MISCELLANEOUS

Butterfly Effect in a System of Quantum Dots in the Sachdev−Ye−Kitaev Model

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A method has been proposed to calculate the out-of-time time ordered correlator in the generalization of the Sachdev−Ye−Kitaev model with a nonzero spatial dimension. The result is applicable not only at small times, when the chaotic properties of the system are developed weakly, but also at large times of about the Erenfest time. It has been shown that information on the applied perturbation, which is described by this correlator, propagates ballistically in the form of a front. The velocity of the front for models of this type has been calculated for the first time.

1. INTRODUCTION

The sensitivity of a system to the initial conditions was noted by H. Poincaré who studied instabilities in the problem of three bodies. Then, this problem was studied by A. Lyapunov. The notion of the butterfly effect was suggested by E. Lorenz, who simulated atmospheric processes and discovered such an instability. This effect is manifested in an exponential increase in the distance between initially very close trajectories of the system; i.e.,

$$\{q(t), p(0)\} = \frac{\partial q(t)}{\partial q(0)} \sim e^{\lambda_L t}. \qquad (1)$$

Here, $\{\ldots,\ldots\}$ is the Poisson bracket for this system; p and q are the canonically conjugate momentum and coordinate, respectively; and λ_L is called the Lyapunov exponent. This formula can also be generalized to the case of quantum systems. In this case, the Lyapunov exponent can be similarly obtained from the following correlator:

$$\langle [q(t), p(0)]^2 \rangle \sim e^{\lambda_L t}. \qquad (2)$$

According to Eq. (2), to calculate the Lyapunov exponent, it is necessary to calculate out-of-time ordered correlators (OTOCs). The relation of such correlators with the chaotic behavior of the system was first mentioned in [1]. In the general case, we can consider not only conjugate pairs of operators but also the correlators

$$F(t_1, t_2, t_3, t_4) = \langle X_1(t_1) X_2(t_2) X_3(t_3) X_4(t_4) \rangle, \qquad (3)$$

where X_i are arbitrary operators and it is assumed that $t_1 \approx t_3 > t_2 \approx t_4$. The study of such correlators allows one to characterize chaotic properties of the system and to understand the propagation of information in the quantum system [2–4].

To experimentally study OTOCs, one should "return" to past, induce perturbation, and observe its consequence compared to the situation observed before. Such an experiment for a classical system was described in the short story "A Sound of Thunder" by Ray Bradbury, where the main characters returned to the far past, randomly killed a butterfly, and, returning back to their present time, observed drastic changes in the world. Modern controlled quantum systems with a large number of degrees of freedom allow one to study such correlators. Completely controlling a system, one can change the sign of the Hamiltonian determining evolution. After such change, the system begins to evolve backward in time. Such an experiment was performed by the Google Quantum AI team [5].

For a sufficiently wide class of systems, the correlator given by Eq. (3) has a universal behavior described as

$$\frac{F(t_1, t_2, t_3, t_4)}{\langle X_1(t_1) X_3(t_3) \rangle \langle X_2(t_2) X_4(t_4) \rangle} \approx 1 - \frac{1}{C} e^{\lambda_L(t_1 - t_2)}, \qquad (4)$$

where λ_L is the Lyapunov exponent of this quantum system and $C \gg 1$. The Lyapunov exponent of such systems can be generally estimated as (below, $\hbar = 1$)

$$\lambda_L \leq 2\pi T. \qquad (5)$$

This estimate was obtained in [6]. One of the examples for which this estimate is saturated is the Sachdev−Ye−Kitaev (SYK) model [7], whose Hamiltonian will be presented below.

The next natural question is: How does this correlator behave in the case $t_1 - t_2 \sim \lambda_{\mathrm{L}}^{-1} \ln C$? The time $t_E = \lambda_{\mathrm{L}}^{-1} \ln C$ is called the Erenfest time. This question in the case of the SYK model was answered in [8], where it was shown that the correlator F vanishes at $t - t_E \gg \lambda_{\mathrm{L}}^{-1}$. The behavior of the OTOC for zero-dimensional fermion systems at times larger than the Erenfest time was considered in [9], where it was shown that this correlator can be calculated by a universal method from the behavior of the correlator at short times given by Eq. (4) and pair correlation functions.

In the general case, the correlator for zero-dimensional systems at times $t \ll t_E$ increases exponentially, but this increase is suppressed by a factor of C^{-1}. Thus, the system weakly demonstrates a chaotic behavior. Chaotic properties become pronounced at times $t - t_E \gg \lambda_{\mathrm{L}}^{-1}$.

