Erratum: “Special Elements of the Lattice of Overcommutative Varieties of Semigroups” [*Mathematical Notes* 70 (5), 608–615 (2001)]

B. M. Vernikov*

Ural State University

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A description of distributive, codistributive, standard, costandard, and neutral elements of the lattice of overcommutative varieties of semigroups was proposed in the paper. In order to state this result more precisely, we need some notation.

Suppose that m and n are natural numbers, $2 \leq m \leq n$. By the *partition of the number n into m parts* we mean a sequence $\lambda = (\ell_1, \ell_2, \dots, \ell_m)$ of natural numbers such that

$$\sum_{i=1}^m \ell_i = n, \quad \ell_1 \geq \ell_2 \geq \dots \geq \ell_m.$$

The set of all partitions of a number n into m parts is denoted by $\Lambda_{n,m}$. Suppose that $\lambda \in \Lambda_{n,m}$. Let $W_{n,m,\lambda}$ denote the set of all words of length n depending on letters $\{x_1, x_2, \dots, x_m\}$ in which x_i occurs ℓ_i times for any $i = 1, 2, \dots, m$. We denote by $\mathcal{SEM}$ the variety of all semigroups, by $\mathcal{S}_n$ the variety of semigroups defined by all balanced identities of length $\geq n$, by $\mathcal{S}_{n,m}$ the subvariety in $\mathcal{S}_{n+1}$, defined (within $\mathcal{S}_{n+1}$) by all balanced identities of length n depending on $\leq m$ letters (here we assume that $\mathcal{S}_{n,1} = \mathcal{S}_{n+1}$), and by $\mathcal{S}_{n,m,\lambda}$ the subvariety in $\mathcal{S}_{n,m-1}$, defined (inside $\mathcal{S}_{n,m-1}$) by all balanced identities of the form $u = v$, where $u \in W_{n,m,\lambda}$. By **OC** we denote the lattice of all overcommutative varieties of semigroups. The main result of the paper cited in the title is the following statement: *For an arbitrary overcommutative variety of semigroups $\mathcal{V}$, the following conditions are equivalent:*

- (i) $\mathcal{V}$ is a distributive element of the lattice **OC**;
- (ii) $\mathcal{V}$ is a codistributive element of the lattice **OC**;
- (iii) $\mathcal{V}$ is a standard element of the lattice **OC**;
- (iv) $\mathcal{V}$ is a costandard element of the lattice **OC**;
- (v) $\mathcal{V}$ is a neutral element of the lattice **OC**;
- (vi) $\mathcal{V}$ is one of the varieties $\mathcal{SEM}$, $\mathcal{S}_n$, $\mathcal{S}_{n,m}$, and $\mathcal{S}_{n,m,\lambda}$, where $2 \leq m \leq n$ and $\lambda \in \Lambda_{n,m}$.

In fact, conditions (i)–(v) are equivalent and the implication (vi) $\Rightarrow$ (v) is valid, but the implication (v) $\Rightarrow$ (vi) is false. Indeed, it is well known that the set of all neutral elements of an arbitrary lattice L constitutes a sublattice in L . At the same time, the set of varieties

$$S = \{\mathcal{SEM}, \mathcal{S}_n, \mathcal{S}_{n,m}, \mathcal{S}_{n,m,\lambda} \mid 2 \leq m \leq n, \lambda \in \Lambda_{n,m}\}$$

is not closed with respect to intersections: if the set $\Lambda_{n,m}$ contains more than k partitions and $\lambda_1, \lambda_2, \dots, \lambda_k$ are pairwise distinct partitions from $\Lambda_{n,m}$, then

$$\bigcap_{i=1}^k \mathcal{S}_{n,m,\lambda_i} \notin S.$$

*E-mail: boris.vernikov@usu.ru

In order to indicate the error in our paper, we introduce some new notation. We set

$$\Lambda = \bigcup_{2 \leq m \leq n} \Lambda_{n,m}.$$

If $\mathcal{V}$ is an overcommutative variety of semigroups, ν is the fully invariant congruence corresponding to this variety on the absolutely free semigroup F of a countable rank, and $\lambda \in \Lambda$, then by ν_λ we denote the restriction of the congruence ν to the set $W_{n,m,\lambda}$. In our paper, it was proved (this part of the arguments holds true) that *each of the properties (i)–(v) is equivalent to the following condition:*

