Far Field of a Three-Dimensional Laminar Wall Jet

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Abstract—We consider a steady submerged laminar jet of viscous incompressible fluid flowing out of a tube and propagating along a solid plane surface. The numerical solution of Navier–Stokes equations is obtained in the stationary three-dimensional formulation. The hypothesis that at large distances from the tube exit the flowfield is described by the self-similar solution of the parabolized Navier–Stokes equations is confirmed. The asymptotic expansions of the self-similar solution are obtained for small and large values of the coordinate in the jet cross-section. Using the numerical solution the self-similarity exponent is determined. An explicit dependence of the self-similar solution on the Reynolds number and the conditions in the jet source is determined.

Keywords: wall jet, submerged jet, self-similar solutions of the first and second kind, conservation law, asymptotics, invariants

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The solution of the two-dimensional problem of a submerged jet blown along a flat plate from an infinitely thin slit is well known [1, 2]. In the boundary layer approximation this flow is self-similar. The self-similarity exponent is determined using a jet invariant, whose dimension is equal to the product of the momentum flux by the flow rate.

At the same time, little is known about a submerged jet flowing out of a tube in parallel to an infinite solid plane surface. The foundation of the mathematical model of this flow was laid in [3]. As viscosity tends to zero, the flow region, where the viscous and inertia terms in Navier–Stokes equations are of the same order, becomes narrow. In this region, the flow is described by the parabolized Navier–Stokes equations, which possess a one-parameter class of solutions, self-similar in the longitudinal coordinate. A hypothesis was put forward [3] that the jet far-field is a solution belonging to this class. For the equations used the local conservation laws of mass, momentum, and angular momentum were calculated. However, the fluxes of these quantities through the wall jet cross-section are not conserved, which led to an incorrect determination of the self-similarity exponent in [3].

The self-similar solutions are asymptotics of the solutions for non-self-similar jets at a large distance from the source. The self-similar far field of a jet depends on the integral characteristics of the source, which are usually determined from the conservation laws. The existence of conserved quantities makes it possible to consider the passage to the point-source limit. In this case, the self-similar solution is the exact solution for the problem of jet outflow from a point source.

The solutions in which the self-similarity exponent is determined using the integral conservation law are conventionally named the self-similar solutions of the first kind [4]. Such are the solution for a two-dimensional wall jet [1, 2] and those for free submerged jets [5–7] in which the momentum flux is conserved. At present, the solution for a three-dimensional wall jet should be attributed to the self-similar solutions of the second kind, since so far the integral invariant of this problem has not been determined.

According to the reviews [8, 9], the numerical and experimental studies of three-dimensional wall jets have been aimed chiefly at an investigation of turbulent jets. Laminar jets were investigated in only few studies [10–12]. In all the studies the jet far-field was taken to be self-similar. At the same time, thorough comparisons of the numerical solutions or experimental data with the self-similar solution [3], which could determine the self-similarity exponent and study the self-similar solution properties in detail, were not performed. This study closes the existing gap.

Fig. 1. Cylindrical coordinate system and boundary conditions.

1. SELF-SIMILAR SOLUTION

We will consider a stationary laminar jet of viscous incompressible fluid flowing out into the ambient space filled with the same fluid of a tube with the characteristic cross-sectional dimension a , parallel to an infinite solid plane surface. Let u_0 be the mean-flow-rate velocity in the tube. We will determine the dimensionless variables as follows:

$$\mathbf{r} = \frac{\mathbf{r}^*}{a}, \quad \mathbf{V} = \frac{\mathbf{V}^*}{u_0}, \quad p = \frac{p^* - p_\infty^*}{\rho u_0^2}, \quad \text{Re} = \frac{u_0 a}{\nu},$$

where $\mathbf{r}$ is the radius-vector of the point in the flowfield having the coordinates (x, r, φ) in the cylindrical coordinate system (Fig. 1); $\mathbf{V} = (u, v, w)$ is the velocity vector and its components in the cylindrical coordinate system; ρ is the constant fluid density; p is the pressure; p_∞^* is the pressure in the undisturbed region; Re is the Reynolds number; and ν is kinematic viscosity. The asterisks refer to dimensional variables. The jet propagates along the x axis. The angle φ is measured from the plane of symmetry; the values $\varphi = \pm\pi/2$ correspond to the solid plane. The origin of the coordinate system is in the plane of the tube exit.

