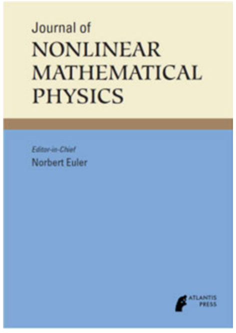


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# Magnetic curves in Sasakian manifolds

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In this paper we classify the magnetic trajectories corresponding to contact magnetic fields in Sasakian manifolds of arbitrary dimension. Moreover, when the ambient is a Sasakian space form, we prove that the codimension of the curve may be reduced to 2. This means that the magnetic curve lies on a 3-dimensional Sasakian space form, embedded as a totally geodesic submanifold of the Sasakian space form of dimension $(2n+1)$.

*Keywords*: contact magnetic field, magnetic curve, Sasakian manifold

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## 1. Introduction

In the electromagnetic theory, a *static magnetic flux density* $\mathbb{B}$ is a divergence-free vector field on an oriented Euclidean 3-space $\mathbb{E}^3$. The *Lorentz force* (derived from $\mathbb{B}$) acting on a charged particle of charge $q$ is a vector field $\mathbb{F} = q\,\dot{\vec{p}}(t) \times \mathbb{B}(\vec{p}(t))$ along the trajectory of the particle. Consequently, the equation of motion (*Newton equation*) for a particle of mass $m$ is $m\,\ddot{\vec{p}}(t) = q\dot{\vec{p}}(t) \times \mathbb{B}(\vec{p}(t))$. In fact, the magnetic flux density $\mathbb{B}$ is often identified with the magnetic field $\mathbb{H}$ as they are related by the equation $\mathbb{B} = \mu\mathbb{H}$, where $\mu$ is the so-called *magnetic permeability* of the medium.

Usually, when $q$ and $\mathbb{B}$ are constant and no other forces are involved, the particle describes a helical trajectory with constant step whose axis is parallel to the magnetic field. However, in case of a variable charge or field, the particle moves along a curve with variable curvature. This strategy was proposed by Xu and Mould in [34] for plotting aesthetic planar curves using simulations of charged particles in a magnetic field. Interestingly, in the design and the production of cartoons, as well as in the description of decay processes, the solutions of the problem describing movement of the charged particle in a magnetic field are used, even for the un-physical time dependence of the charge. Further applications of magnetic curves in CAD systems are described in [33].

In other news, on every oriented Riemannian 3-manifold $(M,g)$, the space $\Lambda^2(M)$ of all smooth 2-forms is identified with the space $\mathfrak{X}(M)$ of all smooth vector fields via the Hodge star operator and the volume form $dv_g$.

In this case, the magnetic fields defined by closed 2-forms are represented by divergence-free vector fields. An important class of magnetic fields consists of Killing vector fields. Studying Killing magnetic curves in 3-dimensional space forms, Barros and Romero showed in [10] that these curves are centerlines of Kirchhoff elastic rods. Further on, Barros et al. in [9] showed that these curves are solitons of the *localized induction equation* (LIE). A more global and geometric view on the connection between solitons of LIE and magnetic curves are provided in [29].

In [13] Cabrerizo proved that in $\mathbb{S}^3$ the Killing magnetic curves are helices as well. It is known from Physics, that when a charged particle enters the Earth's magnetic field, its path changes into a spiral with axis parallel to fields lines due to the Lorentz force. With this in mind, Cabrerizo recalls in [13] a report of an astronomers team concerning the radio emission of the Supernova 1987A's remnants. He concludes that a charged particle "spirals" around the magnetic fields lines independently on the place in the Universe.

The study of magnetic curves in arbitrary Riemannian manifolds was further developed mostly in early 1990's, even though related pioneer works were published much earlier. We can refer to Arnold's problems concerning charges in magnetic fields on Riemannian manifolds of arbitrary dimension, commented by Ginzburg in [22], and references therein.

