$-valued) fields $p_i = p_i(x)$, $q_i = q_i(x)$, $i = 1,\ldots,N$, and the subscript x means derivative with respect to x. For instance, $q_{jxx} = \partial_x^2 q_j(x)$. The parameter $c \in \mathbb{C}$ is a coupling constant and $k \in \mathbb{C}$ is an auxiliary parameter, which can be fixed as $k = 1$ but we keep it as it is. The momenta p_i and coordinates q_j are canonically conjugated fields:

$$\begin{aligned}\{q_i(x), p_j(y)\} &= \delta_{ij}\delta(x - y),\\ \{p_i(x), p_j(y)\} &= \{q_i(x), q_j(y)\} = 0.\end{aligned} \tag{1.2}$$

Equations of motion (the Hamiltonian equations $\dot{f} = \{f, H\}$) take the following form:

$$\dot{q}_i = 2p_i (c - kq_{ix}) - \frac{2}{Nc}\sum_{l=1}^{N} p_l (c - kq_{lx})(c - kq_{ix}), \tag{1.3}$$

$$\begin{aligned}\dot{p}_i &= -2kp_i p_{ix} + \frac{2k}{Nc}\left\{\sum_{l=1}^{N} p_i p_l (c - kq_{lx})\right\}_x + k\left\{\frac{k^3 q_{ixxx}}{2(c - kq_{ix})} + \frac{k^4 q_{ixx}^2}{4(c - kq_{ix})^2}\right\}_x\\ &+ 2\sum_{j:j\neq i}^{N}\left[\frac{k^3 q_{jxxx}}{(q_i - q_j)} - \frac{3k^2 (c - kq_{jx}) q_{jxx}}{(q_i - q_j)^2} - \frac{2(c - kq_{jx})^3}{(q_i - q_j)^3}\right].\end{aligned} \tag{1.4}$$

---

[1] 1 + 1 or 2d means 1 dimension for space variable and 1 dimension for time variable. In this respect mechanics is 0 + 1.
[2] In [1, 2] the elliptic model was considered. In this paper we deal with its rational limit.

The model (1.1) is integrable in the sense that it has algebro-geometric solutions and equations of motion are represented in the Zakharov–Shabat (or Lax or zero curvature) form

$$\partial_t U(z) - k\partial_x V(z) + [U(z), V(z)] = 0, \quad U(z), V(z) \in \mathrm{Mat}(N, \mathbb{C}), \tag{1.5}$$

where U–V pair is a pair $U^{\text{2dCM}}(z)$, $V^{\text{2dCM}}(z)$ of matrix valued functions of the fields $p_j(x)$, $q_j(x)$, $j = 1,\ldots,N$ and their derivatives. They also depend on the spectral parameter z, and (1.5) holds true identically in z (on-shell equations of motion). Explicit expression for U–V pair is as follows:

$$U_{ij}^{\text{2dCM}}(z) = -\delta_{ij}\left(p_i + \frac{\alpha_i^2}{Nz} + \frac{k\alpha_{ix}}{\alpha_i}\right) + (1-\delta_{ij})\alpha_j^2\left(\frac{1}{q_i - q_j} - \frac{1}{Nz}\right), \tag{1.6}$$

$$V_{ij}^{\text{2dCM}}(z) = \delta_{ij}\left[-\frac{q_{it}}{Nz} - \frac{c\alpha_i^2}{Nz^2} + \tilde{m}_i^0 - \frac{\alpha_{it}}{\alpha_i}\right] + (1-\delta_{ij})\alpha_j^2\left[\frac{c}{z}\left(\frac{1}{q_i - q_j} - \frac{1}{Nz}\right) - Nc\left(\frac{1}{q_i - q_j}\right)^2 - \tilde{m}_{ij}\left(\frac{1}{q_i - q_j} - \frac{1}{Nz}\right)\right], \tag{1.7}$$

