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METHODS OF THEORETICAL PHYSICS

A Compatible System of Equations Related to the Lie Superalgebra $\mathfrak{gl}(n|m)$ and Integrable Calogero–Moser Model

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A joint system of equations based on the $\mathfrak{gl}(n|m)$ superalgebra has been determined and its relation to the integrable Calodgero–Moser system of particles has been established.

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INTRODUCTION

Joint systems of differential equations naturally appear in various fields of physics and mathematics. The monodromy of joint systems is specific because it is independent of small deformations of a contour along which it is calculated, and the monodromy matrix itself is very interesting. Aharonov and Bohm [1] proposed an important example of the phase increment of an electron in zero magnetic field beyond a cylindrical region (inaccessible to the electron) but with a nontrivial vector potential.

In modern theoretical/mathematical physics, joint systems of equations occur in the two-dimensional conformal field theory; in particular, correlation functions of vertex operators in the Wess–Zumino–Novikov–Witten model obey the system of Knizhnik–Zamolodchikov equations [2]. Matsuo [3] and Cherednik [4] demonstrate that the Knizhnik–Zamolodchikov equations are closely related to the Calodgero–Moser system. It is also known that the Knizhnik–Zamolodchikov equations are closely related to the Casimir connection discovered by De Concini, Millson, and Toledano Loredo (see [5] and references therein). The Casimir connection commutes with the Knizhnik–Zamolodchikov equations. In this work, the Casimir connection is determined for the $\mathfrak{gl}(n|m)$ superalgebra, and the Matsuo–Cherednik correspondence is established for this case.

SOME PROPERTIES OF THE $\mathfrak{gl}(n|m)$ SUPERALGEBRA

The main properties of the $\mathfrak{gl}(n|m)$ superalgebra are as follows.

Let $\mathcal{J} = \{1, \dots n+m\}$ and $p : \mathcal{J} \to \{0,1\}$

$$\begin{cases} p(a) = 0, & a \le n \,(\text{bosons}), \\ p(a) = 1, & a > n \,(\text{fermions}). \end{cases} \tag{1}$$

The $\mathfrak{gl}(n|m)$ algebra is generated by e_{ab}, where $a, b \in \mathcal{J}$ with the relations

$$\begin{aligned} &e_{ab}e_{cd} - (-1)^{p(e_{ab})p(e_{cd})}e_{cd}e_{ab} \\ &= \delta_{bc}e_{ad} - (-1)^{p(e_{ab})p(e_{cd})}\delta_{da}e_{cb}, \end{aligned} \tag{2}$$

where

$$p(e_{ab}) = p(a) + p(b) \bmod 2. \tag{3}$$

The tensor product $\otimes$ of representations of superalgebras is defined so that eigenoperators of the parity operator that act nontrivially only in the ith and jth tensor factors satisfy the relation

$$A^{(i)}B^{(j)} = (-1)^{p(A)p(B)}B^{(j)}A^{(i)}. \tag{4}$$

In the basis e_a existing in $\mathbb{C}^{n|m}$, $e_{ab}(e_c) = \delta_{bc}e_a$ means that e_{ab} are matrix units.

Let $x, y \in \mathbb{C}^{n|m}$ with certain $p(x)$ and $p(y)$; then, the graded permutation is defined as follows:

$$P_{12}(x \otimes y) = (-1)^{p(x)p(y)}y \otimes x. \tag{5}$$

JOINT SYSTEM OF EQUATIONS

The Knizhnik–Zamolodchikov equations for the function $|\Psi\rangle$ with values in $V = \bigotimes_{i=1}^{k}\mathbb{C}^{n|m}$, where $\mathbb{C}^{n|m}$ is the vector representation of $\mathfrak{gl}(n|m)$, have the form

$$\left(\kappa\partial_{z_i} - \sum_c \lambda_c e_{cc}^{(i)} - \sum_{j,\neq i}\frac{P_{ij}}{z_i - z_j}\right)|\Psi\rangle = 0, \tag{6}$$

where $P_{ij} = \sum_{a,b}(-1)^{p(b)}e_{ab}^{(i)}e_{ba}^{(j)}$ is the graded permutation, $p(a)$ is the parity function defined in the supplementary material, and $e_{ab}^{(j)}$ is the generator of the $\mathfrak{gl}(n|m)$ superalgebra that acts nontrivially only in the ith tensor factor V as a matrix unity in a certain basis (see the supplementary material).