What may we expect for the system with a nonzero spatial dimension? Let the operators X_2 and X_4 be applied at the point $\mathbf{r}_2$ and the OTOC be calculated, varying the point $\mathbf{r}_1$ and the times of application of the operators X_1 and X_3. As a result, at a fixed time difference $t_1 - t_2$, the space can be separated into two regions: (i) the region of undeveloped chaos with $F = \mathrm{const}$ and (ii) the region of developed chaos with $F \to 0$. In particular, sufficiently remote points belong to the first region. The second region appears only at $t_1 - t_2 \geq t_E$. These regions are separated by the boundary (front), which moves with time, so that the second region grows. The velocity of this motion generally depends on the direction because information on perturbation in the system propagates ballistically. A similar scenario of propagation of information for systems with a local Hamiltonian was described in [2], and a similar front was also revealed in other models, e.g., [10, 11].

The aim of this work is to study the behavior of the system of granules, where dynamics is described by the Hamiltonian of the SYK model; tunneling between granules is also allowed in the system. It is shown that the behavior of the OTOC in this system can be described within the scenario mentioned above. The velocity of propagation of the front in the generalization of the SYK model is calculated for the first time. The proposed method of calculation is also applicable to other models based on the SYK model.

The behavior of the OTOC in the generalization of the SYK model was discussed in [12]. However, the behavior of the correlator at long times was not studied in that work; correspondingly, the notion of the front was not defined. Furthermore, the authors of [12] studied only the case of long-wavelength excitations; in this case, the propagation of the front is generally ballistic; i.e., the results of [12] are applicable only in a limited region of parameters. The methods proposed in [12] are significantly based on the form of the interaction between fermions from different granules, which is of the fourth order in fermion operators, similar to the Hamiltonian describing the dynamics inside granules.

This work is organized as follows. The model under study is introduced in Section 2, where the action for its description is obtained. Section 3 describes the main properties of the SYK model. The OTOC is calculated in Section 4, where its main properties are also described.

2. MODEL AND MAIN PROPERTIES

The Hamiltonian of the model under study has the form

$$H = \sum_{\mathbf{r}} \left\{ H_{\mathbf{r}} + i \sum_{\delta\mathbf{r}} \sum_{i,j} w_{\mathbf{r},i;\mathbf{r}+\delta\mathbf{r},j} \chi_{\mathbf{r},i} \chi_{\mathbf{r}+\delta\mathbf{r},j} \right\}, \qquad (6)$$

where

$$H_{\mathbf{r}} = \frac{1}{4!} \sum_{i,j,k,l}^{N} J_{i,j,k,l;\mathbf{r}} \chi_{\mathbf{r},i} \chi_{\mathbf{r},j} \chi_{\mathbf{r},k} \chi_{\mathbf{r},l}.$$

Here, $\chi_{\mathbf{r}}$ are the Majorana fermions satisfying the commutation relations $\{\chi_{\mathbf{r},i}, \chi_{\mathbf{r}',i'}\} = \delta_{i,i'} \delta_{\mathbf{r},\mathbf{r}'}$ and the subscript i runs from 1 to $N \gg 1$. The Hamiltonian (6) describes the lattice of quantum dots having the position vector $\mathbf{r}$, and $\delta\mathbf{r}$ is the vector between an arbitrary point and one of its neighbors. The term $H_{\mathbf{r}}$ describes the dynamics inside the point, and the second term in Eq. (6) describes the tunneling of particles between neighboring dots.

The Hamiltonian $H_{\mathbf{r}}$ is the Hamiltonian of the SYK model [7]. Qualitatively, this Hamiltonian describes a system where there are N degenerate levels; in this case, the effects of interaction become strong and lead to a non-Fermi liquid behavior, which was described, e.g., in [13, 14].

The tensors w and J are antisymmetric in indices and are independent Gaussian random variables with zero mean value and variance determined by the expression

$$\langle J_{i,j,k,l;\mathbf{r}}^2 \rangle = \frac{3! J^2}{N^3}, \qquad \langle w_{\mathbf{r},i;\mathbf{r}+\delta\mathbf{r},j}^2 \rangle = \frac{w^2}{N}. \qquad (7)$$

It is assumed that the characteristic scale of the interaction inside granules is much larger than the characteristic tunneling amplitude, i.e., $J \gg w$.

To calculate the OTOC, we have to use the double Keldysh contour (Fig. 1). Technical details of the calculation were described in [2, 8, 15].
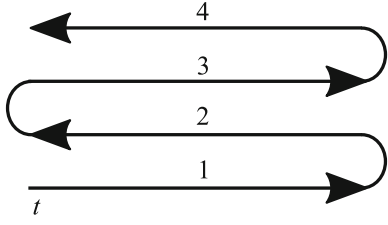
Fig. 1. Contour of integration for the action consisting of four parts two of which are forward in time and the other two are backward in time.

