METHODS
OF THEORETICAL PHYSICS

Theory of Holomorphic Maps of Two-Dimensional Complex Manifolds to Toric Manifolds and Type-A Multi-String Theory

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We study the field theory localizing to holomorphic maps from a complex manifold of complex dimension 2 to a toric target (a generalization of A model). Fields are realized as maps to $(\mathbb{C}^*)^N$ where one includes special observables supported on (1,1)-dimensional submanifolds to produce maps to the toric compactification. We study the mirror of this model. It turns out to be a free theory interacting with N_{comp} topological strings of type A. Here, N_{comp} is the number of compactifying divisors of the toric target. Before the mirror transformation, these strings are vortex (actually, holomortex) strings.

1. INTRODUCTION

The goal of this paper is to generalize to complex dimension 2 the theory of holomorphic maps of Riemann surfaces to toric manifolds (an analog of Gromov−Witten or instanton theory). This may be considered as a first step in the construction of a four-dimensional (4D) quantum field theory of holomorphic maps of complex surfaces to complex manifolds that we will develop in an accompanying paper.

Here, we would like to consider the case of toric targets and proceed as in [1]. In this approach we will be able to represent everything in terms of free field theory, and still find interesting phenomena. In particular, we will see that the corresponding higher-dimensional theory is a gauge theory and that the A−I−B mirror of [1] is replaced by the multi-string theory of type A.

We will start with a brief reminder of the main constructions of [1], then we will show how they are generalized to the case of complex dimension 2, discuss new phenomena and point out interesting lines of further development.

2. BRIEF REVIEW OF A−I−B MIRROR SYMMETRY

In [1], Frenkel and Losev considered holomorphic maps from a Riemann surface to a toric variety (we will consider $\mathbb{C}P^1$ target as an example). The toric structure on $\mathbb{C}P^1$ means that we consider it as $\mathbb{C}^*$ compactified by two points (divisors) that we will call 0 and ∞. The natural linear structure on $\mathbb{C}^* \overset{\log}{\simeq} \mathbb{C}/2\pi i\mathbb{Z}$ allows one to introduce real coordinates R and Φ taking values in $\mathbb{R}$ and S^1, respectively. The complex coordinate on the target is $Z = R + i\Phi$. The main idea of [1] is to

consider each holomorphic map to $\mathbb{C}P^1$ as a holomorphic map from $\Sigma - \{P_{0,1}, P_{\infty,1}, \ldots, P_{0,d}, P_{\infty,d}\}$ to $\mathbb{C}^*$ where Σ is a Riemann surface (we will take Σ also to be $\mathbb{C}P^1$ for simplicity), the set $\{P_{0,1} \ldots P_{0,d}\}$ is the preimage of the divisor 0 on $\mathbb{C}P^1$ and the set $\{P_{\infty,1} \ldots P_{\infty,d}\}$ is the preimage of the divisor ∞. In this case the degree of the map is d. The functional integral contains integration over the configuration space of $2d$ points $\{P_{0,1}, \ldots, P_{\infty,d}\}$ in Σ (points are allowed to collide).[1] Now we may write the Mathai−Quillen representative for holomorphic maps to $\mathbb{C}^*$ and add special observables $\mathcal{O}_0, \mathcal{O}_\infty$ (called *holomortices*) at preimage points that imitate a holomorphic map. Namely, we will write the action as

$$S = -\frac{1}{2\pi} \int_\Sigma p\overline{\partial}Z - \overline{p}\partial\overline{Z} - \pi\overline{\partial}\psi + \overline{\pi}\partial\overline{\psi}, \qquad (1)$$

where p is a $(1, 0)$-form and $\overline{p}$ is a $(0, 1)$-form; p and $\overline{p}$ are Lagrange multipliers for holomorphic maps. Fermions ψ and $\overline{\psi}$ are the superpartners of coordinates Z and $\overline{Z}$ and fermions π and $\overline{\pi}$ are the superpartners of Lagrange multipliers. Then, one has, schematically,