The functions $\mathbf{V}(\mathbf{r})$ and $p(\mathbf{r})$ satisfy the Navier–Stokes equations

$$\nabla \cdot \mathbf{V} = 0, \quad (\mathbf{V} \nabla) \mathbf{V} = -\nabla p + \text{Re}^{-1} \Delta \mathbf{V}. \quad (1.1)$$

The boundary conditions are determined by the velocity distribution in the initial tube section ($x = -L$, where L is the tube length), the no-slip on the solid surfaces, and the decay of all the velocity components at large distances from the tube. The boundary conditions in the plane of symmetry are as follows: $u_\varphi = v_\varphi = w = 0$. Here and in what follows, the subscripts denote partial derivatives.

It is known [3] that in the boundary layer approximation three-dimensional wall jets are described by the parabolized Navier–Stokes equations differing from Eqs. (1.1) by the absence of the terms p_x , u_{xx} , v_{xx} , and w_{xx}

$$(ur)_x + (vr)_r + w_\varphi = 0, \quad (1.2)$$

$$uu_x + vu_r + \frac{wu_\varphi}{r} = \text{Re}^{-1} \left(u_{rr} + \frac{u_r}{r} + \frac{u_{\varphi\varphi}}{r^2} \right), \quad (1.3)$$

$$uv_x + vv_r + \frac{vw_\varphi}{r} - \frac{w^2}{r} = -p_r + \text{Re}^{-1} \left(v_{rr} + \frac{v_r}{r} + \frac{v_{\varphi\varphi}}{r^2} - \frac{v}{r^2} - \frac{2w_\varphi}{r^2} \right), \quad (1.4)$$

$$uw_x + vw_r + \frac{ww_\varphi}{r} + \frac{vw}{r} = -\frac{p_\varphi}{r} + \text{Re}^{-1} \left(w_{rr} + \frac{w_r}{r} + \frac{w_{\varphi\varphi}}{r^2} - \frac{w}{r^2} + \frac{2v_\varphi}{r^2} \right). \quad (1.5)$$

We will present the general form of the solution of system (1.2)–(1.5), self-similar with respect to the longitudinal coordinate

$$u(x, r, \varphi) = \text{Re}^{2\alpha-1} x^{-(2k-1)} U(\eta, \varphi), \quad (1.6)$$

$$v(x, r, \varphi) = \operatorname{Re}^{\alpha-1} x^{-k} V(\eta, \varphi), \quad (1.7)$$

$$w(x, r, \varphi) = \operatorname{Re}^{\alpha-1} x^{-k} W(\eta, \varphi), \quad (1.8)$$

$$p(x, r, \varphi) = \operatorname{Re}^{2\alpha-2} x^{-2k} P(\eta, \varphi), \quad (1.9)$$

$$\eta = r/(\operatorname{Re}^{-\alpha} x^k), \quad (1.10)$$

where α and k are real parameters.

The functions U , V , W , and P satisfy the equations

$$(1-2k)U - k\eta U_\eta + V_\eta + \eta^{-1}V + \eta^{-1}W_\varphi = 0, \quad (1.11)$$

$$((1-2k)U - k\eta U_\eta)U + VU_\eta + \eta^{-1}WU_\varphi - (U_{\eta\eta} + \eta^{-1}U_\eta + \eta^{-2}U_{\varphi\varphi}) = 0, \quad (1.12)$$

$$\begin{aligned} & -k(V + \eta V_\eta)U + VV_\eta + \eta^{-1}WW_\varphi - \eta^{-1}W^2 \\ & + P_\eta - (V_{\eta\eta} + \eta^{-1}V_\eta + \eta^{-2}V_{\varphi\varphi} - \eta^{-2}V - 2\eta^{-2}W_\varphi) = 0, \end{aligned} \quad (1.13)$$

$$\begin{aligned} & -k(W + \eta W_\eta)U + VW_\eta + \eta^{-1}WW_\varphi + \eta^{-1}VW \\ & + \eta^{-1}P_\varphi - (W_{\eta\eta} + \eta^{-1}W_\eta + \eta^{-2}W_{\varphi\varphi} + 2\eta^{-2}V_\varphi - \eta^{-2}W) = 0. \end{aligned} \quad (1.14)$$

We will adopt the hypothesis [3] that at large distances from the tube exit, $x \rightarrow +\infty$, the flowfield is described by the self-similar solution (1.6)–(1.10) of Eqs. (1.2)–(1.5).