The index $\sigma \in \{1,2,3,4\}$ is introduced to specify fields defined in the four parts of the contour. For example, the fermion field defined in the σth part of the contour is denoted as $\chi_{\mathbf{r},i}^{(\sigma)}$. Then, the correlator of interest is represented in the form

$$F(t_1, t_2, t_3, t_4)$$
$$= \frac{1}{N^2} \sum_{i,i'} \langle T_C \chi_{\mathbf{r}_1,i}^{(1)}(t_1) \chi_{\mathbf{r}_2,i'}^{(2)}(t_2) \chi_{\mathbf{r}_1,i}^{(3)}(t_3) \chi_{\mathbf{r}_2,i'}^{(4)}(t_4) \rangle. \tag{8}$$

Here, T_C stands for the ordering of operators along the contour. The action on the contour has the form

$$S = \sum_{\sigma} \varepsilon_{\sigma} \int dt \left\{ \sum_{\mathbf{r},i} \frac{1}{2} \chi_{\mathbf{r},i}^{(\sigma)}(t) \partial_t \chi_{\mathbf{r},i}^{(\sigma)}(t) - H^{(\sigma)}(t) \right\}. \tag{9}$$

Here, $\varepsilon_1 = \varepsilon_3 = 1 = -\varepsilon_2 = -\varepsilon_4$ is the sign of the "time direction" in the corresponding part of the double contour. After averaging over disorder, the action has the form

$$S = \sum_{\sigma} \varepsilon_{\sigma} \int dt_1 dt_0 \delta'(t_1 - t_0) \left\{ \sum_{\mathbf{r},i} \frac{1}{2} \chi_{\mathbf{r},i}^{(\sigma)}(t_1) \chi_{\mathbf{r},i}^{(\sigma)}(t_0) \right\}$$
$$+ \frac{i}{2} \sum_{\mathbf{r}} \sum_{\sigma} \varepsilon_{\sigma_1} \varepsilon_{\sigma_0} \int dt_1 dt_0 \left\{ \frac{J^2}{4N^3} \left[\sum_i \chi_{\mathbf{r},i}^{(\sigma_1)}(t_1) \chi_{\mathbf{r},i}^{(\sigma_0)}(t_0) \right]^4 \right.$$
$$\left. + \sum_{\delta\mathbf{r}} \sum_{i,j} \frac{w^2}{2N} \chi_{\mathbf{r},i}^{(\sigma_1)}(t_1) \chi_{\mathbf{r},i}^{(\sigma_0)}(t_0) \chi_{\mathbf{r}+\delta\mathbf{r},j}^{(\sigma_1)}(t_1) \chi_{\mathbf{r}+\delta\mathbf{r},j}^{(\sigma_0)}(t_0) \right\}.$$

The field G is defined as

$$G_{\sigma_1\sigma_0}(t_1, t_0|\mathbf{r}) = -i \frac{1}{N} \sum_k \chi_{\mathbf{r},k}^{(\sigma_1)}(t_1) \chi_{\mathbf{r},k}^{(\sigma_0)}(t_0). \tag{10}$$

It is also necessary to introduce the field Σ defined as the Lagrange multiplier for the field G, thus making the latter field free of any relations except for symmetries associated with the replacement $t_1 \leftrightarrow t_0$.

Since the amplitude of tunneling between granules is much smaller than the energy scale of the interaction inside granules, the term with tunneling can be considered as a perturbation. Consequently, it is possible to retain only the fields G that depend only on one coordinate because granules in the unperturbed

problem are not coupled with each other. The following convenient notation is introduced:

$$\Sigma_{\sigma_1\sigma_0}^{(\text{free})}(t_1, t_0) = -\varepsilon_{\sigma_1} \delta_{\sigma_1,\sigma_0} \delta'(t_1 - t_0). \tag{11}$$

In terms of new fields and this notation, the action can be represented in the form

$$S = -\sum_{\mathbf{r},\sigma_1,\sigma_0} \frac{iN}{2} \int dt_1 dt_0 \Sigma_{\sigma_1\sigma_0}^{(\text{free})}(t_1, t_0) G_{\sigma_1\sigma_0}(t_1, t_0|\mathbf{r})$$

$$+ \frac{iN}{2} \sum_{\mathbf{r}} \sum_{\sigma} \varepsilon_{\sigma_1} \varepsilon_{\sigma_0} \int dt_1 dt_0 \left\{ \frac{J^2}{4} G_{\sigma_1\sigma_0}^4(t_1, t_0|\mathbf{r}) \right.$$

$$\left. - \sum_{\delta\mathbf{r}} \frac{w^2}{2} G_{\sigma_1\sigma_0}(t_1, t_0|\mathbf{r}) G_{\sigma_1\sigma_0}(t_1, t_0|\mathbf{r} + \delta\mathbf{r}) \right\} \tag{12}$$

$$- \frac{1}{2} \sum_{\mathbf{r}} \sum_{\sigma_1,\sigma_0} \int dt_1 dt_0 \Sigma_{\sigma_1\sigma_0}(t_1, t_0|\mathbf{r})$$

$$\times \left\{ iNG_{\sigma_1\sigma_0}(t_1, t_0|\mathbf{r}) - \sum_k \chi_{\mathbf{r},k}^{(\sigma_1)}(t_1) \chi_{\mathbf{r},k}^{(\sigma_0)}(t_0) \right\}.$$

The correlator given by Eq. (8) can be represented in terms of these variables; after its renormalization, this correlator has the form

$$\tilde{F}(t_1, t_2, t_3, t_4)$$
$$= \frac{\langle T_C G_{13}(t_1, t_3|\mathbf{r}_1) G_{24}(t_2, t_4|\mathbf{r}_2) \rangle}{\langle T_C G_{13}(t_1, t_3|\mathbf{r}_1) \rangle \langle G_{24}(t_2, t_4|\mathbf{r}_2) \rangle}. \tag{13}$$

The correlator in this form was obtained in [8, 9].

