

About Derivatives in Analytic QCD

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The importance of (fractional) derivatives is demonstrated in the case of analytic QCD. As an example, the Bjorken sum rule is considered.

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1. INTRODUCTION

In accordance with the general principles of (local) quantum field theory [1] observables in the spacelike region can have singularities only for negative values of their argument Q^2 . At the same time for large values of Q^2 , we can rewrite these observables as power series expansion by the running coupling constant $\alpha_s(Q^2)$, which, in turn, has a ghost singularity, the so-called Landau pole, for $Q^2 = \Lambda^2$. The only way to restore analyticity is to remove this pole.

An improved expression for the strong coupling constant can be obtained from the renormalization group equation

$$L \equiv \ln \frac{Q^2}{\Lambda^2} = \int \frac{d\bar{a}_s(Q^2)}{\beta(\bar{a}_s)}, \quad (1)$$

$$\bar{a}_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi}, \quad a_s(Q^2) = \beta_0 \bar{a}_s(Q^2),$$

with some boundary condition and the QCD β -function:

$$\beta(a_s) = -\sum_{i=0}^{\infty} \beta_i \bar{a}_s^{i+2} = -\beta_0 \bar{a}_s^2 \left(1 + \sum_{i=1}^{\infty} b_i \bar{a}_s^i \right), \quad (2)$$

$$b_i = \frac{\beta_i}{\beta_0^{i+1}}.$$

So, already at the leading order (LO), when $a_s(Q^2) = a_s^{(1)}(Q^2)$, we have from Eq. (1)

$$a_s^{(1)}(Q^2) = \frac{1}{L}; \quad (3)$$

i.e., $a_s^{(1)}(Q^2)$ does contain a pole at $Q^2 = \Lambda^2$.

In accordance with the approach presented in the papers [2–6], the Landau singularity can be eliminated without extraneous regulators. The idea is based on the dispersion relation, which connects the new analytic coupling constant $A_{\text{MA}}(Q^2)$ with the spectral function $r_{\text{pt}}(s)$, obtained in the framework of perturbation theory (PT). Such approach is called *Minimal Approach* (MA) (see, e.g., [7]) or *Analytical PT* (APT) [2–6]. In the LO, this gives the following

$$A_{\text{MA}}^{(1)}(Q^2) = \int_0^{\infty} \frac{ds}{(s+t)} r_{\text{pt}}^{(1)}(s), \quad t = \frac{1}{z} = \frac{Q^2}{\Lambda^2}, \quad (4)$$

where the LO spectral function

$$r_{\text{pt}}^{(1)}(s) = \frac{1}{\pi} \text{Im} a_s^{(1)}(-s - i\epsilon). \quad (5)$$

A further development of APT is the so-called fractional APT (FAPT), which extends the principles of constructing to non-integer ν -powers of coupling constant, which in the QFT framework arise for many quantities having non-zero anomalous dimensions (see the famous papers [8–10] with some previous one [11] and reviews in [12, 13]).

Following [14, 15], we can introduce also the derivatives

$$\tilde{a}_{n+1}(Q^2) = \frac{(-1)^n d^n a_s(Q^2)}{n! (dL)^n}, \quad (6)$$

which will be very convenient in the case of the analytic QCD.

The series of derivatives \tilde{a}_n can successfully replace the corresponding series of the a_s -powers. Indeed, at LO, the series of derivatives \tilde{a}_n exactly coincide with a_s^n . Beyond LO, the relation between \tilde{a}_n and a_s^n was

established in [15, 16] and extended to the fractional case, where $n \rightarrow$ a non-integer ν , in [17].

For the ν derivative of $A_{\text{MA}}^{(1)}(Q^2)$, i.e., $\tilde{A}_{\text{MA},\nu}^{(1)}(Q^2)$, there is the following equation [17]:

$$\tilde{A}_{\text{MA},\nu}^{(1)}(Q^2) = \frac{(-1)}{\Gamma(\nu)} \int_0^{\infty} \frac{ds}{s} r_{\text{pt}}^{(1)}(s) \text{Li}_{1-\nu}(-sz), \quad (7)$$

where $\text{Li}_{1-\nu}(-sz)$ is the polylogarithmic function:

$$\text{Li}_{-\nu}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^{1-\nu}} = \frac{z}{\Gamma(-\nu)} \int_0^1 \frac{d\xi}{1-z\xi} \ln^{-\nu-1}\left(\frac{1}{\xi}\right). \quad (8)$$

