Some Properties of the Topological Entropy of a Family of Dynamical Systems Defined on an Arbitrary Metric Space

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Abstract—We consider a family of dynamical systems defined on a noncompact metric space and continuously depending on a parameter varying in some metric space. For any such family, the topological entropy of the dynamical systems in the family is studied as a function of the parameter from the viewpoint of the Baire classification of functions.

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The topological entropy of autonomous dynamical systems on an invariant compact metric space was defined in [1]. Later, this concept was extended in [2] to dynamical systems defined on an arbitrary metric space.

1. BAIRE CLASS OF THE TOPOLOGICAL ENTROPY OF A FAMILY OF DYNAMICAL SYSTEMS IN THE CASE OF A NONINVARIANT COMPACT SET

Following [2], let us give a definition needed in what follows. Let (X, d) be a metric space, let $\mathcal{K}(X)$ be the set of compact subsets of X , and let $f : X \rightarrow X$ be a continuous mapping. Along with the original metric d , we define an additional system of metrics d_n^f , $n \in \mathbb{N}$, on X by the formula

$$d_n^f(x, y) = \max_{0 \leq i \leq n-1} d(f^{\circ i}(x), f^{\circ i}(y)), \quad x, y \in X, \quad n \in \mathbb{N},$$

where $f^{\circ i}$, $i \in \mathbb{N}$, is the i th iteration of f and $f^{\circ 0} \equiv \text{id}_X$. Given a $K \in \mathcal{K}(X)$, for any $n \in \mathbb{N}$ and $\varepsilon > 0$ we denote by $N_d(K, f, \varepsilon, n)$ the maximum number of points in K with pairwise d_n^f -distances greater than ε . The numbers

$$\bar{h}_{\text{top}}(K, f) = \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln N_d(K, f, \varepsilon, n) \quad \text{and} \quad \underline{h}_{\text{top}}(K, f) = \lim_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln N_d(K, f, \varepsilon, n) \quad (1)$$

are called the *upper* and *lower topological entropies*, respectively, of f on K .

Note that if the metric d is replaced by a metric generating the same topology as d , then the numbers (1) do not change [3, p. 121 of the Russian translation].

Let us also recall formulas for the upper and lower topological entropies, to be used below. For any $x \in X$, $\varepsilon > 0$, and $n \in \mathbb{N}$, let $B_f(x, \varepsilon, n)$ be the open ball $\{y \in K : d_n^f(x, y) < \varepsilon\}$ of radius ε centered at x in the space (X, d_n^f) . A set $D \subset K$ is called an (f, ε, n) -cover of K if

$$K \subset \bigcup_{x \in D} B_f(x, \varepsilon, n).$$

Let $S_d(K, f, \varepsilon, n)$ be the minimum possible number of elements in an (f, ε, n) -cover of K . Then the upper and lower topological entropies of f on K can be calculated by the formulas [3, p. 122 of the Russian translation]

$$\bar{h}_{\text{top}}(K, f) = \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln S_d(K, f, \varepsilon, n), \quad \underline{h}_{\text{top}}(K, f) = \lim_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln S_d(K, f, \varepsilon, n). \quad (2)$$

Formulas (1) (or (2)) obviously imply the inequality $\bar{h}_{\text{top}}(K, f) \geq \underline{h}_{\text{top}}(K, f)$. If K is f -invariant, i.e., $f(K) \subset K$, then the upper and lower topological entropies of f on K coincide [3, p. 122 of the Russian translation]. The following example shows that the numbers (1) do not necessarily coincide in the general case. Consider the set Ω_2 of sequences $x = (x_1, x_2, x_3, \dots)$, where $x_i \in \{0, 1\}$, with the metric

$$d_{\Omega_2}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1/\min\{i : x_i \neq y_i\} & \text{if } x \neq y. \end{cases}$$

Note that the space (Ω_2, d_{Ω_2}) is homeomorphic to the Cantor set on the interval $[0, 1]$ with the metric induced by the standard metric of the real line. Let $K_0 \subset \Omega_2$ be the compact set defined by the condition

$$(x_1, x_2, x_3, \dots) \in K_0 \Leftrightarrow x_i = 0, \quad i \in \bigcup_{k \in \mathbb{N} \cup \{0\}} \{(2k)!, \dots, (2k+1)!\},$$

and let $\sigma : \Omega_2 \rightarrow \Omega_2$ be the left shift mapping, $\sigma((x_1, x_2, x_3, \dots)) = (x_2, x_3, x_4, \dots)$. Then

$$\bar{h}_{\text{top}}(K_0, \sigma) = \ln 2, \quad \underline{h}_{\text{top}}(K_0, \sigma) = 0.$$