As seen, the action becomes quadratic in the fermion operators; therefore, the integral over them can be calculated and the action can be represented in terms of the fields G and Σ. However, before this, it is convenient to pass to the dimensionless variable $u \equiv 2\pi T t$, where T is the temperature of the system, and to rescale the field as follows:

$$\Sigma(t_1, t_0) \equiv J^2 \left(\frac{2\pi T}{J} \right)^{3/2} \Sigma(u_1, u_0),$$
$$G(t_1, t_0) \equiv \left(\frac{2\pi T}{J} \right)^{1/2} G(u_1, u_0). \tag{14}$$

The action in terms of new variables has the form

$$\frac{2S}{iN} = -\sum_{\mathbf{r}} \sum_{\sigma_1,\sigma_0} \int du_1 du_0 \left\{ -\frac{\varepsilon_{\sigma_1}\varepsilon_{\sigma_0}}{4} G_{\sigma_1\sigma_0}^4(u_1, u_0|\mathbf{r}) \right.$$

$$\times \left[\Sigma_{\sigma_1\sigma_0}^{(\text{free})}(u_1, u_0) + \Sigma_{\sigma_1\sigma_0}(u_1, u_0|\mathbf{r}) \right] G_{\sigma_1\sigma_0}(u_1, u_0|\mathbf{r})$$

$$+ \sum_{\delta\mathbf{r}} \frac{\varepsilon_{\sigma_1}\varepsilon_{\sigma_0}}{2} \left(\frac{w^2}{2\pi TJ} \right) \tag{15}$$

$$\left. \times G_{\sigma_1\sigma_0}(u_1, u_0|\mathbf{r}) G_{\sigma_1\sigma_0}(u_1, u_0|\mathbf{r} + \delta\mathbf{r}) \right\} - \text{Tr} \ln(\Sigma).$$

At a sufficiently low temperature $T \ll T_{FL} \sim \dfrac{w^2}{J}$, the last term in the action given by Eq. (15) becomes dominant and the behavior of the system is described by the convenient theory of Fermi liquid. The temperatures under consideration satisfy the condition $T \gg T_{FL}$. Under this condition, the term with tunneling can be considered perturbatively. The action where this term is neglected coincides with the action of the SYK model. Its properties are considered below.

3. MAIN PROPERTIES OF THE SYK MODEL

Instead of the calculation of the functional integral over the fields G and Σ, the saddle point approximation can be used because $S \propto N$; in this case, $N \gg 1$. As mentioned above, the term with w can be neglected at $T \gg T_{FL}$ when deriving the saddle point equations. The term with $\Sigma^{(\text{free})}$ is important at temperatures about J and at short times. Under the assumption $J \gg T \gg T_{FL}$, the saddle point equations for the fields G and Σ can be used in the form

$$\sum_\sigma \int du \Sigma_{\sigma_1 \sigma}(u_1, u | \mathbf{r}) G_{\sigma \sigma_0}(u, u_0 | \mathbf{r})$$

$$= \delta_{\sigma_1, \sigma_0} \delta(u_1 - u_0),$$

$$\Sigma_{\sigma_1 \sigma_0}(u_1, u_0 | \mathbf{r}) = \varepsilon_{\sigma_1} \varepsilon_{\sigma_2} G^{q-1}_{\sigma_1 \sigma_0}(u_1, u_0 | \mathbf{r}). \quad (16)$$

Here, the parameter $q = 4$ is introduced; in particular calculations, it is appropriate to consider this parameter arbitrary for dimensional regularization. For this reason, the solutions of the saddle point equations are represented below in the general form.

The equations presented above have a large group of symmetries. To demonstrate this, we consider an arbitrary set of monotonic functions $f_{\sigma,\mathbf{r}}(u)$; then, the following transformation can be performed:

$$G_{\sigma_1 \sigma_0}(u_1, u_0 | \mathbf{r}) \mapsto \left[f'_{\sigma_1,\mathbf{r}}(u_1) f'_{\sigma_0,\mathbf{r}}(u_0) \right]^\Delta \quad (17)$$
$$\times G_{\sigma_1 \sigma_0}(f_{\sigma_1,\mathbf{r}}(u_1), f_{\sigma_0,\mathbf{r}}(u_0) | \mathbf{r}),$$