where

$$\alpha_i^2 = kq_{ix} - c, \quad i = 1,\ldots,N \tag{1.8}$$

and

$$\tilde{m}_i^0 = p_i^2 + \frac{k^2\alpha_{ixx}}{\alpha_i} + 2\kappa p_i - \sum_{j:j\neq i}^{N}\left[\frac{2\alpha_j^4 + \alpha_i^2\alpha_j^2}{(q_i - q_j)^2} + \frac{4k\alpha_j\alpha_{jx}}{q_i - q_j}\right],$$
$$\kappa = -\frac{1}{Nc}\sum_{l=1}^{N} p_l\,(c - kq_{lx}),$$
$$\tilde{m}_{ij} = p_i + p_j + 2\kappa + \frac{k\alpha_{ix}}{\alpha_i} - \frac{k\alpha_{jx}}{\alpha_j} - \sum_{k:k\neq i,j}^{N}\alpha_k^2\left(\frac{1}{q_i - q_k} + \frac{1}{q_k - q_j} - \frac{1}{q_i - q_j}\right). \tag{1.9}$$

In what follows we assume the center of mass frame:

$$\sum_{k=1}^{N} q_k = 0. \tag{1.10}$$

Notice that in our previous paper on this topic [4] we used slightly different normalization coefficients and the gauge choice for U–V pair, which was more convenient for the case $N = 2$ when $q_1 = -q_2$.

1.2. Limit to 0 + 1 Mechanics

The finite-dimensional classical mechanics appears in the limit $k \to 0$. All the fields become independent of x, and the field Poisson brackets turn into the ordinary Poisson brackets for mechanical N-body system:

$$\{q_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0. \tag{1.11}$$

The Hamiltonian density (1.1) in this limit provides the ordinary Calogero–Moser model [5, 6]:

$$H^{\text{2dCM}}\Big|_{k=0} = 2cH^{\text{CM}} - \frac{c}{N}\left(\sum_{i=1}^{N} p_i\right)^2 = 2cH^{\text{CM}}, \quad H^{\text{CM}} = \sum_{k=1}^{N}\frac{p_k^2}{2} - \frac{1}{2}\sum_{i\neq j}^{N}\frac{c^2}{(q_i - q_j)^2}, \tag{1.12}$$

where $|_{k=0}$ on the left-hand side assumes also transition to x-independent variables. Similarly, the Zakharov–Shabat equation (1.5) reduces to the Lax equation:

$$\partial_t L^{\text{CM}}(z) + [L^{\text{CM}}(z), M^{\text{CM}}(z)] = 0, \quad L^{\text{CM}}(z), M^{\text{CM}}(z) \in \mathrm{Mat}(N, \mathbb{C}),$$
$$L_{ij}^{\text{CM}}(z) = U_{ij}^{\text{2dCM}}(z)\Big|_{k=0} = \delta_{ij}\left(-p_i + \frac{c}{Nz}\right) - (1-\delta_{ij})c\left(\frac{1}{q_i - q_j} - \frac{1}{Nz}\right), \tag{1.13}$$
$$M^{\text{CM}}(z) = V^{\text{2dCM}}(z)\Big|_{k=0} = (L^{\text{CM}}(z))^2 + M'(z),$$
$$M'_{ij}(z) = -\delta_{ij}\sum_{k:k\neq i}^{N}\frac{2c^2}{(q_i - q_k)^2} + (1-\delta_{ij})\frac{2c^2}{(q_i - q_j)^2}.$$

1.3. Purpose of the Paper

The 1 + 1 field generalizations under consideration are widely known for the Toda chains [7]. For the relativistic models of Ruijsenaars–Schneider type the field generalizations were proposed recently in [8]. In [3] the results of [1, 2] were extended to (multi)spin generalizations of the Calogero–Moser model. It was also explained (using modification of bundles and the symplectic Hecke correspondence) that the field Calogero–Moser system should be gauge equivalent to some model of Landau–Lifshitz type. That is, there exist a gauge transformation $G(z) \in \mathrm{Mat}(N, \mathbb{C})$, which transforms U–V pair for the field Calogero–Moser model to the one for some Landau–Lifshitz type model:

$$U^{\text{LL}}(z) = G(z)U^{\text{2dCM}}(z)G^{-1}(z) + k\partial_x G(z)G^{-1}(z). \tag{1.14}$$

For the $N = 2$ case explicit construction of the matrix $G(z)$ and the change of variables was derived in [4], and the Landau–Lifshitz model for GL_2 rational R-matrix was derived in [9]. The goal of this article is to define the gauge transformation in gl_N case, describe the corresponding Landau–Lifshitz type model and find explicit change of variables using relation (1.14).