Given a metric space $\mathcal{M}$, a compact set $K \subset X$, and a continuous mapping

$$f : \mathcal{M} \times X \rightarrow X, \tag{3}$$

consider the functions

$$\mu \mapsto \bar{h}_{\text{top}}(K, f(\mu, \cdot)), \tag{4}$$

$$\mu \mapsto \underline{h}_{\text{top}}(K, f(\mu, \cdot)). \tag{5}$$

In the present paper, for each mapping (3) the functions (4) and (5) are studied from the viewpoint of the Baire classification of functions. Recall that continuous functions $M \rightarrow \mathbb{R}$ on a metric space $\mathcal{M}$ are called functions of *Baire class 0*, and the functions of *Baire class p* are defined for each positive integer p as the functions that are pointwise limits of sequences of functions of Baire class $p - 1$.

We have already mentioned that the numbers (1) coincide if K is f -invariant; in that case, their common value is called the *topological entropy* of f and is denoted by $h_{\text{top}}(f)$. It was established in the paper [4] that for each mapping (3) the function

$$\mu \mapsto h_{\text{top}}(f(\mu, \cdot)) \tag{6}$$

is of Baire class 2 on $\mathcal{M}$. In [5], a family of homeomorphisms (3) with $X = \mathcal{M} = \Omega_2$ was constructed such that the function (6) is not of Baire class 1; consequently, the functions (4) and (5), generally speaking, are not of Baire class 1 either. It was shown in [4] that if the space $\mathcal{M}$ is metrizable by a complete metric, then the set of points of lower semicontinuity of the function (6) contains a G_δ set everywhere dense in $\mathcal{M}$, while the paper [6] established that the set of points of lower semicontinuity itself is an everywhere dense G_δ set in $\mathcal{M}$. Further, let $X = \Omega_2$, and let $\mathcal{M}$ be an arbitrary complete metric separable zero-dimensional space (for example, Ω_2). In this case, for an arbitrary G_δ set $\mathcal{G}$ everywhere dense in $\mathcal{M}$, the paper [7] presents the construction of a mapping (3) such that the set of points of lower semicontinuity of the corresponding function (6) coincides with $\mathcal{G}$. It turns out that the following assertion holds in the case of a noninvariant compact set K .

Theorem 1. *For any $K \in \mathcal{K}(X)$ and any mapping (3), the function (4) is of Baire class 3 and the function (5) is of Baire class 2 on $\mathcal{M}$. If $\mathcal{M}$ is metrizable by a complete metric, then the set of points of lower semicontinuity of the function (5) is an everywhere dense G_δ set.*

Proof. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, the function $\mu \mapsto n^{-1} \ln N_d(K, f(\mu, \cdot), \varepsilon, n)$ is lower semicontinuous [8] and the function $\mu \mapsto n^{-1} \ln S_d(K, f(\mu, \cdot), \varepsilon, n)$ is upper semicontinuous [4];

consequently [9, Ch. IX, Sec. 37, XI of the Russian translation], there exist sequences of continuous functions $\mu \mapsto \varphi_d^m(K; \mu, \varepsilon, n)$ and $\mu \mapsto \psi_d^m(K; \mu, \varepsilon, n)$, $m \in \mathbb{N}$, on $\mathcal{M}$ such that

$$\frac{1}{n} \ln N_d(K, f(\mu, \cdot), \varepsilon, n) = \sup_{m \in \mathbb{N}} \varphi_d^m(K; \mu, \varepsilon, n), \quad \mu \in \mathcal{M}, \quad (7)$$

$$\frac{1}{n} \ln S_d(K, f(\mu, \cdot), \varepsilon, n) = \inf_{m \in \mathbb{N}} \psi_d^m(K; \mu, \varepsilon, n), \quad \mu \in \mathcal{M}. \quad (8)$$

Hence we obtain the representations

$$\begin{aligned} \bar{h}_{\text{top}}(K, f(\mu, \cdot)) &= \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln N_d(K, f(\mu, \cdot), \varepsilon, n) = \sup_{k \in \mathbb{N}} \overline{\lim}_{n \rightarrow \infty} \sup_{m \in \mathbb{N}} \varphi_d^m(K; \mu, 1/k, n) \\ &= \sup_{k \in \mathbb{N}} \inf_{n \in \mathbb{N}} \sup_{l \geq n} \sup_{m \in \mathbb{N}} \varphi_d^m(K; \mu, 1/k, l) \\ &= \lim_{p \rightarrow \infty} \max_{1 \leq k \leq p} \lim_{r \rightarrow \infty} \min_{1 \leq n \leq r} \lim_{q \rightarrow \infty} \max_{n \leq l \leq q} \max_{1 \leq m \leq q} \varphi_d^m(K; \mu, 1/k, l) \end{aligned}$$