where $\Delta = 1/q$. The substitution of the new function G gives an identity. However, since the initial Hamiltonian is independent of time, the desired Green's function should depend only on the time difference. In this case, the matrix elements of the field G are not independent because they are related through certain relations one of which is the fluctuation–dissipation theorem (which relates the Keldysh Green's function to the retarded and advanced Green's functions). Since the system under study is in equilibrium, the matrix elements of the field G can be indicated only in the first part of the double contour (correspondingly, $\sigma \in \{1, 2\}$ in Eq. (18)). Consequently, the translation-

ally invariant solution of the saddle point equations has the form

$$G^{(0)}_{\sigma_1 \sigma_0}(u_1, u_0 | \mathbf{r}) = b^\Delta g_{\sigma_1 \sigma_0}(u_1 - u_0),$$

$$g_{\sigma_1 \sigma_0}(u > 0) \equiv i \left[\frac{s'_{(0)}(u_1) s'_{(0)}(u_0)}{(s_{(0)}(u_1) - s_{(0)}(u_0))^2} \right]^\Delta \quad (18)$$

$$\times \begin{pmatrix} -e^{-i\pi\Delta} & e^{i\pi\Delta} \\ -e^{-i\pi\Delta} & e^{i\pi\Delta} \end{pmatrix}_{\sigma_1 \sigma_0},$$

where $b = \dfrac{(1 - 2\Delta) \tan(2\pi\Delta)}{2\pi}$ and $s_0(u) = e^u$; this notation is used for the below calculations. Using the property of symmetry $g_{\sigma_1 \sigma_0}(u) = -g_{\sigma_0 \sigma_1}(-u)$ of the Green's function, one can reconstruct the solution at $u < 0$. The group of symmetries of equations exceeds the group of symmetries of this solution, which is specified by the transformations

$$s_0(u) \mapsto s(u) = \frac{a s_0(u) + b}{c s_0(u) + d}. \quad (19)$$

Such a behavior usually indicates the presence of a soft (Goldstone) mode. However, our equations are approximate; in particular, the term $\Sigma^{(\text{free})}$ is neglected even at $w = 0$; hence, the revealed symmetry is asymptotic; this situation takes place when using σ models [16]. In the case under consideration, this means that, instead of the calculation of the entire functional integral over the fields G and Σ, it is necessary to integrate only over the fields that are solutions of the saddle point equations. The set of such fields can be specified as follows: all such fields are obtained by the application of the symmetry transformation given by Eq. (17) to the saddle point solution specified by Eq. (18). Passage from integration over all fields to integration over fields lying on a certain manifold is not accurate; it is applicable because fluctuations in the direction perpendicular to the manifold are suppressed by a factor of βJ [7].

This manifold is parameterized by functions $f_{\sigma,\mathbf{r}}(u)$. The action [7] on this manifold has the form

$$S = -C_J \sum_{\mathbf{r},\sigma} \varepsilon_\sigma \int du \, Sch\{e^{f_{\sigma,\mathbf{r}}}, u\}$$

$$- C_w \sum_{\delta\mathbf{r},\mathbf{r},\sigma_1,\sigma_0} \int du_1 du_0 \varepsilon_{\sigma_1} \varepsilon_{\sigma_0} \quad (20)$$

$$\times g^{(f)}_{\sigma_1 \sigma_0}(u_1, u_0 | \mathbf{r}) g^{(f)}_{\sigma_1 \sigma_0}(u_1, u_0 | \mathbf{r} + \delta\mathbf{r}).$$

Here, $C_J = \alpha_S N \dfrac{2\pi T}{J}$ and $C_w = \dfrac{N w^2}{4\pi J T}$ are the heat capacity of the SYK model and the contribution to the

heat capacity from perturbation, respectively, where $\alpha_S \approx 0.05$; $g^{(f)}$ (at $u_1 - u_2 > 0$) is the field defined as

$$g_{\sigma_1,\sigma_2}^{(f)}(u_1, u_0 | \mathbf{r}) = \left[f_{\sigma_1,\mathbf{r}}'(u_1) f_{\sigma_0,\mathbf{r}}'(u_0) \right]^{\Delta}$$
$$\times g_{\sigma_1\sigma_0}(f_{\sigma_1,\mathbf{r}}(u_1) - f_{\sigma_0,\mathbf{r}}(u_0)); \quad (21)$$

$Sch\{s(u), u\}$ is the Schwarzian derivative defined as

$$Sch\{s(u), u\} \equiv \left(\frac{s''}{s'} \right)' - \frac{1}{2} \left(\frac{s''}{s'} \right)^2. \quad (22)$$

One of the important properties of the Schwarzian derivative is that the replacement of s by a linear fractional transformation of s does not change the derivative. In terms of the introduced notation, the correlator we are going to calculate has the form

$$\tilde{F}(u_1, u_2, u_3, u_4) = \frac{\langle T_{\mathcal{C}} g_{1,3}^{(f)}(u_1, u_3 | \mathbf{r}_1) g_{2,4}^{(f)}(u_2, u_4 | \mathbf{r}_2) \rangle}{\langle g_{1,3}^{(f)}(u_1, u_3 | \mathbf{r}_1) \rangle \langle g_{2,4}^{(f)}(u_2, u_4 | \mathbf{r}_2) \rangle}. \quad (23)$$

The quadratic action under consideration is applicable at $w \gg N/J$ [17, 18]. Otherwise, the consideration is limited by the times $t \ll N/J$ [19]. Further, the OTOC is calculated.