Practically, starting from the 3D case, one can generalize the Newton equation to arbitrary Riemannian manifolds [30]. A closed 2-form $F$ on a Riemannian manifold $(M,g)$ is called a *magnetic field*. The endomorphism field $\phi$ corresponding to $F$ via $g$ is referred to as the *Lorentz force* of $F$. The Newton equation, also called the *Lorentz equation* of a charged particle, is defined as $\nabla_{\gamma'}\gamma' = q\phi(\gamma')$. Here $\nabla$ is the Levi-Civita connection and $q$ is a constant. A curve that satisfies the Lorentz equation is called *magnetic trajectory*. We use magnetostatic theory because we study the magnetic trajectories generated by time-independent magnetic fields, as function only on the spatial coordinates on the manifold. Magnetostatics deals with electric charges moving with constant speeds and their interactions. In particular, if the magnetic field $F$ vanishes, the Lorentz force is zero, so the magnetic trajectories are geodesics and we may regard the magnetic curves as a generalization of the geodesics.

In 2-dimensional Riemannian geometry, (a nonzero constant multiple of) the area form is a standard example of magnetic field. Sunada studied magnetic trajectories on the hyperbolic plane $\mathbb{H}^2$ as well as on compact Riemannian surfaces of genus $\geq 2$. See also [18]. Since every orientable Riemannian 2-manifold can be regarded as a complex 1-dimensional Kähler manifold, one can consider magnetic fields defined by (a nonzero constant multiple of) the Kähler form of a Kähler manifold. Such a magnetic field is called *Kähler magnetic field*. Adachi [1–3] initiated the study on Kähler magnetic curves in non-flat Kähler space forms. Dynamical systems on a Kähler manifold can be investigated having in mind either its Riemannian structure, or its complex structure. In [25], Kalinin studied $H$-planar flows as an important class of Hamiltonian flows on Kähler manifolds. In this approach, trajectories of charged particles corresponding to Kähler magnetic fields on Kähler manifolds of constant holomorphic sectional curvature are investigated. Kalinin proved that the Lorentz equation can be reduced to one ordinary differential equation of second order, because a Kähler manifold of constant holomorphic sectional curvature admits an $H$-projective mapping on a flat Kähler manifold.

On the other hand, if we consider the original situation of the 3-dimensional Riemannian geometry, we notice the following fact. Let $F$ be a magnetic field on an oriented Riemannian 3-manifold $(M, g, dv_g)$. Let us denote the corresponding divergence-free vector field of $F$ by $V$ and the 1-form that is metrically dual to $V$ by $\omega$. If $V$ is a *unitary vector field*, then $(\phi, V, \omega)$ is an almost contact structure on $M$ compatible with the metric $g$. Hence, an oriented Riemannian 3-manifold $(M, g)$ together with a magnetic field $F$ whose corresponding divergence-free vector field has unit length is regarded as an almost contact metric manifold with closed fundamental 2-form, see [9].

This observation motivates us to study Lorentz equations in Riemannian manifolds of arbitrary odd dimension, more particularly in almost contact metric manifolds. Roughly speaking, almost contact metric manifolds are regarded as odd dimensional analogues of almost Hermitian manifolds. As in the case of almost Hermitian manifolds, the fundamental 2-form of an almost contact metric manifold is generally not closed.

From the variational point of view, as a first step, we are interested in almost contact metric manifolds whose fundamental 2-forms are exact, that is, they possess potential 1-forms. The standard examples of almost contact manifolds with exact fundamental 2-form are contact metric manifolds, in particular *Sasakian manifolds*. These have a very particular unitary Killing vector field, called the Reeb vector field, which naturally defines a magnetic field, called *contact magnetic field*. Contact magnetic fields on Sasakian 3-manifolds are investigated in [14, 15]. Generally speaking, Sasakian manifolds may be regarded as the odd dimensional analogue of a Kähler manifold. It is a known fact that the circles are the only magnetic trajectories on Kähler manifolds. Hence, a natural question would be how to find the analogue of this result in the Sasakian case.

The purpose of this paper is to study the trajectories of particles moving under the influence of a contact magnetic field in Sasakian manifolds of arbitrary dimension. The article is organized as follows. After establishing prerequisite knowledge on almost contact structures and magnetic curves in Section 2, we start our investigations of contact magnetic curves in almost contact manifolds in Section 3. More precisely, we give a classification of all contact magnetic trajectories in Sasakian manifolds of arbitrary odd dimension. Further on, we analyze in detail the trajectories corresponding to the contact magnetic fields in Sasakian space forms. For each case, we provide the reduction theorems for trajectories associated to contact magnetic fields.