by (7) and

$$\begin{aligned} \underline{h}_{\text{top}}(K, f(\mu, \cdot)) &= \lim_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln S_d(K, f(\mu, \cdot), \varepsilon, n) = \sup_{k \in \mathbb{N}} \underline{\lim}_{n \rightarrow \infty} \inf_{m \in \mathbb{N}} \psi_d^m(K; \mu, 1/k, n) \\ &= \sup_{k \in \mathbb{N}} \sup_{n \in \mathbb{N}} \inf_{l \geq n} \inf_{m \in \mathbb{N}} \psi_d^m(K; \mu, 1/k, l) \\ &= \lim_{p \rightarrow \infty} \max_{1 \leq k \leq p} \max_{1 \leq n \leq p} \lim_{q \rightarrow \infty} \min_{n \leq l \leq q} \min_{1 \leq m \leq q} \psi_d^m(K; \mu, 1/k, l) \end{aligned}$$

by (8).

Since the maximum and minimum of finitely many functions in a Baire class belong to the same class [9, Ch. IX, Sec. 37, III of the Russian translation], it obviously follows from these representations that the function $\mu \mapsto \bar{h}_{\text{top}}(K, f(\mu, \cdot))$ is of Baire class 3 and the function $\mu \mapsto \underline{h}_{\text{top}}(K, f(\mu, \cdot))$ is of Baire class 2 on $\mathcal{M}$.

Since the function $\mu \mapsto \underline{h}_{\text{top}}(K, f(\mu, \cdot))$ can be represented as the limit of a nondecreasing sequence of functions of Baire class 1, we see that its set of points of lower semicontinuity is an everywhere dense G_δ set [10, Lemma 2]. The proof of the theorem is complete.

Note that if $\mathcal{M}$ is a complete metric space, then, by Baire's theorem [9, Ch. IX, Sec. 39, VI of the Russian translation], Theorem 1 implies that for each mapping (3) there exists an everywhere dense G_δ set $G \subset \mathcal{M}$ such that the restrictions of the functions $\mu \mapsto \underline{h}_{\text{top}}(K, f(\mu, \cdot))$ and $\mu \mapsto \bar{h}_{\text{top}}(K, f(\mu, \cdot))$ to G are continuous.

There arises a natural question about the least Baire class containing the function (4). To answer this question, we construct metric spaces $\mathcal{B}$ and $\mathcal{C}$. By definition, the points of $\mathcal{B}$ are all possible (countable) sequences $\mu = (\mu_k)_{k=1}^\infty$ of positive integers. The distance between two points μ and ν is defined by the formula

$$d_{\mathcal{B}}(\mu, \nu) = \begin{cases} 0 & \text{if } \mu = \nu \\ 1/\min\{k : \mu_k \neq \nu_k\} & \text{if } \mu \neq \nu. \end{cases}$$

Note that the space $(\mathcal{B}, d_{\mathcal{B}})$ is homeomorphic to the set of irrational numbers on the interval $[0, 1]$ with the metric induced by the natural metric of the real line. The points of $\mathcal{C}$ are all possible pairs (x, i) , where $x \in [0, 1]$ and $i \in \mathbb{N}$. The distance between points (x, i) and (y, j) is defined by the formula

$$d_{\mathcal{C}}((x, i), (y, j)) = \begin{cases} |x - y| & \text{if } i = j \\ 1 & \text{if } i \neq j. \end{cases}$$

For each $r \in \mathbb{N}$, let $K_r \in \mathcal{K}(\mathcal{C})$ be the compact set $K_r = [0, 1] \times \{1, \dots, r\}$ in $\mathcal{C}$.

Theorem 2. Let $\mathcal{M} = \mathcal{B}$, $X = \mathcal{C}$, and $K = K_r$; then there exists a mapping (3) such that the function (4) is everywhere discontinuous and does not belong to the second Baire class on $\mathcal{M}$.

Proof. Given a sequence $\mu = (\mu_k)_{k=1}^\infty \in \mathcal{B}$, we construct the sequence $\alpha(\mu)$ with elements $\alpha_k(\mu) = \mu_{\lfloor \log_2(k+1) \rfloor}$ (where $\lfloor \cdot \rfloor$ is the integer part of a number). Consider the sequence (f_k) of mappings of $\mathcal{B} \times [0, 1]$ into $[0, 1]$ given by

$$f_k(\mu, x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 - 1/\alpha_k(\mu) \\ 2x - 1 + 1/\alpha_k(\mu) & \text{if } 1 - 1/\alpha_k(\mu) < x \leq 1 - 1/(2\alpha_k(\mu)) \\ -2x + 3 - 1/\alpha_k(\mu) & \text{if } 1 - 1/(2\alpha_k(\mu)) < x \leq 1. \end{cases}$$

We use this sequence to construct a mapping $f : \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$ by setting

$$f(\mu, (x, k)) = (f_k(\mu, x), k + 1). \quad (9)$$

By definition, the function f is continuous on $\mathcal{B} \times \mathcal{C}$.