One of the main statements of this work is that *the following system of equations is consistent and commutes with* Eqs. (6):

$$\left(\kappa\partial_a - \sum_j z_j e_{aa}^{(j)} - \sum_{b,\neq a}(-1)^{p(b)}\frac{E_{ab}E_{ba} - E_{aa}}{\lambda_a - \lambda_b}\right) \times |\Psi\rangle = 0, \quad (7)$$

where $E_{ab} = \sum_{j=1}^{k} e_{ab}^{(j)}$. This statement is proved in the supplementary material.

MATSUO–CHEREDNIK CORREPONDENCE

In this section, $k = n + m$ is set in the definition of V.

To establish the Matsuo–Cherednik correspondence for Eqs. (7), the following convectors constructed in [6] are needed:

$$\langle\Omega^0| = \sum_{\sigma\in S_{n+m}}\langle e_1\otimes\ldots\otimes e_{n+m}|P_\sigma, \quad (8a)$$

$$\langle\Omega^1| = \sum_{\sigma\in S_{n+m}}\langle e_1\otimes\ldots\otimes e_{n+m}|(-1)^{sgn(\sigma)}P_\sigma. \quad (8b)$$

Here, $P_\sigma = P_{s_{i_1}}\ldots P_{s_{i_l}}$, where $P_{s_{i_j}}$ is the permutation corresponding to the transposition s_{i_j} and $\sigma = s_{i_1}s_{i_2}\ldots s_{i_l}$ is a certain decomposition of the permutation into the product of transpositions. Since $P_{s_{i_j}}$ satisfies the braid group relations, the element P_σ is defined correctly.

To verify that the convectors given by Eqs. (8a) and (8b) are eigenvectors of the operators

$$\langle\Omega^i|(E_{ab}E_{ba} - E_{aa}) = (-1)^{p(b)+i}\langle\Omega^i|, \quad (9)$$

where $i = 0,1$, similar properties can be proved for the vectors $|\Omega^i\rangle$. A simple calculation shows that

$$\begin{aligned}(E_{ab}E_{ba} - E_{aa})|e_1\otimes\ldots\otimes e_{n+m}\rangle \\ = (-1)^{p(b)}P_{ab}|e_1\otimes\ldots\otimes e_{n+m}\rangle.\end{aligned} \quad (10)$$

The symmetrization or antisymmetrization of the right-hand side of Eq. (10) certainly gives Eq. (9).

It is easily seen that the Knizhnik–Zamolodchikov equations and dynamical equations commute with the operators E_{aa}; consequently, the following condition can be imposed on the solution:

$$E_{aa}|\Psi\rangle = |\Psi\rangle. \quad (11)$$

Then, such solutions satisfy the relations

$$\begin{aligned}\left(\kappa^2\sum_{a=1}^{m+n}\partial_{z_a}^2 + \sum_{a\neq b}\frac{(-1)^i\kappa - 1}{(z_a - z_b)^2}\right)\langle\Omega^i|\Psi\rangle \\ = \left(\sum_{a=1}^{n+m}\lambda_a^2\right)\langle\Omega^i|\Psi\rangle,\end{aligned} \quad (12a)$$

$$\begin{aligned}\left(\kappa^2\sum_{a=1}^{m+n}\partial_{\lambda_a}^2 + \sum_{a\neq b}\frac{(-1)^i\kappa - 1}{(\lambda_a - \lambda_b)^2}\right)\langle\Omega^i|\Psi\rangle \\ = \left(\sum_{a=1}^{n+m}z_a^2\right)\langle\Omega^i|\Psi\rangle,\end{aligned} \quad (12b)$$

where $i = 0,1$.

Relation (12a) was proved in [6]. To prove Eq. (12b), it is sufficient to write the following relation for the a-infinitesimal operator D_a in parentheses in Eq. (7):

$$\begin{aligned}0 &= \langle\Omega^i|\sum_{a=1}^{n}D_a^2|\Psi\rangle \\ &= \left(\kappa^2\sum_{a=1}^{m+n}\partial_{\lambda_a}^2 + \sum_{a\neq b}\frac{(-1)^i\kappa - 1}{(\lambda_a - \lambda_b)^2}\right)\langle\Omega^i|\Psi\rangle \\ &\quad - \left(\sum_{a=1}^{n+m}z_a^2\right)\langle\Omega^i|\Psi\rangle.\end{aligned} \quad (13)$$

Details of this calculation are presented in the supplementary material.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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SUPPLEMENTARY INFORMATION

The online version contains supplementary material available at https://doi.org/10.1134/S0021364023600088.

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