$$\delta(\text{hol maps } \Sigma \to \mathbb{C}P^1 \text{ of degree } d)$$
$$= \int_{\Sigma^{2d} \ni (P_{0,1}, \ldots, P_{\infty,d})} \mathcal{D}p \int \mathcal{D}\overline{p}\mathcal{D}\pi\mathcal{D}\overline{\pi}e^{-S} \qquad (2)$$
$$\times \prod_{k=1}^{d} \mathcal{O}_0(P_{0,k})\mathcal{O}_\infty(P_{\infty,k}).$$

Note that the bosonic part of the action can also be rewritten as

$$S_{\text{bos}} = \frac{i}{2\pi} \int_\Sigma -Pd\Phi + PdR, \qquad (3)$$

where $P = p + \overline{p}$.

Consider some paths γ_i connecting points $P_{0,i}$ and $P_{\infty,i}$ and insert into the functional integral the expression

$$\exp\sum_k i \int_{\gamma_k} P. \qquad (4)$$

A simple computation shows that in the presence of such an observable one has the classical solution

$$Z = \log \prod_{k=1}^{d} \frac{(z - P_{0,k})}{(z - P_{\infty,k})} + C \qquad (5)$$

that is exactly a holomorphic map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ of degree d.

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[1] When the genus of the source is zero, the positions of preimages are not restricted, otherwise they are in the kernel of Abel−Jacobi map. In this paper for simplicity we consider simply-connected source.

Now we will describe the mirror map. Integrating over Φ (we assume for simplicity that the functional integral does not contain observables depending on Φ; i.e., all observables are $U(1)$-invariant), we get

$$dP = 0, \qquad (6)$$

which implies (for $\mathbb{C}P^1$ as a source)

$$P = dY, \qquad (7)$$

which defines a mirror coordinate Y taking values on a circle.[2] The case of higher-genus source manifolds will be discussed elsewhere.

Observables $\exp\sum_k i \int_{\gamma_k} P$ that were looking non-local become products of local observables

$$\exp\sum_k i \int_{\gamma_k} P = \prod_{k=1}^{d} \exp(-iY(P_{0,k}))\exp(iY(P_{\infty,k})). \qquad (8)$$

The bosonic part of the action takes the form $\frac{i}{2\pi}\int dYdR$ and integration over positions of the preimages turns into a deformation of the theory by the superpotential $\exp(iY) + \exp(-iY)$, see [1] for further details.

Generalization to general toric manifolds is almost obvious. $\mathbb{C}^*$ is generalized to $(\mathbb{C}^*)^N$ with coordinates $R^a, \Phi^a, a = 1, \ldots, N$. Compactifying divisors are given by N-dimensional integer vectors $\mathbf{D}_\beta \mathbb{Z}^N$ (here, β labels the set of compactifying divisors $\{D_\beta\}$); in the example above $N = 1$, $D_1 = \infty$ with $\mathbf{D}_1 = +1$ and $D_2 = 0$ with $\mathbf{D}_2 = -1$. The mirror superpotential is $\sum_\beta \exp\left(i\sum_{a=1}^{N} D_\beta^a Y_a\right)$.

3. GENERALIZATION TO COMPLEX SURFACES

From the previous section it is clear how to modify the theory for holomorphic maps ϕ of a complex surface X_4 (subscript 4 is the real dimension) to a toric manifold. We will cut out of the surface the preimages of compactifying divisors $\phi^{-1}D_\beta$ (which are holomorphic curves in X_4), and get on the complement the theory of holomorphic maps to $(\mathbb{C}^*)^N$. Below we will consider the case $N = 1$; we will present the case of a general toric target in (17).

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[2] An observable depending on Φ, for instance $e^{i\Phi}$, becomes a vortex for the field Y in the mirror theory.

