ERRATA

Erratum: “On the Distribution of Zeros of Functions from Weighted Bergman Spaces” [*Mathematical Notes* 86 (1), 93–106 (2009)]

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In the authors' paper [1], some well-known characteristics of the zero sets $\{z_n(f)\}$ of functions f from the Bergman spaces A_φ^p with weight $\varphi(t)$ were refined. However, an error was made, which led to false statements in the part of [1] that deals with the sequence $\{n_k\}$ for which the following relation holds:

$$1 - |z_{n_k}(f)| \sim \frac{1}{p} \cdot \frac{\log \Phi(n_k)}{n_k}, \quad k \rightarrow \infty. \quad (1)$$

Let us present the necessary notation, point out the error, and make the correct choice of the sequence $\{n_k\}$. The Bergman space A_φ^p , $0 < p < \infty$, with weight $\varphi(t)$ is the space of functions $f(z)$, $z = x + iy = re^{i\theta}$, analytic in the unit disk $|z| < 1$ such that

$$\left(\frac{1}{\pi} \iint_{|z|<1} |f(z)|^p \varphi(1-r^2) dx dy \right)^{1/p} < \infty.$$

Let $\varphi(t)$ be a positive summable function on $(0, 1]$ satisfying the Δ_2 -condition, i.e., $\varphi(2t) \leq c\varphi(t)$ for $t \in (0, 1/2]$ and some constant $c > 0$; set

$$\Phi(u) = \left(\int_0^{1/u} \varphi(t) dt \right)^{-1}, \quad u \geq 1.$$

Let $\{z_k(f)\}_{k=1}^\infty$ be the sequence of zeros of a function $f \in A_\varphi^p$ arranged in nondecreasing order of their moduli; in this sequence, each zero appears the number of times equal to its multiplicity.

The following statement was proved in [1] (see Lemma 8).

Statement 1. Suppose that $p \geq 2$, $1 = \gamma_0 > \gamma_1 > \dots > \gamma_k > 0$, and $\gamma_0 + \gamma_1 + \dots < \infty$. Set

$$b_1 = 1, \quad b_{k+1} = \gamma_k^{-(p-1)/p} \prod_{j=1}^k b_j, \quad k = 1, 2, \dots \quad (2)$$

Define the numbers $\tilde{n}_k$ for k such that $b_{k+1}^p \geq \Phi(1)$ from the equality

$$\Phi(\tilde{n}_k) = b_{k+1}^p \quad (3)$$

and, for the other k , $k \leq s$, from the definition, $\tilde{n}_0 = \dots = \tilde{n}_s = 1$. Set

$$n_1 = 0, \quad n_k = \max\{\tilde{n}_k, n_{k-1} + 1\}, \quad k = 2, 3, \dots,$$

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and let

$$f(z) = \prod_{k=2}^{\infty} (1 + b_k z^{n_k - n_{k-1}}).$$

Then $f \in A_{\varphi(t)}^p$ and, for all $k > s$, the following relation holds:

$$\log \frac{1}{|z_{n_k}(f)|} = \frac{1}{p} \cdot \frac{1}{n_k - n_{k-1}} \log \Phi(\tilde{n}_k), \quad n_k = [\tilde{n}_k]. \quad (4)$$

It follows from formulas (2) and (3) that

$$\Phi(\tilde{n}_{k-1}) = \left(\left(\frac{\gamma_k}{\gamma_{k-1}} \right)^{p-1} \Phi(\tilde{n}_k) \right)^{1/2}, \quad k > s,$$

so that

$$\log \Phi(\tilde{n}_{k-1}) \sim \frac{1}{2} \log \Phi(\tilde{n}_k), \quad k \rightarrow \infty$$

(if, for example, $\gamma_{k-1}/\gamma_k \leq \text{const}$). It was erroneously stated in [1] that $\log \Phi(\tilde{n}_{k-1}) \sim \log \Phi(n_k)$, $k \rightarrow \infty$ (see the last row in the statement of Lemma 8 from [1]). From this it was derived (see Theorem 4 from [1]) that if the sequence of numbers n_k satisfies

$$\sum_{k=1}^{\infty} \left(\frac{1}{\Phi(n_k)} \prod_{j=1}^{k-1} \Phi(n_j) \right)^{1/(p-1)} < \infty \quad (5)$$

and the terms of the series (5) decrease, then there exists a function $f \in A_{\varphi(t)}^p$, $p \geq 2$ for which relation (1) holds. In fact, condition (5) guarantees the existence of a function $f \in A_{\varphi(t)}^p$ for which

$$1 - |z_{n_k}(f)| \sim \frac{1}{2p} \frac{\log \Phi(n_k)}{n_k}, \quad k \rightarrow \infty.$$

It can be shown that, for $p = 2$, the function f constructed in Statement 1 possesses certain extremal properties, being the so-called “boundary” function in $A_{\varphi(t)}^2$ (see [2]).