## 2. Preliminaries

In this section we collect some fundamental instruments in Sasakian geometry, we briefly mention basic notions about Frenet curves and we recall the general definition and fundamental properties of a magnetic curve.

### 2.1. *Sasakian manifolds*

A smooth manifold $(M^m, g)$ is *Sasakian* if the holonomy group of the metric cone $(C(M^m), dr^2 + r^2 g)$ reduces to a subgroup of $U(\frac{m+1}{2})$, i.e., it is Kähler. Here $m = 2n+1$ is the dimension of the manifold $M^m$. Nevertheless, in this paper we use an equivalent definition for Sasakian manifolds, according to [11], which is more appropriate for our study. We give it shortly.

Let $M^{2n+1}$ be a $(2n+1)$-dimensional orientable differentiable manifold. A *contact form* is a 1-form $\eta$ which satisfies $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M^{2n+1}$. A hyperplane field $D \subset TM^{2n+1}$ with rank $2n$ is said to be a *contact distribution* on $M^{2n+1}$ if for any point $p \in M^{2n+1}$, there exists a contact form $\eta$ defined on a neighborhood $U_p$ of $p$ such that Ker $\eta = D$ on $U_p$. A manifold $M^{2n+1}$ together with a contact structure is called a *contact manifold*.

A differentiable manifold $M^{2n+1}$ is said to have a $(\varphi, \xi, \eta)$ *structure* if it admits a field $\varphi$ of endomorphisms of tangent spaces, a vector field $\xi$ and a 1-form $\eta$ satisfying:

$$\eta(\xi) = 1, \ \ \varphi^2 = -\mathrm{I} + \eta \otimes \xi, \ \ \varphi\xi = 0, \ \ \eta \circ \varphi = 0.$$

Moreover, if $g$ is a Riemannian metric on $M^{2n+1}$ such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(M^{2n+1})$, then $M^{2n+1}$ is said to have an *almost contact metric structure*, and $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called an *almost contact metric manifold*. As a consequence, we see that $g(\xi, \xi) = 1$, and $\eta$ is the dual form of $\xi$, namely $\eta(X) = g(\xi, X)$ for any $X \in \mathfrak{X}(M^{2n+1})$.

An orthonormal basis $\{X_i, \varphi X_i, \xi, \text{ where } i = 1, \ldots, n\}$, constructed in a neighbourhood $U_p$ of a point $p \in M^{2n+1}$, is called a *$\varphi$-basis*.

Next, we define a 2-form $\Omega$ on $(M^{2n+1}, \varphi, \xi, \eta, g)$ by

$$\Omega(X,Y) = g(\varphi X, Y) \tag{2.1}$$

for all $X$, $Y \in \mathfrak{X}(M^{2n+1})$, called *the fundamental* 2*-form* of the almost contact metric structure $(\varphi, \xi, \eta, g)$.

If $\Omega = d\eta$, then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is called a *contact metric manifold*. Here $d\eta$ is defined by

$$d\eta(X,Y) = \frac{1}{2}\big(X\eta(Y) - Y\eta(X) - \eta([X,Y])\big) \text{ for any } X, Y \in \mathfrak{X}(M^{2n+1}).$$

On a contact metric manifold $M$, the 1-form $\eta$ is a contact form. In this situation, the vector field $\xi$ is called the *Reeb vector field* of $M^{2n+1}$ and it is characterized by the properties $\eta(\xi) = 1$ and $d\eta(\xi, \_) = 0$. In analytical mechanics, $\xi$ is traditionally known as the *characteristic vector field* of $M^{2n+1}$.

An almost contact metric manifold $M^{2n+1}$ is said to be *normal* if the normality tensor $S(X,Y) = N_\varphi(X,Y) + 2d\eta(X,Y)\xi$ vanishes, where $N_\varphi$ is the *Nijenhuis torsion* of $\varphi$ defined by

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] + \varphi^2[X,Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] \text{ for any } X,Y \in \mathfrak{X}(M^{2n+1}).$$

A normal contact metric manifold is called a *Sasakian manifold*.