2. RATIONAL TOP AND LANDAU–LIFSHITZ EQUATION

2.1. Rational Integrable Top

In order to explain what kind of Landau–Lifshitz model is expected in (1.14) we first consider its 0 + 1 mechanical analogue. The mechanical version of (1.14) is as follows:

$$L^{\mathrm{top}}(z) = g(z)L^{\mathrm{CM}}(z)g^{-1}(z), \tag{2.1}$$

where $L^{\mathrm{top}}(z)$ is the Lax matrix of some integrable top like model. It is the model, which was introduced in [10] and called the rational top. Equations of motion for top like models are of the form

$$\begin{aligned} \partial_t S &= \{S, H^{\mathrm{top}}\} = 2c[S, J(S)], \\ S &= \sum_{i,j=1}^{N} E_{ij}S_{ij} \in \mathrm{Mat}(N, \mathbb{C}), \end{aligned} \tag{2.2}$$

where S is a matrix of dynamical variables (E_{ij} is the standard matrix basis), $c \in \mathbb{C}$ is a constant and $J(S)$ is some special linear map (see [10]). The Hamiltonian is quadratic, and the Poisson brackets are given by the Poisson–Lie structure on gl_N^* Lie coalgebra:

$$\begin{aligned} H^{\mathrm{top}} &= cN\mathrm{tr}(SJ(S)), \\ \{S_{ij}, S_{kl}\} &= \frac{1}{N}(S_{il}\delta_{kj} - S_{kj}\delta_{il}). \end{aligned} \tag{2.3}$$

It was shown in [10] that in the special case $\mathrm{rk}(S) = 1$ (and $\mathrm{tr}(S) = c$) this model is gauge equivalent (2.1) to the rational Calogero–Moser model. Namely, it was proved by direct evaluation that the expression on the right-hand side of (2.1) is represented in the form

$$\begin{aligned} g(z)L^{\mathrm{CM}}(z)g^{-1}(z) &= \mathrm{tr}_2\left(r_{12}(z)\overset{2}{S}\right), \\ \overset{2}{S} &= 1_N \otimes S, \end{aligned} \tag{2.4}$$

where $S_{ij} = S_{ij}(p_1,\ldots,p_N,q_1,\ldots,q_N,c)$, $r_{12}(z)$ is some classical non-dynamical r-matrix (satisfying the classical Yang–Baxter equation), 1_N is the identity $N \times N$ matrix and tr_2 means trace over the second tensor component in $\mathrm{Mat}(N,\mathbb{C})^{\otimes 2}$. The gauge equivalence means that the Hamiltonians H^{top} (2.3) and H^{CM} (1.12) coincide under a certain change of variables, which will be given below in (2.15).

2.2. Description through R-Matrix

In [11] a construction of Lax pairs with spectral parameter was suggested based on (skew-symmetric and unitary) solution of the associative Yang–Baxter equation [12, 13]:

$$\begin{aligned} R_{12}^{\hbar}R_{23}^{\eta} &= R_{13}^{\eta}R_{12}^{\hbar-\eta} + R_{23}^{\eta-\hbar}R_{13}^{\hbar}, \\ R_{ab}^{x} &= R_{ab}^{x}(z_a - z_b). \end{aligned} \tag{2.5}$$

In fact, a skew-symmetric and unitary solution of (2.5) in the fundamental representation of GL_N Lie group is a quantum R-matrix; i.e., it satisfies also the quantum Yang–Baxter equation $R_{12}^{\hbar}R_{13}^{\hbar}R_{23}^{\hbar} = R_{23}^{\hbar}R_{13}^{\hbar}R_{12}^{\hbar}$. Consider the classical limit expansion of such R-matrix:

$$R_{12}^{\hbar}(z) = \frac{1}{\hbar}1_N \otimes 1_N + r_{12}(z) + \hbar\, m_{12}(z) + O(\hbar^2). \tag{2.6}$$

Then the Lax pair can be written as follows:

$$\begin{aligned} L^{\mathrm{top}}(z) &= \mathrm{tr}_2\left(r_{12}(z)\overset{2}{S}\right), \\ M^{\mathrm{top}}(z) &= -\mathrm{tr}_2\left(m_{12}(z)\overset{2}{S}\right). \end{aligned} \tag{2.7}$$