2. COORDINATE EXPANSION, AS $\eta \rightarrow 0$

From an analysis of the numerical solution given in Section 4 it follows that in the jet under consideration the azimuthal velocity is much smaller than the longitudinal and radial ones. We will construct an asymptotic expansion in the small radial coordinate of the solution of Eqs. (1.11)–(1.14) bearing in mind the symmetry condition at $\varphi = 0$ and the no-slip conditions at $\varphi = \pm\pi/2$

$$U = \eta U_1(\varphi) + \eta^4 U_2(\varphi) + \eta^7 U_3(\varphi) + o(\eta^7), \quad (2.1)$$

$$V = \eta^2 V_1(\varphi) + \eta^5 V_2(\varphi) + O(\eta^8), \quad (2.2)$$

$$W = \eta^5 W_2(\varphi) + O(\eta^8), \quad (2.3)$$

$$P = P_0 + \eta P_1(\varphi) + \eta^4 P_2(\varphi) + O(\eta^7), \quad (2.4)$$

$$U_1(\varphi) = A \cos \varphi,$$

$$U_2(\varphi) = -\frac{1}{24} A^2 \left(k - \frac{1}{3}\right) (\cos 2\varphi + 1)^2,$$

$$U_3(\varphi) = C \cos 7\varphi + \left(\left(\frac{13}{840} A^2 \left(k - \frac{1}{3}\right) \left(k - \frac{4}{39}\right) - \frac{8}{21} B\right) \cos^2 \varphi + \frac{1}{3} B\right) A \cos^5 \varphi,$$

$$V_1(\varphi) = A \left(k - \frac{1}{3}\right) \cos \varphi,$$

$$\begin{aligned} V_2(\varphi) = & -\frac{1}{24} A^2 \left(k - \frac{1}{3}\right) \left(k - \frac{1}{6}\right) (\cos 2\varphi + 1)^2 \\ & - \frac{1}{24} A^2 (k-1) \left(k - \frac{1}{3}\right) \left(\frac{1}{8} \cos 2\varphi + \frac{1}{5} \cos 4\varphi + \frac{3}{40} \cos 6\varphi\right) - \frac{2}{3} B \cos 4\varphi - \frac{2}{3} B \cos 6\varphi, \end{aligned}$$

$$\begin{aligned} W_2(\varphi) = & \frac{1}{16} A^2 (k-1) \left(k - \frac{1}{3}\right) \left(\frac{1}{4} \sin 2\varphi + \frac{1}{5} \sin 4\varphi + \frac{1}{20} \sin 6\varphi\right) \\ & + B \sin 4\varphi + \frac{2}{3} B \sin 6\varphi, \end{aligned}$$

$$P_1(\varphi) = 2A \left(k - \frac{1}{3}\right) \cos \varphi,$$

$$P_2(\varphi) = -\frac{2}{3} \left(\left(k - \frac{1}{3}\right) \left(k - \frac{7}{12}\right) A^2 + 40B\right) \cos^4 \varphi + \frac{80}{3} B \cos^2 \varphi - \frac{10}{3} B,$$

The expansion thus obtained depends on four unknown constants A , P_0 , B , and C . The constants P_0 and B are related with the pressure distribution over the wall, as $\eta \rightarrow 0$, as follows:

$$P = P_0 - \frac{10}{3} B \eta^4, \quad \varphi = \pm \frac{\pi}{2}. \quad (2.5)$$

We note that in the first approximation the azimuthal velocity is absent, while the longitudinal and radial velocities depend on the azimuthal angle in the same fashion.

3. COORDINATE EXPANSION, AS $\eta \rightarrow +\infty$

We will assume that at large values of the radial coordinate, $\eta \rightarrow +\infty$, the longitudinal and azimuthal velocity components decay more rapidly than the radial velocity. Then Eqs. (1.11), (1.13), and (1.14) take the form:

$$\begin{aligned} V_\eta + \eta^{-1} V &= 0, \\ V V_\eta + P_\eta - (V_{\eta\eta} + \eta^{-1} V_\eta + \eta^{-2} V_{\varphi\varphi} - \eta^{-2} V) &= 0, \\ 2V_\varphi - \eta P_\varphi &= 0. \end{aligned}$$

The equations obtained supplemented with the boundary condition $V = 0$ at $\varphi = \pm\pi/2$ correspond to the Hamel problem [13] of flow in a contractor with the semi-angle $\pi/2$. The solution of the problem is as follows:

$$V = \eta^{-1} F(\varphi), \quad (3.1)$$

$$P = 2\eta^{-2} F(\varphi) + \eta^{-2} E,$$

where E is a constant and $F(\varphi)$ satisfies the equation

$$F'' + 4F + F^2 + 2E = 0 \quad (3.2)$$

at $F(\pm\pi/2) = 0$.