Similarly to (1) we may write down the Mathai–Quillen representative for the delta-form on holomorphic maps inside smooth maps

$$S = -\frac{1}{2\pi} \int\limits_{X_4} p\overline{\partial}Z - \overline{p}\partial\overline{Z} - \pi\overline{\partial}\psi + \overline{\pi}\partial\overline{\psi}, \qquad (9)$$

where now p is a $(2, 1)$ form on X_4 and $\overline{p}$ is a $(1, 2)$-form. Similarly, we introduce the 3-form $P = p + \overline{p}$.

We have a new phenomenon: this theory has a gauge symmetry (well-known in the higher-dimensional theory of chiral fields, see [2])

$$p \rightarrow p + \overline{\partial}\nu, \quad \overline{p} \rightarrow \overline{p} + \partial\overline{\nu}, \qquad (10)$$

where ν and $\overline{\nu}$ are $(2, 0)$ and $(0, 2)$ forms on X_4, respectively, and there is a similar gauge symmetry for fermions.

The geometrical meaning of such symmetry in the present case comes from the syzygies of holomorphicity equations; naively, we have twice as many equations as variables, so naively (for almost complex manifolds) the virtual dimension of the space of almost holomorphic maps is $-\infty$. However, in the case of integrable complex structures on both source and target holomorphicity equations are linearly dependent, and this dependence results in syzygies that we see as gauge symmetry for the Lagrange multiplier field. We will discuss the Mathai–Quillen representative for the case of syzygies in a future publication.

Due to the linear structure on the target we may fix the gauge symmetry in the standard way, say, take Lorenz gauge with the help of a Kähler metric on X_4.

Now, generalizing the one-dimensional case, we put in the non-local observable

$$\exp\left(i\int\limits_{\Gamma_3} P\right), \qquad (11)$$

where Γ_3 is a 3-manifold such that its boundary is a collection of holomorphic curves in X_4. Note that this is required by gauge invariance of the observable: under the gauge transformation (10) it changes by the factor

$$\exp\left(i\int\limits_{\partial\Gamma_3} (\nu + \overline{\nu})\right), \qquad (12)$$

which is equal to 1 if the boundary of Γ_3 is of type $(1, 1)$ (that is, the tangent plane to the boundary has type $(1, 1)$ in the tangent space to X_4).

More precisely, Γ_3 can be a $\mathbb{Z}$-valued 3-chain on X_4 with boundary $\partial\Gamma_3 = \Sigma_{(0)} - \Sigma_{(\infty)}$ where $\Sigma_{(0)}$, $\Sigma_{(\infty)}$ are $(1, 1)$-cycles with positive integer coefficients.

Now, it is possible to check that the classical solution Z turns out to be a holomorphic map from the complex surface X_4 to $\mathbb{C}P^1$.

It is possible to construct the mirror map. Indeed, integrating out Φ, we get

$$dP = 0. \qquad (13)$$

If X_4 has no third cohomology, this means that

$$P = d\Omega, \qquad (14)$$

where Ω is a two-form, so the observable takes the form

$$\exp\left(-i\int\limits_{\Sigma_{(0)}} \Omega\right)\exp\left(i\int\limits_{\Sigma_{(\infty)}} \Omega\right)$$

while the bosonic part of the action is

$$\frac{1}{\pi}\int\limits_{X_4} \Omega\partial\overline{\partial}R. \qquad (15)$$

From the equations of motion for R we see that R is the real part of a holomorphic function.

Note that here bosonic gauge degrees of freedom correspond to the $(2, 0)$- and $(0, 2)$-form components of Ω and explicitly decouple from the action. Still, we have a new gauge symmetry in bosonic sector

$$\Omega \rightarrow \Omega + d\Lambda. \qquad (16)$$

This completes the story for $N = 1$. For general N, the field Ω couples to the preimages of compactifying divisors as

$$\prod_\beta \exp\left(i\int\limits_{\Sigma_\beta} \sum_{a=1}^N D_\beta^a \Omega_a\right), \qquad (17)$$

where $\mathbf{D}_\beta$ are again seen as vectors in $\mathbb{Z}^N$; Σ_β is a $\mathbb{Z}$-valued $(1, 1)$-cycle on X_4, i.e., the preimage of D_β.