Analytic analogs of ν -powers of $A_{\text{MA}}^{(1)}(Q^2)$; i.e., $A_{\text{MA},\nu}^{(1)}(Q^2)$, can be expressed [8] as

$$A_{\text{MA},\nu}^{(1)}(Q^2) = \int_0^{\infty} \frac{ds}{(s+t)} r_{\text{pt},\nu}^{(1)}(s), \quad (9)$$

where the LO spectral function of the ν -powers of $a_s^{(1)}(Q^2)$ has the form

$$r_{\text{pt},\nu}^{(1)}(s) = \frac{1}{\pi} \text{Im}\left(a_s^{(1)}(-s - i\epsilon)\right)^{\nu}. \quad (10)$$

Note that $\tilde{A}_{\text{MA},\nu}^{(1)}(Q^2) = A_{\text{MA},\nu}^{(1)}(Q^2)$ [17].

Thus, analytic QCD in its minimal version is a very convenient approach that combines the general (analytical) properties of quantum field quantities and the results obtained within the framework of perturbative QCD, leading to the appearance of the MA coupling constant $A_{\text{MA}}(Q^2)$, close to the usual strong constant $a_s(Q^2)$ in the limit of large values of its argument and completely different at $Q^2 \leq \Lambda^2$.

2. BEYOND THE LEADING ORDER

Equations (4), (7), and (9) are rather simple even in the case of the noninteger ν values. Beyond the LO, the situations changes strongly, the results with ν -powers become to be very complicated (see [18] and discussions therein).

So, beyond the LO, we consider extensions of Eqs. (7) and (9) only for integer values $\nu = n$, and we restrict our investigations for the case $n = 1, 2, 3, 4$, which is needed to study the Bjorken sum rule (see the next section).

2.1. ν -Derivatives

The extension is simple and the final result is (see, e.g., [19]):

$$\tilde{A}_{\text{ma},\nu,k}^{(i+1)}(Q^2) = \frac{(-1)}{\Gamma(\nu)} \int_0^{\infty} \frac{ds}{s} r_{\text{pt}}^{(i+1)}(s) \text{Li}_{1-\nu}(-sz_k), \quad (11)$$

where

$$r_{\text{pt}}^{(i+1)}(s) = r_{\text{pt}}^{(1)}(s) + \sum_{m=1}^i \delta_r^{(m+1)}(s) \quad (12)$$

is the spectral function in the $(i+1)$ -order of perturbation theory and (see [20, 21])

$$y = \ln s, \quad r_{\text{pt}}^{(1)}(y) = \frac{1}{y^2 + \pi^2},$$

$$\delta_r^{(2)}(y) = -\frac{b_1}{(y^2 + \pi^2)^2} [2yf_1(y) + (\pi^2 - y^2)f_2(y)],$$

$$\delta_r^{(3)}(y) = \frac{b_1^2}{(y^2 + \pi^2)^3}$$

$$\times [(3y^2 - \pi^2)f_3(y) - y(y^2 - 3\pi^2)f_4(y)],$$

$$\delta_r^{(4)}(y) = \frac{b_1^3}{(y^2 + \pi^2)^4} \quad (13)$$

$$\times [4y(y^2 - \pi^2)f_5(y) + [4\pi^2y^2 - (y^2 - \pi^2)^2]f_6(y)],$$

with

$$f_1(y) = \frac{1}{2} \ln(y^2 + \pi^2), \quad f_2(y) = \frac{1}{2} - \frac{1}{\pi} \arctan\left(\frac{y}{\pi}\right), \quad (14)$$

and

$$f_3(y) = f_1(y)(f_1(y) - 1) - \pi^2 f_2^2(y) - 1 + \frac{b_2}{b_1^2},$$

$$f_4(y) = f_2(y)(2f_1(y) - 1),$$

$$f_5(y) = f_1(y) \left(3\pi^2 f_2^2(y) - f_1^2(y) + 2 - 3\frac{b_2}{b_1^2} \right) \quad (15)$$

$$+ \frac{5}{2} (f_1^2(y) - \pi^2 f_2^2(y)) + \frac{1}{2} \left(\frac{b_3}{b_1^3} - 1 \right),$$

$$f_6(y) = f_2(y)(\pi^2 f_2^2(y) - 3f_1^2(y) + 5f_1(y) - 3\frac{b_2}{b_1^2} + 2).$$

For the case $\nu = n$ and $n = 1, 2, 3, 4$, we have also

$$\text{Li}_0(z) = \frac{z}{1-z}, \quad \text{Li}_{-1}(z) = \frac{z}{(1-z)^2},$$

$$\text{Li}_{-2}(z) = \frac{z(1+z)}{(1-z)^3}, \quad \text{Li}_{-3}(z) = \frac{z(1+4z+z^2)}{(1-z)^4}, \quad (16)$$

$$\text{Li}_{-4}(z) = \frac{z(1+z)(1+10z+z^2)}{(1-z)^5},$$

which greatly simplifies the calculations.