4. LARGE-TIME BEHAVIOR OF THE OTOC

The main idea of the work is to use the following ansatz for the field $f(u)$:

$$e^{f(u)} = \frac{e^u + a(u)}{b(u)e^u + 1}, \quad (24)$$

where $a(u)$ and $b(u)$ are slowly varying functions. This ansatz allows one to obtain an expression for the field g following from the expression

$$4\sinh^2 \left(\frac{u_1 - u_0}{2} \right) \frac{f_{\sigma_1,\mathbf{r}}'(u_1) f_{\sigma_0,\mathbf{r}}'(u_0)}{4\sinh^2 \left(\frac{f_{\sigma_1,\mathbf{r}}(u_1) - f_{\sigma_0,\mathbf{r}}(u_0)}{2} \right)}$$

$$\approx \left[1 + 2 \frac{a_{\sigma_1,\mathbf{r}}(u_1) - a_{\sigma_0,\mathbf{r}}(u_0)}{e^{u_1} - e^{u_0}} \right. \quad (25)$$

$$\left. + 2 \frac{b_{\sigma_1,\mathbf{r}}(u_1) - b_{\sigma_0,\mathbf{r}}(u_0)}{e^{-u_1} - e^{-u_0}} \right]^{-1}.$$

Here, the derivatives of the fields a and b are neglected because these fields are slowly varying and only linear terms in a or b with exponentially increasing coefficients are retained because these fields are small ($a, b \propto N^{-1/2}$). However, their multiplication by exponentially large factors gives a combination that is not

small. The further consideration requires the auxiliary integral

$$a^{-\gamma} = \int_0^{\infty} s^{\gamma-1} e^{-sa} \frac{ds}{\Gamma(\gamma)}. \quad (26)$$

Using Eq. (26) twice, we can represent the correlator of interest in the form

$$\tilde{F}(u_1, u_2, u_3, u_4) = \int_0^{\infty} \frac{ds_o ds_e}{\Gamma(2\Delta)^2} (s_o s_e)^{(2\Delta)-1} e^{-s_e - s_o} \langle e^{iS_J} \rangle, \quad (27)$$

$$S_J = \sum_r \int du \left(\hat{j}_{a,\mathbf{r}}^T(u) \hat{a}_{\mathbf{r}}(u) + \hat{j}_{b,\mathbf{r}}^T(u) \hat{b}_{\mathbf{r}}(u) \right).$$

Here, $\hat{a}$ and $\hat{b}$ are columns consisting of the fields a and b, respectively. These columns have four components according to the parts of the contour. The column $\hat{j}_a$ has the form

$$\hat{j}_a = -i \begin{pmatrix} \dfrac{-2s_o \delta(u - u_1) \delta_{\mathbf{r},\mathbf{r}_1}}{e^{u_1} - e^{u_3}} \\ \dfrac{-2s_e \delta(u - u_2) \delta_{\mathbf{r},\mathbf{r}_2}}{e^{u_2} - e^{u_4}} \\ \dfrac{2s_o \delta(u - u_3) \delta_{\mathbf{r},\mathbf{r}_1}}{e^{u_1} - e^{u_3}} \\ \dfrac{2s_e \delta(u - u_4) \delta_{\mathbf{r},\mathbf{r}_2}}{e^{u_2} - e^{u_4}} \end{pmatrix}. \quad (28)$$

The vector $\hat{j}_b$ is defined similarly.

Now, it is possible to consider the action for the fields a and b. Since the chosen field f (24) is locally close to the saddle-point field, the quadratic action can be used to derive the action for the fields a and b. It is convenient to write the quadratic action introducing the field δf defined as $f = u + \delta f(u)$. Such a quadratic action was obtained in [14, 20] and has the form

$$S_2 = \frac{1}{2} \int \frac{d\Omega d^d \mathbf{p}}{(2\pi)^{d+1}} \delta \hat{f}_{\mathbf{p}}^{\dagger}(\Omega) \left[\hat{\mathcal{G}}(\Omega, \mathbf{p}) \right]^{-1} \delta \hat{f}_{\mathbf{p}}(\Omega),$$
$$\left[\hat{\mathcal{G}}(\Omega, \mathbf{p}) \right]^{-1} = \left[\hat{\mathcal{G}}^0(\Omega) \right]^{-1} - \hat{\Sigma}(\Omega, \mathbf{p}), \quad (29)$$