Let $\mathcal{E} \subset \mathcal{B}$ be the set of sequences tending to infinity. Let us find the upper topological entropy of the mapping (9) for $\mu \in \mathcal{E}$.

Lemma 1. If $\mu \in \mathcal{E}$, then

$$\bar{h}_{\text{top}}(K_r, f(\mu, \cdot)) = 0$$

for the mapping (9) for any $r \in \mathbb{N}$.

Proof. Fix an $\varepsilon \in (0, 1)$ and a $\mu \in \mathcal{E}$. Then there exists a number $k_0(\varepsilon) > r$ such that $1/\alpha_k(\mu) < \varepsilon/2$ for each $k \geq k_0(\varepsilon)$.

Let $A_{k_0(\varepsilon)}$ be an $(f(\mu, \cdot), \varepsilon/2, k_0(\varepsilon))$ -cover of K_r with minimum number of elements. Let us prove that $A_{k_0(\varepsilon)}$ is an $(f(\mu, \cdot), \varepsilon, k_0(\varepsilon) + i)$ -cover of K_r for each $i \in \mathbb{N} \cup \{0\}$.

By the definition of $A_{k_0(\varepsilon)}$, for each point $(x, l) \in K_r$ there exists an element $(x_0, l) \in A_{k_0(\varepsilon)}$ such that $(x, l) \in B_{f(\mu, \cdot)}((x_0, l), \varepsilon/2, k_0(\varepsilon))$.

If $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x, l)), f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0, l)) \in [0, 1 - \varepsilon/2] \times \mathbb{N}$, then

$$\begin{aligned} d_{\mathcal{C}}\left(f^{\circ(k_0(\varepsilon)+i)}(\mu, (x, l)), f^{\circ(k_0(\varepsilon)+i)}(\mu, (x_0, l))\right) \\ = d_{\mathcal{C}}\left(f^{\circ(k_0(\varepsilon)-1)}(\mu, (x, l)), f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0, l))\right) < \varepsilon/2 \end{aligned} \quad (10)$$

for each $i \in \mathbb{N} \cup \{0\}$, because the interval $[0, 1 - \varepsilon/2]$ is invariant with respect to the mapping $f_{k_0(\varepsilon)+i}(\mu, \cdot)$ for all $i \in \mathbb{N} \cup \{0\}$.

If $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x, l)), f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0, l)) \in [1 - \varepsilon/2, 1] \times \mathbb{N}$, then

$$d_{\mathcal{C}}\left(f^{\circ(k_0(\varepsilon)+i)}(\mu, (x, l)), f^{\circ(k_0(\varepsilon)+i)}(\mu, (x_0, l))\right) \leq \varepsilon/2 \quad (11)$$

for each $i \in \mathbb{N} \cup \{0\}$, because the interval $[1 - \varepsilon/2, 1]$ is invariant with respect to the mappings $f_{k_0(\varepsilon)+i}(\mu, \cdot)$ for all $i \in \mathbb{N} \cup \{0\}$.

If either $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x, l)) \in [0, 1 - \varepsilon/2] \times \mathbb{N}$ and $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0, l)) \in [1 - \varepsilon/2, 1] \times \mathbb{N}$ or $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0, l)) \in [0, 1 - \varepsilon/2] \times \mathbb{N}$ and $f^{\circ(k_0(\varepsilon)-1)}(\mu, (x, l)) \in [1 - \varepsilon/2, 1] \times \mathbb{N}$, then

$$\begin{aligned} d_{\mathcal{C}}\left(f^{\circ(k_0(\varepsilon)+i)}(\mu, (x, l)), f^{\circ(k_0(\varepsilon)+i)}(\mu, (x_0, l))\right) \\ \leq d_{\mathcal{C}}\left(f^{\circ(k_0(\varepsilon)-1)}(\mu, (x, l)), f^{\circ(k_0(\varepsilon)-1)}(\mu, (x_0, l))\right) + \varepsilon/2 < \varepsilon \end{aligned} \quad (12)$$

for each $i \in \mathbb{N} \cup \{0\}$.