Remark 1. If $H^{1,1}(X_4) \neq 0$, then there exists another global symmetry given by a shift by cohomology:

$$\Omega \rightarrow \Omega + \omega^{(1,1)}, \quad \omega^{(1,1)} \in H^{1,1}(X_4). \qquad (18)$$

This symmetry leads to a selection rule: a correlator including an observable as in (17) vanishes unless

$$\sum_\beta \mathbf{D}_\beta[\Sigma_\beta] = 0 \in H_2(X_4) \otimes \mathbb{Z}^N. \qquad (19)$$

4. BIRD'S-EYE VIEW ON THE EMERGING THEORY

(i) We constructed the theory of holomorphic maps from simply-connected $2_\mathbb{C}$-dimensional complex manifolds to toric targets as a QFT.

(ii) This provides a strong evidence for existence of the theory of holomorphic maps between complex manifolds in all dimensions of both source and target. We will construct such theory for a $2_\mathbb{C}$-dimensional complex source in a forthcoming paper. This theory will be a generalization of 2D BF theory, namely, the

theory $\int p \bar{\partial} X$ + c.c. + fermions with gauge symmetry $p \to p + \bar{D}\varepsilon$.

This generalization is two-fold: (a) $\int BdA$ is considered as $\int AdB$, i.e., as a Mathai–Quillen representative of the delta-form on constant maps, where gauge symmetries come from syzygies of equations ($\int d\varepsilon dB = 0$). We will also explain the generalization of flat supersymmetric BF theory to general curved targets.[3]

(b) In going to complex dimension 2, the exterior derivative d gets replaced by the Dolbeault operator $\bar{\partial}$.

Suppose that X_4 is a Calaby–Yau manifold with Calabi–Yau form ω_{CY}. Then we can switch to new variables

$$p = \omega_{CY}\bar{A}, \quad \pi = \omega_{CY}\psi_{DW},$$
$$\nu = \omega_{CY}\varepsilon_{Maxwell}, \quad H = \omega_{CY}Z, \quad \lambda = \omega_{CY}\psi. \quad (20)$$

Then, the action takes the form

$$\int H\bar{\partial}\bar{A} + \lambda\bar{\partial}\psi_{DW}, \quad (21)$$

which corresponds to the Mathai–Quillen representative for the Donaldson–Witten abelian theory of holomorphic bundles modulo complex gauge group that is equivalent (on Kähler manifolds) to the theory of self-dual connections modulo the compact group.[4]

However, despite the theories being the same, their Q-differentials are different. While the differential Q_{HM} of the theory of holomorphic maps acts as

$$Q_{HM}(Z) = \psi, \quad Q_{HM}(\pi) = p \quad (22)$$

the differential of Donaldson–Witten theory acts in the opposite way

$$Q_{DW}\bar{A} = \psi_{DW}, \quad Q_{DW}\lambda = H \quad (23)$$

because in the theory of holomorphic maps Z is a field and p is a Lagrange multiplier, while in the Donaldson–Witten theory $\bar{A}$ (proportional to p) is a field and H (proportional to Z) is a Lagrange multiplier.

We find a similar story in supersymmetric Poisson sigma model in real dimension 2. Since Donaldson–Witten theory may be generalized to non-abelian case and the theory of holomorphic maps may be generalized to non-toric targets, we conjecture the existence

of the universal generalized gauge theory that contains Donaldson–Witten and the theory of holomorphic maps as its particular limits. We expect such theory to contain two differentials: one that generalizes Donaldson–Witten and another that generalizes the de Rham differential of the holomorphic maps theory.