We will reduce the order of Eq. (3.2)

$$F' = \pm \left(-4F^2 - \frac{2F^3}{3} - 4EF + C_1 \right)^{1/2}. \quad (3.3)$$

We will search a symmetric solution $F(\varphi) < 0$ having a minimum in the plane of symmetry $F(0) = -F_0$. From the condition $F'(0) = 0$ it follows that $C_1 = 4F_0^2 - 2F_0^3/3 - 4EF_0$. Then $-4F^2 - 2F^3/3 - 4EF + C_1 = (F + F_0)(4(F_0 - F) - 2(F^2 - F_0F + F_0^2)/3 - 4E)$.

For an arbitrary quantity $F_0 > 0$ the solution takes the form:

$$\varphi = \text{sign}(\varphi) \int_{-F_0}^{F(\varphi)} \left[(F + F_0) \left(4(F_0 - F) - \frac{2(F^2 - F_0F + F_0^2)}{3} - 4E \right) \right]^{-1/2} dF.$$

The constant E is determined from the integral relation

$$\frac{\pi}{2} = \int_{-F_0}^0 \left[(F + F_0) \left(4(F_0 - F) - \frac{2(F^2 - F_0F + F_0^2)}{3} - 4E \right) \right]^{-1/2} dF.$$

We will now derive the limiting solution, as $F_0 \rightarrow +\infty$. In this case, the outer solution is a constant: $F(\varphi) = -F_0$. From Eq. (3.2) we obtain that $E = -F_0^2/2$.

Let us measure the angle θ from the wall

$$\varepsilon\theta = \frac{\pi}{2} - \varphi,$$

$$F = F_0 f(\theta),$$

where ε is a small angle confining the boundary layer. The variables θ and f are of the order of unity.

We will write Eq. (3.3) in the new variables, as $F_0 \rightarrow +\infty$

$$\varepsilon^{-1} F_0 \frac{df}{d\theta} = -F_0^{3/2} \left[(f+1) \left(-\frac{2(f^2 - f + 1)}{3} + 2 \right) \right]^{1/2}.$$

Thence we determine the angular dimension of the boundary layer $\varepsilon = 1/\sqrt{F_0}$. We have

$$\frac{df}{d\theta} = -\sqrt{\frac{2}{3}}(f+1)\sqrt{2-f}.$$

This equation can be integrated

$$f = 2 - 3 \tanh^2 \left(\operatorname{arctanh} \left(\sqrt{\frac{2}{3}} \right) + \frac{\theta}{\sqrt{2}} \right).$$

The solution depends on one unknown constant F_0 .

4. ANALYSIS OF THE NUMERICAL SOLUTION

Numerical solutions of the Navier–Stokes equations (1.1) were obtained for the problem of laminar jet propagation along a solid wall in the case of jet outflow from a tube of rectangular or circular cross-section at different Reynolds numbers. The method of finite volume was used. The convective terms were approximated according to the second-order scheme with upwind differences. The diffusion terms and the pressure were approximated according to the second-order scheme with central differences. The stationary solution was determined using the method of steady state attainment. The convergence criterion was the establishment of a constant flow rate through a far cross-section of the computation domain at a maximum value of the residuals less than 10^{-5} .

The calculations of jet outflow from a tube of circular cross-section were performed on a grid used in [14]. The longitudinal dimension of the computation domain was 550 diameters, while the dimensions of the rectangular cross-section of the computation domain could change linearly with increase in the longitudinal coordinate from 60×30 to 536×266 , where the former dimension is given in the z direction (Fig. 1) and the latter in the y direction. The tube length was 10. The distance from the tube axis to the solid plane surface was equal to two diameters. The number of the grid cells was 10.3×10^6 . The problem was solved at the Reynolds numbers 100, 150, and 200, where the Reynolds number is based on the tube diameter and the mean-flow-rate velocity in the tube. The symmetry condition was not used, which made it possible to ascertain that a symmetric flow was realized in all the cases considered.

The calculations of jet outflow from a tube with a rectangular cross-section measuring 1×0.5 , where the smaller dimension is given in the direction of the normal to the solid plane, were also performed. The problem was solved with the plane of symmetry. The longitudinal dimension of the computation domain was 700, while the dimensions of the square cross-section of the computation domain varied linearly with increase in the longitudinal coordinate from 50×50 to 1750×1750 . The tube, 14 in length, lies on the solid plane surface. The number of the grid cells was 5.7×10^6 . The longitudinal grid dimension amounts to 600 cells, 100 cells of them being located along the tube. The grid dimension in the cross-section is 100×100 cells, 50×50 cells of this number corresponding to the jet. The grid in a tube cross-section contains 30×30 cells. The flow was studied at the Reynolds number 200; it is based on the mean-flow-rate velocity and the greater side of the rectangular cross-section.

In both calculated cases a uniform distribution of the longitudinal velocity was preassigned at the tube entry. Zero normal derivatives of all velocity components were preassigned on all boundaries of the computation domain, except from the tube entry section and the plane of symmetry.