It follows from inequalities (10)–(12) that $(x, l) \in B_{f(\mu, \cdot)}((x_0, l), \varepsilon, k_0(\varepsilon) + i)$ for each $i \in \mathbb{N} \cup \{0\}$, and hence the set $A_{k_0(\varepsilon)}$ is an $(f(\mu, \cdot), \varepsilon, k_0(\varepsilon) + i)$ -cover of K_r for each $i \in \mathbb{N} \cup \{0\}$. Thus, the estimate

$$S_{d_c}(K_r, f(\mu, \cdot), \varepsilon, n) \leq S_{d_c}(K_r, f(\mu, \cdot), \varepsilon/2, k_0(\varepsilon))$$

holds for $n \geq k_0(\varepsilon)$, which implies that

$$\begin{aligned} \bar{h}_{\text{top}}(K_r, f(\mu, \cdot)) &= \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln S_{d_c}(K_r, f(\mu, \cdot), \varepsilon, n) \\ &\leq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln S_{d_c}(K_r, f(\mu, \cdot), \varepsilon/2, k_0(\varepsilon)) = 0. \end{aligned}$$

The proof of the lemma is complete.

Now let us estimate the upper topological entropy of the mapping (9) for $\mu \notin \mathcal{E}$.

Lemma 2. *If $\mu \notin \mathcal{E}$, then*

$$\bar{h}_{\text{top}}(K_1, f(\mu, \cdot)) \geq \frac{1}{4} \ln 2$$

for the mapping (9).

Proof. Let $\mu \notin \mathcal{E}$. Then there exists a subsequence $(\mu_{k_j})_{j=1}^\infty \subset (\mu_k)_{k=1}^\infty$ and a positive integer q such that $\mu_{k_j} = q$ for all $j \in \mathbb{N}$.

Further, $f_k(\mu, x) = f_{2^{k_j}-1}(\mu, x) = t_q(x)$ for all $j \in \mathbb{N}$, $k \in \{2^{k_j} - 1, \dots, 2^{k_j+1} - 2\}$, and $x \in [0, 1]$, where

$$t_q(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 - 1/q \\ 2x - 1 + 1/q & \text{if } 1 - 1/q < x \leq 1 - 1/(2q) \\ -2x + 3 - 1/q & \text{if } 1 - 1/(2q) < x \leq 1. \end{cases}$$

The affine order-preserving transformation φ mapping the interval $I_q = [1 - 1/q, 1]$ onto the interval $[0, 1]$ takes the mapping $t_q|_{I_q} : I_q \rightarrow I_q$ to the mapping $g = \varphi \circ t_q|_{I_q} \circ \varphi^{-1} : [0, 1] \rightarrow [0, 1]$ given by the formula

$$g(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2 - 2x & \text{if } 1/2 < x \leq 1. \end{cases}$$

It was established in the monograph [3, p. 502 of the Russian translation] that the topological entropy of g is $\ln 2$; consequently, there exists an $\varepsilon_0 < 1/q$ such that

$$\varliminf_{n \rightarrow \infty} \frac{1}{n} \ln N_d([0, 1], g, \varepsilon, n) \geq \frac{1}{2} \ln 2, \quad d(x, y) = |x - y|,$$

for every $\varepsilon < \varepsilon_0$. For each $n \in \mathbb{N}$, consider a set $\{a_1, \dots, a_{N_d([0, 1], g, \varepsilon, n)}\}$ of points on the interval $[0, 1]$ with pairwise d_n^g -distances greater than $\varepsilon > 0$.

Let $\varepsilon < \varepsilon_0$ and $n = 2^{k_j+1} - 2^{k_j} - 1$. Then the $d_{2^{k_j}+n-1}^{f(\mu, \cdot)}$ -distance between any preimages of two arbitrary points $(\varphi^{-1}(a_i), 2^{k_j} - 1)$ and $(\varphi^{-1}(a_m), 2^{k_j} - 1)$, $i \neq m$, under the mapping $f^{\circ(2^{k_j}-2)}(\mu, \cdot)$ is greater than $\varepsilon/q > 0$, and consequently,

$$N_{d_c}(K_1, f(\mu, \cdot), \varepsilon/q, 2^{k_j+1} - 2^{k_j}) \geq N_d([0, 1], g, \varepsilon, 2^{k_j+1} - 2^{k_j}).$$

Hence we obtain the estimates

$$\bar{h}_{\text{top}}(K_1, f(\mu, \cdot)) \geq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{j \rightarrow \infty} \frac{(2^{k_j+1} - 2^{k_j})}{2^{k_j+1}} \frac{1}{(2^{k_j+1} - 2^{k_j})} \ln N_d([0, 1], g, \varepsilon, 2^{k_j+1} - 2^{k_j}) \geq \frac{1}{4} \ln 2.$$

The proof of the lemma is complete.