Denoting by $\nabla$ the Levi-Civita connection associated to $g$, a Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is characterized by

$$(\nabla_X \varphi)Y = -g(X,Y)\xi + \eta(Y)X \tag{2.2}$$

for any $X,Y \in \mathfrak{X}(M^{2n+1})$. As a consequence, we have

$$\nabla_X \xi = \varphi X \text{ for all } X \in \mathfrak{X}(M^{2n+1}). \tag{2.3}$$

In Eq. (2.2) we use the sign convention of [32]. See also [23, 31].

Due to the formula (2.3) and the skew-symmetry of $\varphi$ with respect to $g$, it follows that a Sasakian manifold is *K-contact*, that is $\xi$ is a Killing vector field. The converse is generally not true. Yet, a 3-dimensional contact metric manifold is Sasakian if and only if it is $K$-contact.

A plane section $\Pi$ at $p \in M^{2n+1}$ is called a *$\varphi$-section* if there exists a vector $X$ orthogonal to $\xi$ such that $\{X, \varphi X\}$ span the section. The sectional curvature $k(\Pi)$ of a $\varphi$-section is called the *$\varphi$-sectional curvature* of $M^{2n+1}$ at $p$. A Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be a *Sasakian space form* if it has constant $\varphi$-sectional curvature.

### 2.2. *Frenet Curves*

Let $\gamma : I \to (M^3, g)$ be a Frenet curve parametrized by arc length in a Riemannian 3-manifold $M^3$ with Frenet frame field $(T, N, B)$. Here $T$, $N$, $B$ are the tangent, principal normal, and binormal vector fields, respectively. If we denote by $\nabla$ the Levi-Civita connection of $(M^3, g)$, then the Frenet frame satisfies the following *Frenet-Serret* equations:

$$\nabla_T T = \kappa N, \ \ \nabla_T N = -\kappa T + \tau B, \ \ \nabla_T B = -\tau N, \tag{2.4}$$

where $\kappa = |\nabla_T T|$ and $\tau$ are the *curvature* and *torsion* of $\gamma$, respectively.

Extending this theory to curves in a Riemannian manifold $(M, g)$ of arbitrary dimension, let us recall the notion of *Frenet curve of osculating order r*, where $r \geq 1$, according to [11]. This means that there exists an orthonormal frame $\{T = \dot\gamma, \nu_1, \ldots, \nu_{r-1}\}$ of rank $r$ along $\gamma$, such that

$$\begin{aligned}
&\nabla_T T = \kappa_1 \nu_1, \\
&\nabla_T \nu_1 = -\kappa_1 T + \kappa_2 \nu_2, \\
&\nabla_T \nu_j = -\kappa_j \nu_{j-1} + \kappa_{j+1} \nu_{j+1} \ \text{ for } j = 2, \ldots, r-2, \\
&\nabla_T \nu_{r-1} = -\kappa_{r-1} \nu_{r-2},
\end{aligned} \tag{2.5}$$

where $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are positive $C^\infty$ functions of $s$. Moreover, $\kappa_j$ is called the *j-th curvature of* $\gamma$.

For example, a *geodesic* in $(M, g)$ is a Frenet curve of osculating order 1, and a *circle* is a Frenet curve of osculating order 2 with constant curvature $\kappa_1$. A *helix of order r* is defined as a Frenet curve of osculating order $r$, such that all curvatures $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are constant.

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact metric manifold, and $\gamma$ a Frenet curve of osculating order $r$ on $M^{2n+1}$.

A curve of osculating order $r \geq 3$ is called a *$\varphi$-curve* in $M^{2n+1}$ if the space spanned by $T, \nu_1, \ldots, \nu_{r-1}$ is $\varphi$-invariant. A curve of osculating order 2 is called a *$\varphi$-curve* if $\{T, \nu_1, \xi\}$ spans a $\varphi$-invariant space. Furthermore, a *$\varphi$-helix of order $r$* is defined as a $\varphi$-curve of osculating order $r$, such that $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are all constant.