The sensitivity of the numerical solution with respect to grid refinement was investigated using the calculations on a coarser grid with double cell dimensions in all directions. The profiles of the longitudinal and radial velocities were compared in the cross-section $x = 400$ at $\varphi = 0$ and $\pi/4$. The difference of the profiles calculated on different grids is less than 1% in the high-velocity jet region $r \leq 100$.

In Fig. 2 the dependences of maximum values of the longitudinal and radial velocities at $x = \text{const}$ on the x coordinate are plotted. It can be seen that at $x > 100$ these dependences can be well approximated by power-law functions

$$u_{\max}(x) = 108.771x^{-5/3}, \quad (4.1)$$

$$v_{\max}(x) = 1.195x^{-4/3}. \quad (4.2)$$

Fig. 2. Tube of rectangular cross-section, $Re = 200$. Dependence of the maximum longitudinal (a) and radial (b) velocity components over the cross-section on the longitudinal coordinate. The solid lines are approximations (4.1) and (4.2).

The difference of the numerical solution from formulas (4.1) and (4.2) is less than 2% at $160 \leq x \leq 640$.

The functions (4.1), (4.2) are associated with the self-similarity exponent $k = 4/3$ in solution (1.6)–(1.10). The error in determining this quantity from the numerical data is of the order of 1%.

We will show that the dependence of solution (1.6)–(1.10) on the Reynolds number corresponds to $\alpha = k$. In Fig. 3 the curves $U(\eta, 0)$ and $V(\eta, 0)$ are plotted at $\alpha = k$ in the cross-section $x = 390$ for the Reynolds numbers 100, 150, and 200. Their agreement indicates the correctness of determining the exponent α .

Good agreement between the self-similar velocity profiles $U(\eta, \varphi)$, $V(\eta, \varphi)$ plotted according to the numerical solution in the cross-sections $x = 200, 300, 400, 500$, and 600 at $\varphi = 0, \pi/4$ can be observable. For the sake of illustration, in Fig. 4 we have plotted the velocity profiles obtained in the sections $x = 200$ and 500.

In Figs. 3 and 4 the self-similar velocity profiles are compared with the first two terms of expansions (2.1) and (2.2) in which $A = 0.0146$ for the tube with the circular cross-section, $A = 0.00173$ for the tube with the rectangular cross-section, and $B = 0$ in both cases. The constant B is determined using Eq. (2.5) from the numerical solution. With increase in the coordinate η the radial velocity V first increases, reaches a maximum, and then decreases and becomes negative. The longitudinal velocity U is everywhere positive.

The numerical solution makes it possible to establish certain approximate properties of the self-similar solution. In Fig. 5 the streamlines of the self-similar solution are plotted in the (η, φ) plane. It can be seen that almost everywhere the streamlines are radially directed; therefore, the azimuthal velocity is considerably smaller than the radial one. The radial streamlines are everywhere directed toward the limiting streamline, which is similar with a half-circle. The point $\eta = 0$ is a source, while the intersection of the plane of symmetry and the half-circle is a sink. The deviation of the streamlines from the radial direction is observable only in the immediate vicinity of the half-circle. In Fig. 5c the field of the radial-to-longitudinal velocity ratio $V(\eta, \varphi)/U(\eta, \varphi)$ is plotted against the background of the streamlines of the self-similar solution. It is clearly visible that this field is almost independent of the azimuthal angle φ .

5. THE HIDDEN CONSERVATION LAW

We will introduce the Cartesian coordinates (x, y, z) with the y axis directed normal to the solid wall (Fig. 1). In the case of the two-dimensional (z -independent) flow the parabolized Navier–Stokes equa-

Fig. 3. Tube of circular cross-section. Self-similar profiles of the longitudinal (a) and radial (b) velocity components in the plane of symmetry of cross-section $x = 390$ at different Reynolds numbers: 1, $Re = 100$; 2, $Re = 150$; 3, $Re = 200$; and 4, two terms of asymptotics (2.1) and (2.2) at $A = 0.0146$ and $B = 0$.