We expect that the two-dimensional version of such theory is given by the supersymmetric Poisson sigma model that we will study in a separate paper.

(iii) For toric targets we constructed the analog of a mirror theory where the gas of points (leading to the superpotential) is replaced by a multi-string theory.

Namely, holomorphic maps from X_4 to a toric manifold $\mathcal{T}$ are expressed by N_{comp} types of topological strings of type A with the target X_4. Here, N_{comp} is the number of compactifying divisors in $\mathcal{T}$.

Moreover, consider natural evaluation observables in the theory of holomorphic maps $\phi: X_4 \to \mathcal{T}$, $\int_C \mathcal{O}_{D_\beta}^{(\dim C)}$, where C is a cycle in X_4. Namely, consider a map $f: \{1,\ldots,k\} \to \{1,\ldots,N_{comp}\}$, sending $i \mapsto \beta = f(i)$, and consider the correlator $\left\langle \prod_{i=1}^k \int_{C_i} \mathcal{O}_{D_{f(i)}}^{(\dim C_i)} \mathcal{O}_{p_\mathcal{T}}^{(0)}(p_{X_4}) \right\rangle$ that is the number of maps ϕ such that $\phi(C_i)$ intersects $D_{f(i)}$ for all i and such that $\phi(p_{X_4}) = p_\mathcal{T}$ (here, p_{X_4}, $p_\mathcal{T}$ are some fixed points in the source and the target; this fixes $(\mathbb{C}^*)^N$ action on the space of holomorphic maps corresponding to a set of given preimages of compactifying divisors, cf. the shift by a constant in (5)).

For each β, consider a type A correlator in the theory of maps $\Sigma \to X_4$, $n_\beta = \left\langle \prod_{i\in f^{-1}(\beta)} \int_\Sigma \mathcal{O}_{C_i}^{(2)} \right\rangle$ is the number of holomorphic maps passing through cycles C_i, with $f(i) = \beta$. In n_β we are also integrating over complex structures on Σ and summing over topological types of Σ (possibly disconnected, with connected components of any genus).

The relation between the theory of holomorphic maps $X_4 \to \mathcal{T}$ and maps $\Sigma \to X_4$ is schematically given by

$$\left\langle \prod_{i=1}^k \int_{C_i} \mathcal{O}_{D_{f(i)}}^{(\dim C_i)} \mathcal{O}_{p_\mathcal{T}}^{(0)}(p_{X_4}) \right\rangle = \prod_{\beta=1}^{N_{comp}} n_\beta. \quad (24)$$

In some sense, the theory of holomorphic maps $X_4 \to \mathcal{T}$ looks like a second quantized field theory for N_{comp} types of strings, see Fig. 1. We are planning to study this unexpected phenomenon in the future.

(iv) One is tempted to make the following speculation. The mirror theory is a theory of several types of strings coupled to free field theory. It looks similar to compactifications of M-theory that are, roughly speaking, Lagrangian field theories coupled to a set of
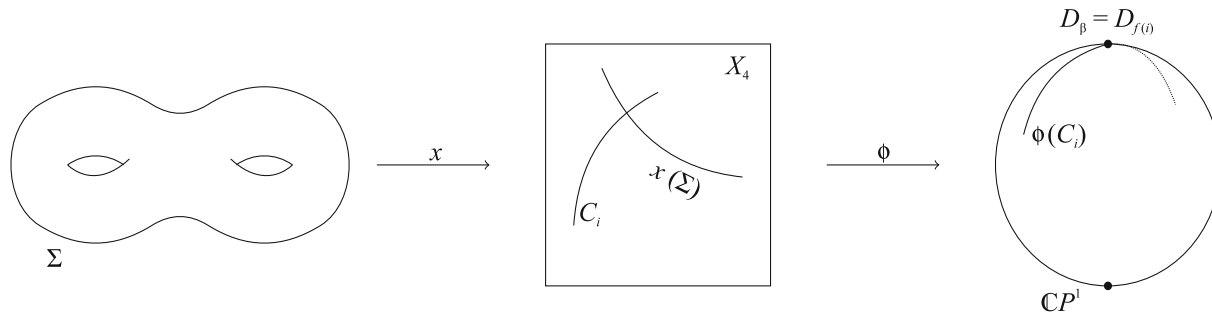
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[3] Abelian BF with curved target, a.k.a. Poisson sigma model with zero Poisson bivector was studied from the viewpoint of formal geometry of the target [3]. It seems that a simpler approach can be used for supersymmetric Poisson sigma model. An analog for such simplification is explained in instanton theory approach [4], on the other hand non-supersymmetric curved $\beta\gamma$-system requires formal geometry approach (see [5–8]).