It generates the Euler–Arnold equation (2.2) with

$$J(S) = \mathrm{tr}_2\left(m_{12}(0)\overset{2}{S}\right). \tag{2.8}$$

2.3. Rational R-Matrix

In this paper we will use the rational R-matrix calculated in [14]. In the $N = 2$ case it reproduces the 11-vertex R-matrix found by I. Cherednik [15]:

$$R_{12}^{\hbar}(z) = \begin{pmatrix} 1/\hbar + 1/z & 0 & 0 & 0 \\ -z - \hbar & 1/\hbar & 1/z & 0 \\ -z - \hbar & 1/z & 1/\hbar & 0 \\ -z^3 - \hbar^3 - 2z^2\hbar - 2z\hbar^2 & z + \hbar & z + \hbar & 1/\hbar + 1/z \end{pmatrix}. \tag{2.9}$$

For $N > 2$ all its properties, different possible forms and explicit expressions for the coefficients of expansions (2.6) and (2.18) can be found in [16].

2.4. Rational IRF-Vertex Transformation

Following [10] introduce the matrix $g(z) \in \mathrm{Mat}(N,\mathbb{C})$:

$$\begin{aligned} g(z) = g(z, q_1, \ldots, q_N) &= \Xi(z,q)D^{-1}(q), \\ \Xi(z,q), D(q) &\in \mathrm{Mat}(N,\mathbb{C}), \end{aligned} \tag{2.10}$$

where

$$D_{ij}(q) = \delta_{ij}\prod_{k\neq i}^{N}(q_i - q_k),$$
$$\Xi_{ij}(z,q) = (z+q_j)^{\varrho(i)}, \qquad (2.11)$$
$$\sum_{k=1}^{N} q_k = 0,$$

with

$$\varrho(i) = \begin{cases} i-1 & \text{for} \quad 1 \le i \le N-1, \\ i & \text{for} \quad i = N, \end{cases}$$

$$\varrho^{-1}(i) = \begin{cases} i+1 & \text{for} \quad 0 \le i \le N-2, \\ i & \text{for} \quad i = N. \end{cases} \qquad (2.12)$$

The matrix $\Xi(z)$ is degenerated at $z = 0$: $\det \Xi(z,q) = Nz\prod_{i>j}^{N}(q_i - q_j)$. It plays the role of IRF-Vertex transformation for rational R-matrices [16]. The inverse of matrix $g(z,q)$ is as follows:

$$g^{-1}_{kj}(z,q) = (-1)^{\varrho(j)}\left(\frac{\sigma_{\varrho(j)}(x)}{Nz} - \overset{k}{\sigma}_{\varrho(j)}(x)\right),$$
$$x_j = z + q_j, \qquad (2.13)$$

where $\sigma_j(x)$ and $\overset{k}{\sigma}_j(x)$ are symmetric functions (for variables $x_1, \ldots, x_N$) defined as

$$\prod_{m=1}^{N}(\zeta - x_m) = \sum_{k=0}^{N}(-1)^k\zeta^k\sigma_k(x_1,\ldots,x_N),$$
$$\prod_{m:m\neq k}^{N}(\zeta - x_m) = -\sum_{s=0}^{N-1}(-1)^s\zeta^s\overset{k}{\sigma}_s(x). \qquad (2.14)$$

Details can be found in [10, 16]. The latter formula provides via (2.1), (2.4) explicit change of variables in 0 + 1 mechanics between the Calogero–Moser model given by Eq. (1.13) and the rational top specified by Eqs. (2.2), (2.3), (2.7), and (2.8):

$$S_{ij} = \frac{(-1)^{\varrho(j)}}{N}$$
$$\times \sum_{m=1}^{N}\frac{-(q_m)^{\varrho(i)}\breve{p}_m + c\varrho(i)(q_m)^{\varrho(i)-1}}{\prod_{l\neq m}(q_m - q_l)}\sigma_{\varrho(j)}(q),$$
$$\breve{p}_j = p_j + \sum_{l:l\neq j}\frac{c}{q_j - q_l}. \qquad (2.15)$$

Similar results are known for trigonometric [17] and elliptic [3, 8] models.

