ERRATA

Erratum to: Real-Imaginary Conjugacy Classes and Real-Imaginary Irreducible Characters in Finite Groups*

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Let G be a finite group. A character χ of G is said to be *real-imaginary* if its values are real or purely imaginary. A conjugacy class C of a in G is *real-imaginary* if and only if $\chi(a)$ is real or purely imaginary for all irreducible characters χ of G . And a finite group G is called *real-imaginary* if all of its irreducible characters are real-imaginary.

In [1], the author described real-imaginary conjugacy classes and real-imaginary characters. In Theorem 1 of [1], he showed that C is a real-imaginary conjugacy class of G if and only if $C^2 = (C^{-1})^2$. In the next example, we show that this statement is not true.

Example. Let G be a non-Abelian group of order 21, i.e.

$$G = \langle a, b \mid a^7 = b^3 = 1, b^{-1}ab = a^2 \rangle.$$

We can check easily from the character table of G that the conjugacy class $C = \{a, a^2, a^4\}$ satisfies $C^2 = (C^{-1})^2$, but it is not real-imaginary.

The mistake in the proof of Theorem 1 of [1] comes from the last paragraph of p. 252, in which it was asserted that if C is the conjugacy class of a in G such that $C^2 = (C^{-1})^2$, then $\chi(a)\chi(a)$ is real for each irreducible character χ of G . As a counterexample to this claim, take the above example and the character $\chi \in \text{Irr}(G)$ that takes the following values: $\chi(1) = 3$, $\chi(a) = \alpha + \alpha^2 + \alpha^4$ ($\alpha^7 = 1$); $\chi(a^6) = \alpha^3 + \alpha^5 + \alpha^6$; and $\chi(g) = 0$ for every element $g \in G$ of order 3. It follows from Problem 3.12 of [2] that

$$\chi(a)\chi(a) = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(aa^x) = \frac{\chi(1)}{|G|} (7\chi(a^3) + 7\chi(a^5) + 7\chi(a^2)),$$

which is not real.

The following theorem is the revised version of Theorem 1 of [1].

Theorem 1. *Let G be a finite group, let C be a conjugacy class of G , and let $\mathfrak{C}$ denote the class sum of C in the complex group algebra $\mathbb{C}[G]$. Then C is a real-imaginary conjugacy class if and only if $\mathfrak{C}^2 = (\mathfrak{C}^{-1})^2$.*

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Proof. Assume that $C = a^G$ is a real-imaginary conjugacy class, i.e., $\chi(a)$ is real or purely imaginary for all $\chi \in \text{Irr}(G)$. By Theorem 2.4 of [2], we see that $\mathfrak{C}^2 = \sum_j a_j \mathfrak{C}_j$ and $(\mathfrak{C}^{-1})^2 = \sum_j b_j \mathfrak{C}_j$, where $\mathfrak{C}$, $\mathfrak{C}^{-1}$, and $\mathfrak{C}_j$ are the class sums of the conjugacy classes C , $C^{-1} = (a^{-1})^G$, and C_j in G . By Problem 3.9 from [2], we can write

$$a_j = \frac{|C|^2}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(a)^2 \overline{\chi(c_j)}}{\chi(1)} = \frac{|C^{-1}|^2}{|G|} \sum_{\chi \in \text{Irr}(G)} \frac{\chi(a^{-1})^2 \overline{\chi(c_j)}}{\chi(1)} = b_j$$

where $c_j \in C_j$ for every j . Thus, the equality $\mathfrak{C}^2 = (\mathfrak{C}^{-1})^2$ is proved.

For the converse assertion, suppose that $\mathfrak{C}^2 = (\mathfrak{C}^{-1})^2$; let $\chi \in \text{Irr}(G)$, and let $\mathfrak{X}$ be a complex irreducible representation associated to χ . Following the notation of [2], we know that $\mathfrak{X}$ can be linearly extended to $\mathbb{C}[G]$ and that $\mathfrak{X}(\mathfrak{C}) = w_\chi(\mathfrak{C})I$, where

$$w_\chi(\mathfrak{C}) = \frac{|C|\chi(a)}{\chi(1)}$$

and I is the identity matrix. Since $\mathfrak{C}^2 = (\mathfrak{C}^{-1})^2$, we have $w_\chi(\mathfrak{C})w_\chi(\mathfrak{C}) = w_\chi(\mathfrak{C}^{-1})w_\chi(\mathfrak{C}^{-1})$, and hence

$$\frac{|C|\chi(a)}{\chi(1)} \frac{|C|\chi(a)}{\chi(1)} = \frac{|C^{-1}|\chi(a^{-1})}{\chi(1)} \frac{|C^{-1}|\chi(a^{-1})}{\chi(1)}.$$

This gives

$$\chi(a)^2 = \chi(a^{-1})^2 = \overline{\chi(a)}^2$$

for every $\chi \in \text{Irr}(G)$, and this is certainly equivalent to the fact that $\chi(a)$ is real or purely imaginary for every $\chi \in \text{Irr}(G)$. Therefore, C is a real-imaginary class, as wanted. $\square$

Remark. The equality $(\mathfrak{C})^2 = (\mathfrak{C}^{-1})^2$ trivially implies the set equality $C^2 = C^{-2}$, but the converse is not necessarily true. Consequently, the condition $C^2 = C^{-2}$ is not enough for a conjugacy class C to be real-imaginary. In the example, the conjugacy class C satisfies $C^2 = C^{-2}$, but $(\mathfrak{C})^2 \neq (\mathfrak{C}^{-1})^2$. Indeed, we have

$$(\mathfrak{C})^2 = \mathfrak{C} + 2\mathfrak{C}^{-1}, \quad (\mathfrak{C}^{-1})^2 = 2\mathfrak{C} + \mathfrak{C}^{-1},$$

and C is not real-imaginary.

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