To complete the proof of Theorem 2, we use the following assertion established in [11]: if a function $\mu \mapsto \bar{h}_{\text{top}}(K_r, f(\mu, \cdot))$ is of Baire class 2, then the intersection of closures of the sets $\bar{h}_{\text{top}}(K_r, f(\mathcal{E}, \cdot))$ and $\bar{h}_{\text{top}}(K_r, f(\mathcal{B} \setminus \mathcal{E}, \cdot))$ is nonempty. By Lemmas 1 and 2,

$$\bar{h}_{\text{top}}(K_r, f(\mathcal{E}, \cdot)) = 0 < \frac{1}{4} \ln 2 \leq \bar{h}_{\text{top}}(K_1, f(\mathcal{B} \setminus \mathcal{E}, \cdot)) \leq \bar{h}_{\text{top}}(K_r, f(\mathcal{B} \setminus \mathcal{E}, \cdot)),$$

and consequently, the function $\mu \mapsto \bar{h}_{\text{top}}(K_r, f(\mu, \cdot))$ is not of Baire class 2. Since both $\mathcal{E}$ and $\mathcal{B} \setminus \mathcal{E}$ are everywhere dense in $\mathcal{B}$, it follows that this function is everywhere discontinuous on $\mathcal{B}$. The proof of the theorem is complete.

2. BAIRE CLASS OF THE TOPOLOGICAL ENTROPY OF A FAMILY OF DYNAMICAL SYSTEMS ON A NONCOMPACT METRIC SPACE

Following [2], we define the *upper* and *lower topological entropies* of a mapping $f : X \rightarrow X$ as the numbers

$$\bar{h}_{\text{top}}(f) = \sup_{K \in \mathcal{K}(X)} \bar{h}_{\text{top}}(K, f), \quad \underline{h}_{\text{top}}(f) = \sup_{K \in \mathcal{K}(X)} \underline{h}_{\text{top}}(K, f), \quad (13)$$

respectively. The following example shows that the two numbers (13) may be distinct. Let us construct a space $\mathcal{A}$ as follows. The points of $\mathcal{A}$ are all possible pairs (x, i) , where $x \in \Omega_2$ and $i \in \mathbb{N}$, and the metric is defined by the formula

$$d_{\mathcal{A}}((x, i), (y, j)) = \begin{cases} d_{\Omega_2}(x, y) & \text{if } i = j \\ 1 & \text{if } i \neq j. \end{cases}$$

Consider the sequence

$$f_n = \begin{cases} \text{id}_{\Omega_2} & \text{if } t_{2k} \leq n \leq t_{2k+1} - 1 \\ \sigma & \text{if } t_{2k+1} \leq n \leq t_{2k+2} - 1, \end{cases} \quad t_s = \sum_{m=0}^s m!, \quad k = 0, 1, 2, \dots,$$

of continuous self-mappings of Ω_2 and define a continuous mapping $f_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ by setting

$$f_{\mathcal{A}}(x, n) = (f_n(x), n + 1).$$

Lemma 3. $\underline{h}_{\text{top}}(f_{\mathcal{A}}) < \bar{h}_{\text{top}}(f_{\mathcal{A}})$.

Proof. For an arbitrary $r \in \mathbb{N}$, let $H_r \in \mathcal{K}(\mathcal{A})$ be the compact set $H_r = \Omega_2 \times \{1, \dots, r\}$. Each compact set $K \in \mathcal{K}(\mathcal{A})$ is contained in H_r for some r ; consequently,

$$\underline{h}_{\text{top}}(f_{\mathcal{A}}) = \lim_{r \rightarrow \infty} \underline{h}_{\text{top}}(H_r, f_{\mathcal{A}}), \quad \bar{h}_{\text{top}}(f_{\mathcal{A}}) = \lim_{r \rightarrow \infty} \bar{h}_{\text{top}}(H_r, f_{\mathcal{A}}).$$

Let $p \in \mathbb{N}$, and let Q be an $(f_{\mathcal{A}}, 1/p, t_{2k})$ -cover of H_r with minimum number of elements. Then Q is an $(f_{\mathcal{A}}, 1/p, t_{2k+1})$ -cover of H_r by the definition of $f_{\mathcal{A}}$. Since the points $(x, i) \in \mathcal{A}$, where $x = (x_1, \dots, x_{t_{2k}+p}, 0, 0, \dots)$ and $i \in \{1, 2, \dots, r\}$, form an $(f_{\mathcal{A}}, 1/p, t_{2k})$ -cover of H_r , it follows that the number of elements in Q does not exceed $r2^{t_{2k}+p}$. Therefore,

$$\underline{h}_{\text{top}}(H_r, f_{\mathcal{A}}) \leq \lim_{p \rightarrow \infty} \lim_{k \rightarrow \infty} \left(\frac{t_{2k} + p}{(2k + 1)!} \ln 2 + \frac{\ln r}{(2k + 1)!} \right) \leq \lim_{k \rightarrow \infty} \frac{\ln 2}{2k + 1} \left(2 + \frac{p}{(2k)!} \right) = 0,$$

and hence $\underline{h}_{\text{top}}(f_{\mathcal{A}}) = 0$.