2.2. ν -Powers

An analytic analog of the ν -power of $A_{\text{MA}}^{(i+1)}(Q^2)$, i.e., $A_{\text{MA},\nu}^{(i+1)}(Q^2)$, can be expressed as

$$A_{\text{MA},\nu}^{(i+1)}(Q^2) = \int_0^\infty \frac{ds}{(s+i)} r_{\text{pt},\nu}^{(i+1)}(s), \quad (17)$$

where the corresponding spectral function

$$r_{\text{pt},\nu}^{(i+1)}(s) = \frac{1}{\pi} \text{Im} \left(a_s^{(i+1)}(-s - i\varepsilon) \right)^\nu. \quad (18)$$

Finding the exact values of $r_{\text{pt},\nu}^{(i+1)}(s)$ requires very cumbersome research (see, e.g., [18]). The situation for $\nu = n$ is greatly simplified and $r_{\text{pt},n}^{(i+1)}(s)$ can be expressed in terms of the real $R_{\text{pt}}^{(i+1)}(s)$ and the imaginary $r_{\text{pt}}^{(i+1)}(s)$ part of the coupling constant itself:

$$a_s^{(i+1)}(-s \pm i\varepsilon) = R_{\text{pt}}^{(i+1)}(s) \mp i\pi r_{\text{pt}}^{(i+1)}(s). \quad (19)$$

For the cases $n = 1, 2, 3, 4$, we have (see [20, 21])

$$\begin{aligned} r_{\text{pt},n=1}^{(i+1)}(s) &\equiv r_{\text{pt}}^{(i+1)}(s), \\ r_{\text{pt},n=2}^{(i+1)}(s) &= 2r_{\text{pt}}^{(i+1)}(s)R_{\text{pt}}^{(i+1)}(s), \\ r_{\text{pt},n=3}^{(i+1)}(s) &= r_{\text{pt}}^{(i+1)}(s) \\ &\times \left[3\left(R_{\text{pt}}^{(i+1)}(s)\right)^2 - \pi^2 \left(r_{\text{pt}}^{(i+1)}(s)\right)^2 \right], \\ r_{\text{pt},n=4}^{(i+1)}(s) &= 4r_{\text{pt}}^{(i+1)}(s)R_{\text{pt}}^{(i+1)}(s) \\ &\times \left[\left(R_{\text{pt}}^{(i+1)}(s)\right)^2 - \pi^2 \left(r_{\text{pt}}^{(i+1)}(s)\right)^2 \right], \end{aligned} \quad (20)$$

where $r_{\text{pt}}^{(i+1)}(s)$ are given above in Eqs. (12)–(15) and

$$R_{\text{pt}}^{(i+1)}(s) = R_{\text{pt}}^{(1)}(s) + \sum_{m=1}^i \delta_{\text{R}}^{(m+1)}(s), \quad (21)$$

with

$$\begin{aligned} R_{\text{pt}}^{(1)}(y) &= \frac{y}{y^2 + \pi^2}, \\ \delta_{\text{R}}^{(2)}(y) &= -\frac{b_1}{(y^2 + \pi^2)^2} [(y^2 - \pi^2)f_1(y) + 2\pi^2 y f_2(y)], \\ \delta_{\text{R}}^{(3)}(y) &= \frac{b_1^2}{(y^2 + \pi^2)^3} \\ &\times [y(y^2 - 3\pi^2)f_3(y) + \pi^2(3y^2 - \pi^2)f_4(y)], \\ \delta_{\text{R}}^{(4)}(y) &= \frac{b_1^3}{(y^2 + \pi^2)^4} \\ &\times [(y^2 - \pi^2)^2 - 4\pi^2 y^2]f_5(y) + 4\pi^2 y(y^2 - \pi^2)f_6(y). \end{aligned} \quad (22)$$