where $\mathcal{G}$ is the Green's function of soft modes for our action, $\mathcal{G}^0$ is the Green's function of soft modes in the SYK model, and $\Sigma \sim w^2$ is the self-energy part. The matrix notation is again used, joining four fields δf from different parts of the contour in one vector, and the frequency–momentum representation is also used. All matrices above are 4×4 matrices, but they are calculated for the equilibrium system taking into account the relation $\mathcal{G}_R(\Omega) = \mathcal{G}_A(-\Omega)$ between the retarded and advanced Green's functions and the fluctuation–dissipation theorem. We can write only

the retarded Green's function (all others are reconstructed from it):

$$\left[\mathcal{G}_R(\Omega,\mathbf{p})\right]^{-1} = \left[\mathcal{G}_R^0(\Omega)\right]^{-1} - \hat{\Sigma}_R(\Omega,\mathbf{p}),$$

$$\Sigma_R(\Omega,\mathbf{p}) = \frac{C_w}{2}\sum_{\delta r}\left\{\Omega^2\psi(\Omega)\right.$$

$$\left. + \frac{1}{2}\sin^2\left(\frac{\mathbf{p}\delta\mathbf{r}}{2}\right)\left(2(1+\Omega^2)+(1+2\Omega^2)\psi(\Omega)\right)\right\}, \quad (30)$$

$$\psi(\Omega) = \Psi\left(\frac{1}{2}-i\Omega\right) - \Psi\left(-\frac{1}{2}\right),$$

$$\Psi(z) = \partial_z\ln\Gamma(z), \quad \mathcal{G}_R^0(\Omega) = \frac{1}{C_J\Omega^2(\Omega^2+1)}.$$

The action for the fields $a(\Omega)$ and $b(\Omega)$ is defined for $\Omega \ll 1$ because of their slowness and has the form

$$S_{2,ab} = \int\frac{d\Omega d^d\mathbf{p}}{(2\pi)^{d+1}}\hat{a}_{\mathbf{p}}^{\dagger}(\Omega)\left[\hat{\mathcal{G}}(i+\Omega,\mathbf{p})\right]^{-1}\hat{b}_{\mathbf{p}}(\Omega). \quad (31)$$

For the quadratic action for the fields a and b, averaging can be performed:

$$\langle e^{iS_j}\rangle \approx \exp\{-s_e s_o z\}, \quad (32)$$

$$z \equiv \frac{i}{4}\frac{f_\alpha(\mathbf{r}_1-\mathbf{r}_2)}{C_J+\sum_{\delta r}C_w\left(\frac{\pi^2}{8}-1\right)}$$

$$\times\frac{\sinh\left(\frac{u_4+u_2-u_1-u_3}{2}\right)}{\sinh\left(\frac{u_4-u_2}{2}\right)\sinh\left(\frac{u_3-u_1}{2}\right)}, \quad (33)$$

$$f_\alpha(\mathbf{r}) \equiv \int_{BZ}\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{e^{i\mathbf{r}\mathbf{p}}}{1+\alpha\sum_{\delta r}\sin^2\left(\frac{\mathbf{p}\delta\mathbf{r}}{2}\right)}, \quad (34)$$

$$\alpha \equiv \frac{\sum_{\delta r}C_w\frac{\pi^2}{8}}{C_J+\sum_{\delta r}C_w\left(\frac{\pi^2}{8}-1\right)}$$

$$= \left[\frac{8\alpha_S}{d\pi^2}\left(\frac{2\pi T}{w}\right)^2+\left(1-\frac{8}{\pi^2}\right)\right]^{-1}. \quad (35)$$

The sign $\approx$ in Eq. (32) means that the terms proportional to s_e^2 and s_o^2 are omitted because they are "local," i.e., depend on the pair of times u_1 and u_3 or u_2 and u_4, do not increase exponentially, and are thereby small. As above, d is the dimension of the lattice. On the one hand, the function z increases exponentially with $u_1 - u_2$, and on the other hand, it decreases exponentially at large r. Finally, the explicit formula for the correlator can be written in the form

$$\tilde{F}(u_1,u_2,u_3,u_4) = \int_0^\infty\frac{ds_o ds_e}{\Gamma(2\Delta)^2}(s_o s_e)^{(2\Delta)}e^{-s_e-s_o}e^{-s_e s_o z}$$

$$= \int_0^\infty\frac{s^{2\Delta-1}}{\Gamma(2\Delta)}\frac{e^{-s}}{(1+sz)^{2\Delta}}ds = \frac{U\left(2\Delta,1,\frac{1}{z}\right)}{z^{2\Delta}} \quad (36)$$

$$= \begin{cases}1-4z\Delta^2 & z \ll 1, \\ \dfrac{\ln(z)}{z^{2\Delta}\Gamma(2\Delta)} & z \gg 1.\end{cases}$$

Here, U is the confluent hypergeometric function certainly defined by its asymptotic behavior. This expression together with Eq. (33) for z is the main result of this work. Importance of the result is as follows.