2.5. Landau–Lifshitz Equation

Recently the 1 + 1 field generalization of the Lax pair (2.7) to U–V pair was suggested in [18]. In the field case the Poisson brackets (2.3) are replaced with

$$\{S_{ij}(x), S_{kl}(y)\}$$
$$= \frac{1}{N}(S_{il}(x)\delta_{kj} - S_{kj}(x)\delta_{il})\delta(x-y). \qquad (2.16)$$

The construction of U–V pair is again based on R-matrix satisfying the associative Yang–Baxter equation (2.5). For this purpose, the following relation is used (it can be deduced from (2.5)):

$$r_{12}(z)r_{13}(z) = r_{23}^{(0)}r_{12}(z) - r_{13}(z)r_{23}^{(0)}$$
$$-\partial_z r_{13}(z)P_{23} + m_{12}(z) + m_{23}(0) + m_{13}(z), \qquad (2.17)$$

where P_{12} is the matrix permutation operator and $r_{12}^{(0)}$ is the coefficient in the expansion

$$r_{12}(z) = z^{-1}P_{12} + r_{12}^{(0)} + O(z). \qquad (2.18)$$

Suppose $\operatorname{rank}(S) = 1$, so that $S^2 = cS$, $c = \operatorname{tr}(S)$. Then the Landau–Lifshitz equation reads

$$\partial_t S = \frac{k^2}{c}[S, \partial_x^2 S] + 2c[S, J(S)] - 2k[S, E(\partial_x S)], \qquad (2.19)$$

where

$$E(S) = \operatorname{tr}_2(r_{12}^{(0)}\overset{2}{S}), \qquad \overset{2}{S} = 1_N \otimes S,$$
$$S \in \operatorname{Mat}(N, \mathbb{C}). \qquad (2.20)$$

Then the U–V pair generating equations of motion (2.19) through the Zakharov–Shabat equation (1.5) has the form

$$U^{\text{LL}}(z) = L^{\text{top}}(S,z) = \operatorname{tr}_2(r_{12}(z)\overset{2}{S}),$$
$$V^{\text{LL}}(z) = V_1(z) + V_2(z), \qquad (2.21)$$

$$V_1(z) = -c\partial_z L^{\text{top}}(S,z) + L^{\text{top}}(E(S)S, z),$$
$$V_2(z) = -cL^{\text{top}}(T,z), \qquad T = -\frac{k}{c^2}[S, \partial_x S]. \qquad (2.22)$$

Equations (2.19) are Hamiltonian with the Hamiltonian function

$$H^{\text{LL}} = \oint dy(cN\operatorname{tr}(SJ(S))$$
$$-\frac{Nk^2}{2c}\operatorname{tr}(\partial_y S\partial_y S) + kN\operatorname{tr}(\partial_y S E(S))), \qquad (2.23)$$
$$S = S(y),$$

so that (2.19) is reproduced as $\partial_t S(x) = \{S(x), H^{\text{LL}}\}$ with the Poisson brackets (2.16).

3. GAUGE EQUIVALENCE AND CHANGE OF VARIABLES

Introduce the matrix $G(z,q) = b(x,t)g(z,q)$, where $b(x,t)$ is the function

$$G(z,q) = b(x,t)\Xi(z,q)D^{-1} \in \mathrm{Mat}(N,\mathbb{C}),$$
$$b(x,t) = \prod_{a<b}^{N}(q_b - q_a)^{1/N}\prod_{m=1}^{N}\left(kq_{m,x} - c\right)^{1/(2N)}. \qquad (3.1)$$

The statement is that by applying the gauge transformation with the matrix (3.1) we obtain the desired relation (1.14).[3] Calculations are performed similarly to those in 0 + 1 mechanics [10]. As a result, we obtain explicit change of variables:

$$S_{ij} = \frac{(-1)^{\varrho(j)+1}}{N}$$
$$\times \sum_{m=1}^{N} \frac{(q_m)^{\varrho(i)}\left(\tilde{p}_m + \dfrac{k\alpha_{mx}}{\alpha_m}\right) + \alpha_m^2\varrho(i)(q_m)^{\varrho(i)-1}}{\prod_{l\neq m}(q_m - q_l)} \qquad (3.2)$$
$$\times \sigma_{\varrho(j)}(q), \quad \tilde{p}_j = p_j - \sum_{l\neq j}^{N}\frac{\alpha_j^2}{q_j - q_l}$$

with the properties

$$\mathrm{Spec}(S) = (0,\ldots,0,c), \quad \mathrm{rk}(S) = 1,$$
$$\mathrm{tr}(S) = c, \quad S^2 = cS. \qquad (3.2)$$