[4] An observation that holomorphic BF theory with matter with 4-dimensional source leads to holomorphic maps to a toric target was made in [9] (Introduction, p. 10).

Fig. 1. Holomorphic map ϕ is parametrized by the preimages of compactifying divisors. These preimages are, in turn, strings in X_4 images of Riemannian surfaces.

extended objects. It is desirable, but not known, how to understand compactifications of M-theory as a field theory in Segal's sense. The theory that we consider seems to provide a much simpler example of this phenomenon. One can go even further. Among similar examples modeling M-theory are instantonic strings (instantons exist in codimension 4) in 6D gauge theories. Interestingly, such strings can be considered as fundamental strings, bound to a 6D NS5-brane in type IIB theory (that is why they are called "little strings" in [10]). More recently people considered vortex strings. Since vortices have codimension 2, they exist in 4D theories and there are strong arguments that they are also bound states of fundamental strings [11, 12]. Strings that we consider are also vortex strings (in [1] the corresponding object was called "holomortex") and seem to be the topological sector of Shifman−Yung's little strings. Certainly, we are planning to investigate this relation further.

(v) It seems very interesting to study tropicalization [13, 14] of the constructed theory. Not only it allows computations and turns algebraic geometry into combinatorics: tropicalization of a string theory is also a field theory. Thus, after tropicalization, a 4D theory with strings becomes a 2D theory with particles that may be related to Feynman diagrams in conventional 2D theory, despite not being Lorentz-invariant. We plan to study this tropicalization elsewhere. Tropicalization may lead to an approach to a higher-dimensional analog of Witten−Dijkgraaf−Verlinde−Verlinde (WDVV) theory. Tropicalization of the theory on a toric manifold is a field theory on its moment polytope. In particular, for the source of complex dimension 2, we will get a field theory on a convex polygon. Feynman diagrams will be tropical curves that are graphs with straight edges. Summing up these graphs will allow to replace tropical strings by "integrating in" another field (in the terminology of Seiberg); this is an analog of non-perturbative summation of instantons in complex dimension 1. A theory of this type was constructed in [15].

(vi) It is clear from the discussion above that consideration of holomorphic maps is not restricted to source manifolds of complex dimension 2. However, it may be technically more complicated. For instance, in tropicalization graphs with straight edges are replaced by polyhedral complexes. That is why we restrict to complex dimension 2.

(vii) *Disclaimer A.* Actually, we are dealing not with holomorphic maps but rather with Drinfeld's holomorphic quasi-maps (since, e.g., for $\mathbb{C}P^1$ target the preimages of $\{0\}$ and $\{\infty\}$ are allowed to intersect).

Disclaimer B. The situation becomes more complicated when the source is not simply-connected. In particular, formulas (7) and (14) have corrections due to 1st and 3rd cohomology of the source. We will address this issue in the nearest future.

(viii) Holomorphic quasi-maps between two toric varieties of any dimension can be effectively described, and an analog of Givental−Nekrasov theory [16, 17] can be constructed. It may lead to another approach to higher dimensional version of WDVV theory.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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