Let us establish the inequality $\bar{h}_{\text{top}}(f_{\mathcal{A}}) \geq 0.5 \ln 2$, whence Lemma 3 will follow. In the space Ω_2 , consider the set R_k , $k \in \mathbb{N} \cup \{0\}$, of points of the form

$$(x_1, \dots, x_{(2k+2)!}, 0, 0, \dots).$$

Take one point $(y_x, 1) \in H_1$ in the preimage of every point (x, t_{2k+1}) , $x \in R_k$, under the mapping $f_{\mathcal{A}}^{\circ(t_{2k+1}-1)}$. If $x' \neq x''$, $x', x'' \in R_k$, then

$$d_{t_{2k+2}}^{f_{\mathcal{A}}}((y_{x'}, 1), (y_{x''}, 1)) \geq \max_{0 \leq i \leq (2k+2)!-1} d_{\mathcal{A}}(f_{\mathcal{A}}^{\circ i}(x', t_{2k+1}), f_{\mathcal{A}}^{\circ i}(x'', t_{2k+1})) = 1.$$

Thus, $N_{d_{\mathcal{A}}}(H_1, f_{\mathcal{A}}, \varepsilon, t_{2k+2})$ is not less than the cardinality of R_k , which is $2^{(2k+2)!}$, for each $\varepsilon < 1$, and hence

$$\bar{h}_{\text{top}}(f_{\mathcal{A}}) \geq \bar{h}_{\text{top}}(H_1, f_{\mathcal{A}}) \geq \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{k \rightarrow \infty} \frac{1}{t_{2k+2}} \ln N_{d_{\mathcal{A}}}(H_1, f_{\mathcal{A}}, \varepsilon, t_{2k+2}) \geq \lim_{k \rightarrow \infty} \frac{(2k+2)!}{t_{2k+2}} \ln 2 \geq \frac{\ln 2}{2}.$$

The proof of the lemma is complete.

For the mapping (3), consider the functions

$$\mu \mapsto \bar{h}_{\text{top}}(f(\mu, \cdot)), \quad (14)$$

$$\mu \mapsto \underline{h}_{\text{top}}(f(\mu, \cdot)). \quad (15)$$

If X is a compact metric space, then the numbers (13) are equal to the topological entropy of f . Therefore, the functions (14) and (15) are of Baire class 2 by [4] but in general not of Baire class 1 by [5].

Theorem 3. *If $\mathcal{M} = \mathcal{B}$ and $X = \mathcal{C}$, then there exists a mapping (3) such that the function (14) is everywhere discontinuous and does not belong to the second Baire class on $\mathcal{M}$.*

Proof. Each compact set $K \in \mathcal{K}(\mathcal{C})$ is contained in K_r for some r , and consequently, the topological entropy of an arbitrary continuous mapping $f : \mathcal{C} \rightarrow \mathcal{C}$ satisfies the relation

$$\bar{h}_{\text{top}}(f) = \sup_{K \in \mathcal{K}(\mathcal{C})} \bar{h}_{\text{top}}(K, f) = \sup_{r \in \mathbb{N}} \bar{h}_{\text{top}}(K_r, f).$$

By Lemmas 1 and 2, we obtain the chain of inequalities

$$\bar{h}_{\text{top}}(f(\mathcal{E}, \cdot)) = 0 < \frac{1}{4} \ln 2 \leq \bar{h}_{\text{top}}(f(\mathcal{B} \setminus \mathcal{E}, \cdot))$$

for the family (9). Consequently, the function $\mu \mapsto \bar{h}_{\text{top}}(f(\mu, \cdot))$ is not of Baire class 2 [11], and since the sets $\mathcal{E}$ and $\mathcal{B} \setminus \mathcal{E}$ are everywhere dense in $\mathcal{B}$, it follows that this function is everywhere discontinuous on $\mathcal{B}$. The proof of the theorem is complete.

Recall that a metric space X is said to be *locally compact* if each of its points has a compact neighborhood [12, p. 315 of the Russian translation]. A locally compact space X is said to be *countable at infinity* [12, p. 316 of the Russian translation] if it is a union of a countably many compact sets. The space $\mathbb{R}^n$, the above-defined space $\mathcal{C}$, and, in general, any locally compact space with countable base are examples of such spaces [12, p. 316; 13, p. 254].