Fig. 4. Tube of rectangular cross-section, $Re = 200$. Self-similar profiles of the longitudinal (a) and radial (b) velocity components in different cross-sections: 1, $x = 200$, $\varphi = 0$; 2, $x = 200$, $\varphi = \pi/4$; 3, $x = 500$, $\varphi = 0$; 4, $x = 500$, $\varphi = \pi/4$; 5, two terms of asymptotics (2.1) and (2.2) with the parameters $A = 0.00173$ and $B = 0$ at $\varphi = 0$; and 6, same as 5 at $\varphi = \pi/4$.

tions (1.2)–(1.5) transform to the boundary layer equations. Within the framework of these equations an exact self-similar solution [1, 2] was obtained for a wall jet flowing out of an infinitely thin slit

$$u = 1/x^{2k-1}U(y/x^k). \quad (5.1)$$

Fig. 5. Tube of rectangular cross-section, $Re = 200$, $x = 600$. Self-similar fields of the longitudinal (a) and radial (b) velocity components and the radial-to-longitudinal velocity ratio (c). The streamlines of the self-similar flow are plotted.

Moreover, for this jet the conservation law [1, 2] was derived; it is also fulfilled in the general case of a non-self-similar wall jet

$$E = \int_0^{+\infty} u^2 \psi dy = \text{const}, \quad \psi(x, y) = \int_0^y u(x, y_1) dy_1. \quad (5.2)$$

Let us substitute solution (5.1) into integral (5.2). We obtain that the integral E is conserved on the self-similar solution for the self-similarity exponent $k = 3/4$.

We will now consider a non-self-similar jet flowing out of a slit of finite width a at a velocity u_0 . From the conservation of integral (5.2) it follows that $E \sim u_0^3 a^2 \sim u^3 y^2$ (in the dimensionless variables used in this study $a = 1$ and $u_0 = 1$). Here, u and y are the characteristic velocity and jet-width scales. From the longitudinal momentum equation we obtain $u^2/x \sim Re^{-1} u/y^2$. The estimates presented above lead to a solution that takes account for the dependence on the Reynolds number and the initial velocity distribution in the slit (in terms of the invariant E)

$$u = E^{1/2} \left(\frac{x}{Re} \right)^{-1/2} U \left(E^{1/4} \left(\frac{x}{Re} \right)^{-3/4} y \right). \quad (5.3)$$

The problem of hidden invariants for swirling and nonswirling axisymmetric jets is discussed in [15–22]. In the case of axisymmetric flow, from the system of equations (1.2), (1.3) there follows the conservation law [21]

$$\int_0^{+\infty} u^2 (\psi - Re^{-1} x) r dr = \text{const}, \quad \psi(x, r) = \int_0^r u(x, r_1) r_1 dr_1. \quad (5.4)$$

This quantity is conserved in free submerged jets and is used for deriving the second coordinate approximation in the jet far-field [22].

The dimensions of integrals (5.2) and (5.4) are the same as the dimension of the product of the momentum flux by the flow rate. It would be nice to obtain the analogous conservation law for the three-dimensional case. The direct generalization of integrals (5.2) and (5.4) to the three-dimensional case is hardly possible. Of course, one stream function in the three-dimensional case can be replaced by two independent components of the vector potential. However, in this case it is difficult to imagine an integral which would turn in (5.2) in the two-dimensional case and in (5.4) in the axisymmetric case. On the other

hand, even if the conservation law could exist for a quantity with the dimension $u^3\delta^4$, it could not be fulfilled for the three-dimensional wall jet, since it would lead to an incorrect self-similarity exponent $k = 3/2$, rather than $k = 4/3$. In analyzing this problem in [3] the conservation law for the angular momentum was used, which also led to an incorrect self-similarity exponent $k = 2$.

We will assume that the solution depends on a unique invariant of the order of $I \sim u^4\delta^5$, where u is the longitudinal velocity in the jet and δ is the jet width. The chosen invariant dimension ensures the correct self-similarity exponent $k = 4/3$. Then, on the basis of this invariant, together with the equations of momentum $u^2/x \sim \nu u/\delta^2$ and $uv/x \sim p/\delta$ and continuity $u/x \sim v/\delta \sim w/\delta$, we can determine the form of the solution:

$$u(x, r, \varphi) = I^{2/3} \xi^{-5/3} \tilde{U}(I^{1/3} \eta, \varphi), \quad (5.5)$$

$$v(x, r, \varphi) = I^{1/3} \text{Re}^{-1} \xi^{-4/3} \tilde{V}(I^{1/3} \eta, \varphi), \quad (5.6)$$

$$w(x, r, \varphi) = I^{1/3} \text{Re}^{-1} \xi^{-4/3} \tilde{W}(I^{1/3} \eta, \varphi), \quad (5.7)$$

$$p(x, r, \varphi) = I^{2/3} \text{Re}^{-2} \xi^{-8/3} \tilde{P}(I^{1/3} \eta, \varphi), \quad (5.8)$$

$$\xi = x/\text{Re}, \quad \eta = r/\xi^{4/3},$$

where the functions $\tilde{U}$, $\tilde{V}$, $\tilde{W}$, and $\tilde{P}$ are universal for any (fairly high) Reynolds numbers and any initial conditions (tube geometry, velocity profile in the tube, and tube/wall spacing).