(vii) *for any partition $\lambda \in \Lambda$, the relation ν_λ is either the universal relation or the equality relation.*

Further, it was asserted in the paper that the variety $\mathcal{V}$ possesses property (vii) if and only if it belongs to the set S . This assertion was not justified in any way, but was proclaimed as an obvious one.¹ To verify that this statement is false, it suffices to give an example of a variety possessing property (vii), but not belonging to the set S .

Let $\mathcal{K}$ denote the variety of semigroups given by all balanced non-permutation identities. Taking into account the fact that the varieties $\mathcal{S}_n$, $\mathcal{S}_{n,m}$, and $\mathcal{S}_{n,m,\lambda}$ satisfy any permutation identity of length $n+1$, and, in $\mathcal{K}$, no permutation identity is valid, we find that $\mathcal{K} \notin S$. At the same time, the variety $\mathcal{K}$ possesses property (vii), because if ξ is the fully invariant congruence on the semigroup F corresponding to the variety $\mathcal{K}$, and $\lambda \in \Lambda_{n,m}$, then the relation ξ_λ is the equality relation for $m = n$ and the universal relation for $m < n$.

The author, together with V. Yu. Shaprynskii, obtained a valid description of varieties satisfying conditions (i)–(v). In order to state this result, we need some new notation. Suppose that

$$\lambda = (\ell_1, \ell_2, \dots, \ell_m) \in \Lambda_{n,m}.$$

The numbers $\ell_1, \ell_2, \dots, \ell_m$ are called the *components* of the partition λ . Let $q(\lambda)$ denote the number of components of the partition λ equal to 1 (if $\ell_m > 1$, then $q(\lambda) = 0$), and let $r(\lambda)$ denote the sum of all components of the partition λ , greater than 1 (if $\ell_1 = 1$, then $r(\lambda) = 0$). We set

$$s(\lambda) = \max\{r(\lambda) - q(\lambda) - \delta, 0\},$$

where $\delta = 0$ if $n = 3$, $m = 2$, and $\lambda = (2, 1)$, and $\delta = 1$ in all other cases. For any $\lambda = (\ell_1, \ell_2, \dots, \ell_m) \in \Lambda_{n,m}$ and $k \geq 0$, we set

$$\lambda^k = (\ell_1, \ell_2, \dots, \ell_{m+k}), \quad \text{where } \ell_{m+1} = \dots = \ell_{m+k} = 1$$

(in particular, $\lambda^0 = \lambda$) and

$$\mathcal{U}_\lambda = \bigwedge_{i=0}^{s(\lambda)} \mathcal{W}_{n+i, m+i, \lambda^i},$$

where $\mathcal{W}_{k,\ell,\mu}$ is the variety defined by all identities of the form $u = v$ for $u, v \in W_{k,\ell,\mu}$. The following assertion holds. (Its proof will be given elsewhere.)

Statement. *The variety of semigroups $\mathcal{V}$ possesses any one of the properties (i)–(v) if and only if either $\mathcal{V} = \mathcal{SEM}$, or $\mathcal{V} = \bigwedge_{i=1}^k \mathcal{U}_{\lambda_i}$ for certain partitions $\lambda_1, \lambda_2, \dots, \lambda_k \in \Lambda$.*

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¹ Actually, this was done in the following sentence in the proof of Theorem 2 of the paper: “The implications to be proved are, obviously, equivalent to the fact that, for any $\lambda \in \Lambda$, the congruence ν_λ is either the universal relation or the equality relation on W_λ ”. It is this sentence that is erroneous.