Theorem 4. *Let X be a locally compact space countable at infinity. Then, for any space $\mathcal{M}$ and any mapping (3), the function (14) is of Baire class 3 on $\mathcal{M}$ and the function (15) is of Baire class 2 on $\mathcal{M}$. If $\mathcal{M}$ is metrizable by a complete metric, then, for each mapping (3), the set of points of lower semicontinuity of the function (15) is a G_δ set everywhere dense in $\mathcal{M}$.*

Proof. Since X is countable at infinity, it follows that there exists an increasing sequence $\{U_s\}_{s=1}^\infty$ of relatively compact open sets that form a cover of X and satisfy $\bar{U}_s \subset U_{s+1}$ for all $s \in \mathbb{N}$ [12, p. 316 of the Russian translation]. Every compact set $K \subset X$ is contained in $\bar{U}_{s_0}$ for some s_0 , because otherwise the cover of K by the sequence $\{U_s\}_{s=1}^\infty$ would not contain a finite subcover, which contradicts the compactness of K . Thus,

$$\begin{aligned} \bar{h}_{\text{top}}(f) &= \sup_{K \in \mathcal{K}(X)} \bar{h}_{\text{top}}(K, f) = \sup_{s \in \mathbb{N}} \bar{h}_{\text{top}}(\bar{U}_s, f), \\ \underline{h}_{\text{top}}(f) &= \sup_{K \in \mathcal{K}(X)} \underline{h}_{\text{top}}(K, f) = \sup_{s \in \mathbb{N}} \underline{h}_{\text{top}}(\bar{U}_s, f) \end{aligned}$$

for each continuous mapping $f : X \rightarrow X$. Using formula (7), we obtain

$$\begin{aligned}\overline{h}_{\text{top}}(f(\mu, \cdot)) &= \sup_{s \in \mathbb{N}} \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln N_d(\overline{U}_s, f(\mu, \cdot), \varepsilon, n) \\ &= \sup_{s \in \mathbb{N}} \sup_{k \in \mathbb{N}} \overline{\lim}_{n \rightarrow \infty} \sup_{m \in \mathbb{N}} \varphi_d^m(\overline{U}_s; \mu, 1/k, n) = \sup_{s \in \mathbb{N}} \sup_{k \in \mathbb{N}} \inf_{n \in \mathbb{N}} \sup_{l \geq n} \sup_{m \in \mathbb{N}} \varphi_d^m(\overline{U}_s; \mu, 1/k, l) \\ &= \lim_{p \rightarrow \infty} \max_{1 \leq s \leq p} \max_{1 \leq k \leq p} \lim_{r \rightarrow \infty} \min_{1 \leq n \leq r} \lim_{q \rightarrow \infty} \max_{n \leq l \leq q} \max_{1 \leq m \leq q} \varphi_d^m(\overline{U}_s; \mu, 1/k, l).\end{aligned}$$

By (8), we have

$$\begin{aligned}\underline{h}_{\text{top}}(f(\mu, \cdot)) &= \sup_{s \in \mathbb{N}} \lim_{\varepsilon \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \ln S_d(\overline{U}_s, f(\mu, \cdot), \varepsilon, n) \\ &= \sup_{s \in \mathbb{N}} \sup_{k \in \mathbb{N}} \underline{\lim}_{n \rightarrow \infty} \inf_{m \in \mathbb{N}} \psi_d^m(\overline{U}_s; \mu, 1/k, n) = \sup_{s \in \mathbb{N}} \sup_{k \in \mathbb{N}} \sup_{n \in \mathbb{N}} \inf_{l \geq n} \inf_{m \in \mathbb{N}} \psi_d^m(\overline{U}_s; \mu, 1/k, l) \\ &= \lim_{p \rightarrow \infty} \max_{1 \leq s \leq p} \max_{1 \leq k \leq p} \max_{1 \leq n \leq p} \lim_{q \rightarrow \infty} \min_{n \leq l \leq q} \min_{1 \leq m \leq q} \psi_d^m(\overline{U}_s; \mu, 1/k, l).\end{aligned}$$

Since the maximum and minimum of finitely many functions in some Baire class belong to the same class [9, Ch. IX, Sec. 37, III of the Russian translation], it follows that the function $\mu \mapsto \overline{h}_{\text{top}}(f(\mu, \cdot))$ is of Baire class 3 and the function $\mu \mapsto \underline{h}_{\text{top}}(f(\mu, \cdot))$ is of Baire class 2 on $\mathcal{M}$.

Since the function $\mu \mapsto \underline{h}_{\text{top}}(f(\mu, \cdot))$ can be represented as the limit of a nondecreasing sequence of functions of Baire class 1, it follows that its set of points of lower semicontinuity is an everywhere dense G_δ set provided that $\mathcal{M}$ is metrizable by a complete metric [10, Lemma 2]. The proof of the theorem is complete.

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