From the agreement between Eqs. (5.5)–(5.8) and the asymptotic expansions (2.1)–(2.4) it follows that the hidden invariant I is proportional to the constant A .

In Fig. 6 the numerical solutions for the tubes of circular and rectangular cross-sections are compared in the universal variables $A^{-2/3} \xi^{5/3} u$, $A^{-1/3} \text{Re} \xi^{4/3} v$, and $A^{1/3} \eta$ at $\varphi = 0$. Although the initial conditions and constants A are considerably different for the two cases, the self-similar profiles so represented coincide.

From an analysis of the universal profile of the radial velocity the constant F_0 in expansion (3.1) was evaluated: $F_0 \approx 4$ – 6 . The weak accuracy is due to the errors of the numerical solution at large values of r . From Eq. (5.6) it follows that this constant does not depend on I or A , that is, it is the same for all three-dimensional wall jets considered.

The constant P_0 in expansion (1.9) was determined from an analysis of the universal pressure profile at $\varphi = 0$, which is obtained, when the numerical data are represented in the variables $A^{-2/3} \text{Re} \xi^{8/3} p$, $A^{1/3} \eta$. It was determined that $P_0 = 0.2A^{2/3}$.

The constant C in expansion (2.1) could not be determined using the numerical solution.

We will now consider the range of applicability of the proposed model (5.5)–(5.8) of the far field of a three-dimensional wall jet. It is known that the boundary layer approximation (1.2)–(1.5) holds only at large Reynolds numbers. The comparison of the numerical solution (5.5)–(5.8) with the numerical solution of Navier–Stokes equations allows us to conclude that the boundary layer approximation is fulfilled with a sufficient accuracy even at Reynolds numbers of the order of one hundred. At these Reynolds numbers the jet is unstable against small perturbations. However, at $\text{Re} \sim 100$ small perturbations in the tube flow increase very slowly, which ensures the laminar flow field at large distances from the source. At $\text{Re} \sim 1000$ it might be expected that the flow would become turbulent before the self-similar regime (5.5)–(5.8) could be attained. From the form of the self-similar variables in Eqs. (5.5)–(5.8) it follows that the jet becomes self-similar at distances from the source of the order of the Reynolds number. The further increase in the longitudinal coordinate leads, even under the assumption that the flow is laminar, to the violation of the conditions of the boundary layer approximation, owing to fairly rapid jet expansion $\delta \sim (x/\text{Re})^{4/3}$. According to Eqs. (5.5)–(5.8), the longitudinal velocity is of the order of the radial velocity at distances $x \sim \text{Re}^4$, which provides an estimate for the range of applicability of the boundary layer approximation.

Fig. 6. Universal profiles of the longitudinal (a) and radial (b) velocity components in the plane of symmetry: 1, tube of circular cross-section, $Re = 150$, $x = 390$; 2, tube of rectangular cross-section, $Re = 200$, $x = 400$; and 3, approximations (6.1) and (6.2).

6. APPROXIMATION OF THE NUMERICAL SOLUTION

On the basis of solution (5.5)–(5.8), the principal terms of the asymptotic expansions (2.1)–(2.3) and (3.1), and the numerically-established properties of the self-similar jet far-field the following approximation of the velocities in the jet was derived

$$u = A^{2/3} \left(\frac{x}{Re} \right)^{-5/3} \frac{A^{1/3} \eta \cos \varphi}{1 + 0.07(A^{1/3} \eta)^4}, \quad (6.1)$$

$$v = Re^{-1} A^{1/3} \left(\frac{x}{Re} \right)^{-4/3} \frac{(A^{1/3} \eta)^2 (1 - (A^{1/3} \eta / 3.8)^{1.6}) \cos \varphi}{1 + 0.02(A^{1/3} \eta)^{4.6}}, \quad (6.2)$$

$$w = 0,$$

$$\eta = \left(\frac{x}{Re} \right)^{-4/3} r.$$

This approximation describes the longitudinal and radial velocities at $\varphi = 0$ with an error with respect to the numerical solution not greater than 10% (Fig. 6) and ensures qualitative agreement for the other values of the azimuthal angle.