3. BJORKEN SUM RULE

The polarized Bjorken sum rule is defined as the difference between the proton and neutron polarized structure functions g_1 integrated over the whole x interval (see, e.g., [22–31])

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 dx [g_1^p(x, Q^2) - g_1^n(x, Q^2)]. \quad (23)$$

The quantity can be written as

$$\Gamma_1^{p-n}(Q^2) = \left| \frac{g_A}{g_V} \right| \frac{1}{6} (1 - D_{\text{BS}}(Q^2)) + \frac{\tilde{\mu}_4}{Q^2 + M^2}, \quad (24)$$

where $|g_A/g_V| = 1.2723 \pm 0.0023$ is the ratio of the nucleon axial charge, $(1 - D_{\text{BS}}(Q^2))$ is the perturbation expansion for the twist-two contribution, and $\tilde{\mu}_4/(Q^2 + M^2)$ is the “massive” twist-four contribution (see [32–34]). The values of $\tilde{\mu}$ and M^2 were fitted in [29] as

$$M^2 = 0.439, \quad \tilde{\mu}_4 = -0.082. \quad (25)$$

The perturbative part has the form

$$D_{\text{BS}}(Q^2) = \frac{4}{\beta_0} a_s (1 + d_1 a_s + d_2 a_s^2 + d_3 a_s^3), \quad (26)$$

$$= \frac{4}{\beta_0} (a_s + \tilde{d}_1 \tilde{a}_2 + \tilde{d}_2 \tilde{a}_3 + \tilde{d}_3 \tilde{a}_4), \quad (27)$$

where

$$\begin{aligned} \tilde{d}_1 &= d_1, \quad \tilde{d}_2 = d_2 - b_1 d_1, \\ \tilde{d}_3 &= d_3 - \frac{5}{2} b_1 d_2 + \left(\frac{5}{2} b_1^2 - b_2 \right) d_1. \end{aligned} \quad (28)$$

For the case of three active quarks, we have

$$\begin{aligned} d_1 = \tilde{d}_1 &= 1.59, \quad d_2 = 3.99, \quad \tilde{d}_2 = 2.51, \\ d_3 &= 15.42, \quad \tilde{d}_3 = 10.58. \end{aligned} \quad (29)$$

In APT, the perturbative part in the first ($k = 1$), second ($k = 2$), third ($k = 3$), and fourth ($k = 4$) orders of perturbation theory has the form

$$\begin{aligned} D_{\text{MA,BS}}(Q^2) &= \frac{4}{\beta_0} (A_{\text{MA}}^{(k)}(Q^2) + \sum_{m=2}^k \tilde{d}_{m-1} \tilde{A}_{\text{MA},m}^{(k)}(Q^2)) \end{aligned} \quad (30)$$

$$= \frac{4}{\beta_0} (A_{\text{MA}}^{(k)}(Q^2) + \sum_{m=2}^k d_{m-1} A_{\text{MA},m}^{(k)}(Q^2)). \quad (31)$$

The results of calculations are shown in Fig. 1 and Table 1. Here we use the Q^2 -independent M and $\tilde{\mu}_4$ values taken from Eq. (25) and the twist-two part shown in Eqs. (26), (27), (30), and (31). The Λ values are different in each PT order and can be taken in [37] (see also [19]).

As seen in Fig. 1, results obtained using usual coupling constants are good only at LO and deteriorate as

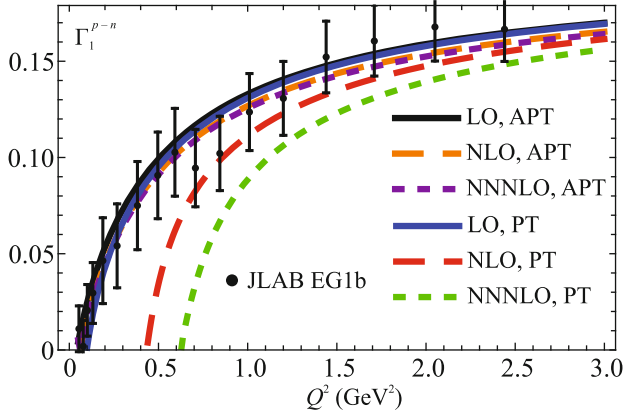


Fig. 1. Results for $\Gamma_1^{p-n}(Q^2)$ in the first, second, and fourth orders of APT and usual PT obtained with Eqs. (27) and (30) for the twist-two part. Experimental points are taken from [35, 36].

the PT order increases. The good agreement at LO is due to the use of Λ_{LO} , which is small, and therefore the investigated range of Q^2 is higher than Λ_{LO}^2 . Visually, the results are close to those obtained in [24–26], where the standard form of the twist-four terms has been used. Thus, the usage of the massive twist-four form (24) does not improve the results, since at $Q^2 \rightarrow \Lambda^2$ coupling constants become to be singular, that leads to large and negative results for the twist-two part $D_{BS}(Q^2)$.