First, the OTOC for the SYK model is obtained at $C_w = 0$ and coincides with Eq. (6.10) from [8] and with the result of application of the method from [9] to the SYK model. Since the method of calculation proposed in this work differs from the methods in [8, 9], coincidence of the results indicates that the ansatz (24) is applicable for the problem with $w \neq 0$.

Second, the integral $f_\alpha(\mathbf{r})$ specifies the dependence on the distance between operators in the correlator. This integral decreases exponentially with the distance, but the form of the decrease depends not only on the magnitude of $\mathbf{r}$ but also on its direction at not overly large α values (the contribution from small p values dominates in the integral at large α values and this dependence disappears). Thus, information on perturbation in the system propagates ballistically, qualitatively coinciding with the behavior of other systems [2, 10, 11].

Third, according to Eq. (36), the correlator in the region with $z \gg 1$ is small, $\tilde{F} \approx 0$; i.e., applied perturbation "is known" at points in this region; perturbation "is unknown" in the region with $z \ll 1$, where $\tilde{F} \approx 1$. The region with $z \sim 1$ requires a more detailed consideration. In a fixed direction, $z \propto e^{\lambda_L\left(|t_{12}|-\frac{r_{12}}{v}\right)-\ln(N)}$, where $r_{12} = |\mathbf{r}_1-\mathbf{r}_2| \gg 1$, $t_{12} = t_1 - t_2 \approx t_3 - t_4$, and v is a parameter with the dimension of velocity. This expression shows that the region with $z \sim 1$ propagates in time along the fixed direction at the velocity v.

The velocities for various cases are presented below.

In the general case (arbitrary dimension), the universal behavior of the function f given by Eq. (34) is observed only at $\dfrac{r^2}{a^2} \gg \alpha \gg 1$, where a is the length of the edge of the lattice. In this case, $f \sim \exp\left\{-\sqrt{\dfrac{2}{\alpha}\dfrac{r}{a}}\right\}$; i.e., the front propagation velocity is given by the expression

$$v = 2\pi T \sqrt{\dfrac{\alpha}{2}} a, \qquad (37)$$

which is independent of the direction.

The asymptotic behavior of the function f in the one-dimensional system ($\mathbf{r} = (na)$) is described by the expression

$$f_\alpha^{(1D)}(r) = \begin{cases} \left(\dfrac{\alpha}{2}\right)^{|n|} & \alpha \ll 1, \\[2mm] \dfrac{1}{\sqrt{2\alpha}}\left(1 - \sqrt{\dfrac{2}{\alpha}}\right)^{|n|} & \alpha \gg 1. \end{cases} \qquad (38)$$

In the two-dimensional case, $\alpha \ll 1$ and

$$f_{\alpha \ll 1}^{(2D)}(\mathbf{r}) = \left(\dfrac{\alpha}{2}\right)^{|n|+|m|} C_{|n|+|m|}^{|m|}. \qquad (39)$$

Here, $\mathbf{r} = (na, ma)$; it is seen that this function strongly depends on the direction. At $\alpha \gg 1$, the function f has the form

$$f_{\alpha \gg 1}^{(2D)}(\mathbf{r}) = \dfrac{K_0\left(\sqrt{\dfrac{2}{\alpha}\dfrac{r}{a}}\right)}{\pi\alpha}. \qquad (40)$$

Let us consider the temperature dependence of the parameter α. According Eq. (35), α is independent of J and is determined only by the parameter T/w. At sufficiently high temperatures $T \gg w$, $\alpha \sim (w/T)^2$ depends on the temperature. In the opposite case $w \gg T$, the parameter $\alpha \approx \alpha_0 = \dfrac{\pi^2}{\pi^2 - 8}$ becomes universal; i.e., it is independent of the parameters of the system and of the temperature. Since $\alpha_0 \approx 5.27$, it can be assumed that $\alpha_0 \gg 1$.

5. CONCLUSIONS

To summarize, the behavior of the out-of-time time ordered correlator for the system of quantum dots has been investigated. The dynamics inside dots is described by the Sachdev−Ye−Kitaev model with a characteristic energy scale J. Tunneling with a characteristic amplitude w occurs between dots. It has been shown that the Lyapunov exponent of this system has the maximum value $\lambda_L = 2\pi T$, similar to the Sachdev−Ye−Kitaev model. Since the dimension is nonzero, the studied correlator has a spatial structure: the space

can be divided into two regions. In the first region, the applied perturbation with $\tilde{F} \approx 0$ "is known," whereas in the second region, the applied perturbation with $F \approx 1$ "is unknown." The boundary separating these regions propagates in time. Details of propagation generally depend on the parameters of the system and lattice. However, the boundary at $w \gg T$ can be considered as spherical and its propagation velocity $v_f = 2\pi T \sqrt{\dfrac{\alpha_0}{2}} a$ depends only on the temperature and the length of the edge of the lattice. It is also noteworthy that the properties of the system change strongly at $T \sim w$, although the contribution with w to the saddle point equation is insignificant at $T \gg T_{FL} \sim \dfrac{w^2}{J}$. A similar effect was also revealed in [14, 20].

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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