SUMMARY

The self-similar solution (5.5)–(5.8) for the far field of a three-dimensional laminar wall jet is derived in the boundary layer approximation. Using the numerical solution of the non-self-similar problem the self-similarity exponent $k \approx 4/3$ is determined, together with the dependence on the Reynolds number. Moreover, the relation between this solution and the conditions in the jet source is established. This relation is determined by the constant A in the principal term of the asymptotic expansion of the jet far-field (2.1) in the small radial coordinate. The problem of constructing the integral invariant of the non-self-similar jet, which could relate this constant with the velocity distribution in the source, remains open. The coordinate expansions of the self-similar solution are obtained for small and large values of the radial

coordinate. In the first approximation of the expansion in the small coordinate, the azimuthal velocity is absent, while the longitudinal and radial velocities depend on the azimuthal angle in the same fashion. It is shown that the Hamel solution for flow in a confuser is the principal term of the expansion in large values of the radial coordinate. It is shown numerically that the locus of the points of zero radial velocity in a cross-section is similar with a semicircle. The range of applicability of solution (5.5)–(5.8) is as follows: $Re \sim 100$, $Re \leq x \ll Re^4$.

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DECLARATION OF CONFLICTING INTERESTS

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REFERENCES

1. N. I. Akatnov, "Propagation of a plane laminar jet of viscous fluid along a solid wall," *Trudy Leningr. Politekh. Inst.* No. 5, 24–31 (1953).
2. M. B. Glauert, "The wall jet," *J. Fluid Mech.* **1**, 625–643 (1956).
3. R. Krechetnikov and I. Lipatov, "Hidden invariances in problems of two-dimensional and three-dimensional wall jets for Newtonian and non-Newtonian fluids," *SIAM J. Appl. Math.* **62** (6), 1837–1855 (2002).
4. G. I. Barenblatt, *Self-Similar Phenomena – Dimensional Analysis and Scaling* (Intellect, Dolgoprudny, 2009) [in Russian].
5. H. Schlichting, "Laminare Strahlausbreitung," *Z. Angew. Math. Mech.* **13** (4), 260–263 (1933).
6. L. D. Landau, "On an exact solution of Navier–Stokes equations," *Doklady Akad. Nauk SSSR* **43** (7), 299–301 (1944).
7. H. B. Squire, "Some viscous fluid flow problems. I: Jet emerging from a hole in a plane wall," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **43** (344), 942–945 (1952).
8. B. E. Launder and W. Rodi, "The turbulent wall jet," *Progr. Aerospace Sci.* **19**, 81–128 (1979).
9. B. E. Launder and W. Rodi, "The turbulent wall jet measurements and modeling," *Annu. Rev. Fluid Mech.* **15** (1), 429–459 (1983).
10. T. J. Craft and B. E. Launder, "On the spreading mechanism of the three-dimensional turbulent wall jet," *J. Fluid Mech.* **435**, 305–326 (2001).
11. K. K. Adane and M. F. Tachie, "Numerical investigation of three-dimensional laminar wall jet of Newtonian and non-Newtonian fluids," *AIAA J.* **46** (11), 2868–2880 (2008).
12. K. K. Adane and M. F. Tachie, "Experimental and numerical study of laminar round jet flows along a wall," *J. Fluids Engng.* **132** (10), 101203 (2010).
13. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).
14. A. M. Gaifullin and V. V. Zhvick, "Interaction between two oppositely swirled submerged jets," *Fluid Dynamics* **54** (3), 339–348 (2019).
15. M. A. Gol'dshtik, "On swirling jets," *Fluid Dynamics* **14** (1), 19–26 (1979).
16. M. A. Gol'dshtik and N. I. Yavorskii, "On submerged jets," *J. Appl. Math. Mech.* **50** (4), 438–445 (1986).

17. M. A. Gol'dshtik, V. N. Shtern, and N. I. Yavorskii, *Viscous Flows with Paradoxical Properties* (Nauka, Novosibirsk, 1989) [in Russian].
18. M. A. Goldshtik, "Viscous-flow paradoxes," *Annu. Rev. Fluid Mech.* **22** (1), 441–472 (1990).
19. N. I. Yavorskii, "On the hidden conservation integral in theory of submerged jets," in: *XII All-Russian Congress of the Fundamental Problems of Theoretical and Applied Mechanics. Abstracts of the Reports. August 19–24, 2019* (Ufa, 2019), p. 92 [in Russian].
20. V. V. Zhvick, "Invariants and asymptotics of axisymmetric swirling submerged jets," *J. Appl. Mech. Techn. Phys.* **61** (2), 235–249 (2020).
21. R. Naz, "Conservation laws for laminar axisymmetric jet flows with weak swirl," *Applicable Analysis. An International Journal* **91** (5), 1045–1052 (2012).
22. A. M. Gaifullin and V. V. Zhvick, "The connection of the asymptotics of the far field of a jet with the velocity profile at an orifice," *Doklady Physics* **65** (11), 387–389 (2020).

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