It is the 1 + 1 field generalization of the change of variables in mechanics (2.15). It can be also verified that the Poisson brackets for $S_{ij}(p,q,c)$ (3.2) calculated through the canonical brackets (1.2) indeed reproduce the linear Poisson structure (2.16), so that (3.2) is a Poisson map. The Hamiltonian (1.1) of 1 + 1 field Calogero–Moser model coincides with the one (2.23) for the Landau–Lifshitz equation under the change of variables (3.2): $H^{\mathrm{LL}}[S(p(x),q(x))] = H^{\mathrm{2dCM}}[p(x),q(x)]$.

FUNDING

This work was supported by the Russian Science Foundation (project no. 21-41-09011, https://rscf.ru/en/project/21-41-09011/).

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

[3] Let us also remark that V-matrices of 1 + 1 Calogero–Moser and the Landau–Lifshitz models are also related by the gauge transformation $V^{\mathrm{LL}}(z) = G(z)V^{\mathrm{2dCM}}(z)G^{-1}(z) + \partial_t G(z)G^{-1}(z)$ up to additional scalar (i.e., proportional to 1_N) term. The latter can be removed by applying additional gauge transformation with the matrix $G = \exp\left(-\int_0^t f(x,t')dt'\right)1_N$, where $f(x,t) = \frac{1}{N}\sum_{i=1}^{N}\tilde{m}_i^0$.

OPEN ACCESS

REFERENCES

1. I. Krichever, Commun. Math. Phys. **229**, 229 (2002); arXiv: hep-th/0108110.
2. A. A. Akhmetshin, I. M. Krichever, and Y. S. Volvovski, Funct. Anal. Appl. **36**, 253 (2002); arXiv: hep-th/0203192.
3. A. Levin, M. Olshanetsky, and A. Zotov, Commun. Math. Phys. **236**, 93 (2003); arXiv: nlin/0110045.
4. K. Atalikov and A. Zotov, J. Geom. Phys. **164**, 104161 (2021); arXiv: 2010.14297 [hep-th].
5. F. Calogero, Lett. Nuovo Cim. **13**, 411 (1975).
6. J. Moser, *Surveys in Applied Mathematics* (Academic, Amsterdam, 1976), p. 235.
7. A. V. Mikhailov, M. A. Olshanetsky, and A. M. Perelomov, Commun. Math. Phys. **79**, 473 (1981).
8. A. Zabrodin and A. Zotov, J. High Energ. Phys. **2022**, 23 (2022); arXiv: 2107.01697 [math-ph].
9. A. Levin, M. Olshanetsky, and A. Zotov, Nucl. Phys. B **887**, 400 (2014); arXiv: 1406.2995 [math-ph].
10. G. Aminov, S. Arthamonov, A. Smirnov, and A. Zotov, J. Phys. A: Math. Theor. **47**, 305207 (2014); arXiv: 1402.3189 [hep-th].
11. A. Levin, M. Olshanetsky, and A. Zotov, J. Phys. A: Math. Theor. **49**, 395202 (2016); arXiv: 1603.06101.
12. S. Fomin and A. N. Kirillov, *Advances in Geometry,* Vol. 172 of *Progress in Mathematics Book Series* (Springer, New York, 1999), p. 147.
13. A. Polishchuk, Adv. Math. **168**, 56 (2002); arXiv: math/0008156 [math.AG].
14. A. Levin, M. Olshanetsky, and A. Zotov, J. High Energy Phys., No. 07, 012 (2014); arXiv: 1405.7523 [hep-th].
15. I. V. Cherednik, Theor. Math. Phys. **43**, 356 (1980).
16. K. Atalikov and A. Zotov, arXiv: 2303.02391 [math-ph].
17. T. Krasnov and A. Zotov, Ann. Henri Poincaré **20**, 2671 (2019); arXiv: 1812.04209 [math-ph].
18. K. Atalikov and A. Zotov, JETP Lett. **115**, 757 (2022); arXiv: 2204.12576 [math-ph].