Using results from [1], let us point out conditions on the sequence $\{n_k\}$ ensuring relation (1) and refining it (for $p \geq 2$).

Statement 2. Suppose that $p \geq 2$, $1 > \gamma_0 > \gamma_1 > \dots > \gamma_k > 0$, $\gamma_0 + \gamma_1 + \dots < \infty$, $a_k \in (0, 1/2]$, $k = 1, 2, \dots$, $\tilde{n}_0 \in \mathbb{N}$, and $\Phi(\tilde{n}_0) \geq 1$. Define $\tilde{n}_k$, $k = 1, 2, \dots$, from the equality

$$\Phi(\tilde{n}_k)^{a_k} = \left(\frac{\gamma_{k-1}}{\gamma_k} \right)^{p-1} \Phi(\tilde{n}_{k-1}), \quad (6)$$

and let

$$\frac{\Phi(\tilde{n}_k)}{\Phi(\tilde{n}_{k-1})} \uparrow \infty, \quad k \rightarrow \infty. \quad (7)$$

Set

$$n_k = \max\{[\tilde{n}_k], n_{k-1} + 1\}, \quad k = 1, 2, \dots,$$

and let

$$b_k = \Phi(\tilde{n}_k)^{(1-a_k)/p}, \quad (8)$$

$$f(z) = \prod_{s=1}^{\infty} (1 + b_s z^{n_s - n_{s-1}}). \quad (9)$$

Then $f \in A_{\varphi(t)}^p$ and, for sufficiently large k , the following equality holds:

$$\log \frac{1}{|z_{n_k}(f)|} = \frac{1 - a_k}{p} \cdot \frac{1}{n_k - n_{k-1}} \log \Phi(\tilde{n}_k). \quad (10)$$

Proof. It follows from condition (7) (see Lemma 4 from [1]) that, for all sufficiently large k ,

- 1) $n_k > 2n_{k-1}$;
- 2) $\Phi(n_k) \leq c_\Phi \Phi(n_k - n_{k-1})$.

Equality (8) and the relations $\log \Phi(u) = o(u)$, $u \rightarrow \infty$ following from the Δ_2 -condition imposed on the function $\varphi(t)$ imply that

- 3) $(n_k - n_{k-1})^{-1} \log b_k \rightarrow 0$.

Therefore,

$$b_k^{1/(n_k - n_{k-1})} \rightarrow 1$$

and the product (9) converges in the disk $|z| < 1$. Note that the following property also holds:

- 4) $\sum b_k^{-p/(p-1)} < \infty$.

Indeed, in view of (8), (6) as well as of the inclusion $a_k \in (0, 1/2]$, we can write

$$\left(\frac{b_{k-1}}{b_k}\right)^{p/(p-1)} = \left(\frac{\gamma_k}{\gamma_{k-1}}\right)^{(1-a_k)/a_k} \Phi(\tilde{n}_{k-1})^{(2-a_{k-1}-1/a_k)/(p-1)} \leq \frac{\gamma_k}{\gamma_{k-1}}.$$

Hence, using the condition $\sum \gamma_k < \infty$, we obtain 4).

In the proof of Lemma 7 from [1], it was established that if conditions 1)–4) hold, then the Taylor coefficients $a_k(g)$ of the function

$$g(z) = \prod_{k=r}^{\infty} (1 + b_k z^{n_k - n_{k-1}}),$$

where r is sufficiently large, satisfy the inequality

$$\begin{aligned} & \sum_{k=1}^{\infty} |a_k(g)|^{p/(p-1)} \Phi(k)^{-1/(p-1)} \\ & \leq \Phi(n_r - n_{r-1})^{-1/(p-1)} |a_{n_r - n_{r-1}}|^{p/(p-1)} + c_\Phi(p) \sum_{s=r+1}^{\infty} \Phi(\tilde{n}_s)^{-1/(p-1)} \prod_{k=r}^s b_k^{p/(p-1)} \end{aligned} \quad (11)$$

(see formulas (5.7)–(5.9) from [1]).