The results obtained with APT are close to those of [29, 30], which is not surprising since we used the parameters (25) obtained in [29]. Moreover, we see that the results based on different orders are close to each other.

In Table 1 we see that our results for $\Gamma_1^{p-n}(Q^2)$ in the framework of derivatives and powers of coupling constants are very similar. Indeed, consider the ratio

$$\Delta_\Gamma(Q^2) = (\Gamma_{1,D}^{p-n}(Q^2) - \Gamma_{1,P}^{p-n}(Q^2)) / \Gamma_{1,P}^{p-n}(Q^2) \times 100, \quad (32)$$

where $\Gamma_{1,D}^{p-n}(Q^2)$ and $\Gamma_{1,P}^{p-n}(Q^2)$ are the results obtained using Eqs. (27), (30) and (26), (31), respectively.

In the case of APT, $\Gamma_{1,D}^{p-n}(Q^2)$ and $\Gamma_{1,P}^{p-n}(Q^2)$ agree with an accuracy better than 1%. Moreover, the discrepancy between them decreases as Q^2 increases. In

the usual PT case at $Q^2 \geq 1.5 \text{ GeV}^2$, where applicable, the difference between the results based on derivatives and powers is larger, but strongly decreases with increasing Q^2 values and PT order.

4. CONCLUSIONS

To summarize, we have considered ν derivatives and (analytic analogs of) ν powers in the case of usual and MA QCD. We used the case of the integer $\nu = n = 1, 2, 3, 4$ and applied it to studying the Bjorken sum rule. All results have been presented up to the fourth order of perturbation theory, where the corresponding Wilson coefficients d_i ($i = 1, 2, 3$) for the Bjorken sum rule are known.

We have shown that the results based on the usual perturbation theory disagree with the experimental data at $Q^2 \leq 1.5 \text{ GeV}^2$. APT in the minimal version leads to good agreement with the experimental data when we used the massive version (24) for high-twist contributions. The results based on derivatives and (analytic analogs of) powers of usual and MA coupling constants are very similar to each other. In the case of MA QCD, the application based on the derivatives strongly simplifies the study.

In the future, we plan to apply the obtained results to study the processes of deep-inelastic scattering in order to extract $\alpha_s(M_Z^2)$ including the experimental data at low Q^2 values. One of the most important issues is the approximations (fits) of experimental data for the deep-inelastic scattering structure function $F_2(x, Q^2)$ (see, e.g., [38–46]). This is one of the main ways for obtaining precision values of $\alpha_s(M_Z)$, the strong constant normalization.

We plan to use (the ν derivatives of) the MA coupling constant $\tilde{A}_{MA,\nu}^{(i)}(Q^2)$ in our approximations, which is indeed possible, because in the fits we use the SF Mellin moments (following [47–49]) and reconstruct SFs themselves at the end. In this case, the Q^2 -dependence of the SF moments is known [50] exactly in analytic form. Thus, it can be expressed in terms of ν derivatives $\tilde{A}_{MA,\nu}^{(i)}(Q^2)$, where the corresponding ν variable becomes n -dependent (here, n is the Mellin moment number), and the use of ν deriva-

Table 1. Ratio $\Delta_\Gamma(Q^2)$ in APT (usual PT) for various Q^2 values, $Q^2 = N \text{ GeV}^2$ and k numbers of the used perturbation theory orders

k	$N = 0.5$	$N = 1$	$N = 1.5$	$N = 2$	$N = 2.5$
2	-0.32	-0.28	-0.26 (-7.81)	-0.25 (-5.88)	-0.24 (-4.80)
3	0.51	0.38	0.31 (-3.12)	0.27 (-2.34)	0.24 (-1.96)
4	-0.61	-0.17	-0.07 (-7.14)	-0.03 (-0.19)	-0.001 (-0.01)

tives should be crucial. Beyond LO, to obtain complete analytic results for Mellin moments, we will use their analytic continuation [51–53].

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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