Now note that

$$\begin{aligned} & \Phi(\tilde{n}_s)^{-1/(p-1)} \prod_{k=1}^s b_k^{p/(p-1)} = \Phi(\tilde{n}_s)^{1/(1-p)} \prod_{k=1}^s \Phi(\tilde{n}_k)^{(1-a_k)/(p-1)} \\ & = \left(\frac{1}{\Phi(\tilde{n}_s)^{a_s}} \prod_{k=1}^{s-1} \Phi(\tilde{n}_k)^{1-a_k} \right)^{1/(p-1)} \\ & = \left(\frac{\Phi(\tilde{n}_{s-1})}{\phi(\tilde{n}_s)^{a_s}} \cdot \frac{\Phi(\tilde{n}_{s-2})}{\Phi(\tilde{n}_{s-1})^{a_{s-1}}} \cdots \frac{\Phi(\tilde{n}_1)}{\Phi(\tilde{n}_2)^{a_2}} \cdot \frac{1}{\Phi(\tilde{n}_1)^{a_1}} \right)^{1/(p-1)} \\ & = \frac{\gamma_s}{\gamma_{s-1}} \cdot \frac{\gamma_{s-1}}{\gamma_{s-2}} \cdots \frac{\gamma_2}{\gamma_1} \Phi(\tilde{n}_1)^{-a_1/(p-1)} = c_1 \gamma_s, \end{aligned} \quad (12)$$

where

$$c_1 = \gamma_1^{-1} \cdot \Phi(\tilde{n}_1)^{-a_1/(p-1)}.$$

Combining this with (11), and the condition $\sum \gamma_k < \infty$, we obtain the convergence of the series

$$\sum_{k=1}^{\infty} |a_k(g)|^{p/(p-1)} \Phi(k)^{-1/(p-1)}.$$

By Lemma 5 from [1], we have $g \in A_{\varphi}^p$ and, therefore, also $f \in A_{\varphi}^p$. Besides,

$$|z_{n_{s-1}+1}(f)| = \cdots = |z_{n_s}(f)| = \left(\frac{1}{b_s}\right)^{1/(n_s-n_{s-1})},$$

$$\log \frac{1}{|z_{n_s}(f)|} = \frac{1}{n_s - n_{s-1}} \log b_s = \frac{1 - a_s}{p} \cdot \frac{\log \Phi(\tilde{n}_s)}{n_s - n_{s-1}}.$$

The statement is proved. $\square$

In view of (12), we can write

$$\gamma_s = \frac{1}{c_1} \left(\frac{1}{\Phi(\tilde{n}_s)} \prod_{k=1}^s \Phi(\tilde{n}_k)^{1-a_k} \right)^{1/(p-1)},$$

Hence Statement 2 implies the following assertion, which must replace Theorem 4 in [1] (accordingly, in [1], the statement in the case $\varphi(t) = t^\alpha$ must also be changed; see Remark 5 from [1]).

Statement 3. *For any weight $\varphi(t) \in \Delta_2$ any sequence $a_k \rightarrow +0$, and any sequence of numbers n_k such that*

$$\sum_{k=1}^{\infty} \left(\frac{1}{\Phi(n_k)^{a_k}} \prod_{j=1}^{k-1} \Phi(n_j)^{1-a_j} \right)^{1/(p-1)} < \infty, \quad p \geq 2, \quad (13)$$

and for which the terms of the series (13) decrease, there exists a function $f \in A_{\varphi(t)}^p$, $f(0) = 1$, such that, for sufficiently large k ,

$$\log \frac{1}{|z_{n_k}(f)|} > \frac{1 - a_k}{p} \cdot \frac{\log \Phi(n_k)}{n_k}; \quad (14)$$

in particular,

$$1 - |z_{n_k}(f)| \sim \frac{1}{p} \cdot \frac{\log \Phi(n_k)}{n_k}, \quad k \rightarrow \infty.$$

Note that here, for any function $f \in A_{\varphi(t)}^p$,

$$1 - |z_n(f)| = \frac{1}{p} \cdot \frac{\log(\varepsilon_n(f) \Phi(n))}{n},$$

where $\varepsilon_n(f) \rightarrow 0$ as $n \rightarrow \infty$ (see Theorem 1 from [2]). Results from [2] also imply that, for $p = 2$, condition (13) is sharp if inequality (14) is valid.

REFERENCES

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