We consider the *$\varphi$-torsions* of $\gamma$ given by $\tau_{ij} = g(v_i, \varphi v_j)$ for $0 \leq i < j \leq r-1$, where $v_0 = T$, and $v_i = \nu_i$ for $i = 1, \ldots, r-1$.

Finally, we recall that the *contact angle $\theta$* of $\gamma$ is a function defined as the angle between the tangent and the characteristic vector field $\xi$, that is $\cos\theta(s) = g(\dot\gamma(s), \xi_{|\gamma(s)})$, where $s$ denotes the arc length parameter of $\gamma$. The curves of contact angle $\pi/2$ are traditionally called *Legendre curves*. The Reeb flow is a curve of contact angle 0.

### 2.3. *Magnetic Curves*

Magnetic curves represent, in Physics, the trajectories of charged particles moving on a Riemannian manifold under the action of magnetic fields. A *magnetic field $F$* on a Riemannian manifold $(M,g)$ is a closed 2-form $F$ and the *Lorentz force* associated to $F$ is an endomorphism field $\phi$ such that

$$F(X,Y) = g(\phi X, Y) \text{ for any } X,Y \in \mathfrak{X}(M). \tag{2.6}$$

The *magnetic trajectories* of $F$ are curves $\gamma$ in $M$ that satisfy the *Lorentz equation* (called also the *Newton equation*)

$$\nabla_{\dot\gamma}\dot\gamma = \phi(\dot\gamma), \tag{2.7}$$

which generalizes the equation of geodesics under arc length parametrization, namely, $\nabla_{\dot\gamma}\dot\gamma = 0$. Here $\nabla$ denotes the Levi-Civita connection associated to the metric $g$.

A magnetic field $F$ is said to be *uniform* if $\nabla F = 0$.

It is well-known that the magnetic trajectories have constant speed. When the magnetic curve $\gamma(s)$ is parametrized by the arc length, it is called *normal magnetic curve*.

The 3-dimensional case is rather special since on any 3-dimensional oriented Riemannian manifold $(M^3,g)$ the 2-forms may be identified with vector fields using the Hodge $\star$ operator and the volume form $dv_g$. In this manner, the magnetic fields, i.e., closed 2-forms, may be thought as divergence-free vector fields. Taking a divergence-free vector field $V$ on $M^3$ we define a magnetic field $F_V$, associated to $V$, by

$$F_V(X,Y) = dv_g(V,X,Y) \tag{2.8}$$

for any vector fields $X$, $Y$ on $M^3$. Moreover, on $(M^3,g)$ one may define the *cross product* $\times$ through the formula

$$g(X \times Y, Z) = dv_g(X,Y,Z)$$

for all vector fields $X,Y,Z \in \mathfrak{X}(M^3)$. Then, the Lorentz force of $F_V$ is $\phi(X) = V \times X$ for all $X \in \mathfrak{X}(M^3)$. See, e.g., [15, 20, 26]. Consequently, the Lorentz equation (2.7) becomes

$$\nabla_{\dot\gamma}\dot\gamma = V \times \dot\gamma. \tag{2.9}$$

A particular class of divergence-free vector fields is represented by the Killing vector fields. Magnetic fields $F_V$ obtained from Killing vector fields are named *Killing magnetic fields*. Their trajectories, called *Killing magnetic curves*, are of great importance since they are related to the Kirchhoff elastic rods. See, e.g., [9, 10].

## 3. Magnetic curves in Sasakian manifolds

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a contact metric manifold and $\Omega$ the fundamental 2-form defined by (2.1). Since $\Omega = d\eta$, we can define a magnetic field on $M^{2n+1}$ by

$$F_q(X,Y) = q\Omega(X,Y), \tag{3.1}$$

where $X, Y \in \mathfrak{X}(M^{2n+1})$ and $q$ is a real constant. We call $F_q$ the *contact magnetic field* with *strength* $q$. Notice that if $q = 0$, then the contact magnetic field vanishes identically and the magnetic curves are the geodesics of $M^{2n+1}$. In the sequel we assume $q \neq 0$.

The *Lorentz force* $\phi_q$ associated to the contact magnetic field $F_q$ may be easily determined combining (2.1) and (2.6), that is

$$\phi_q = q\varphi, \tag{3.2}$$

where $\varphi$ is the field of endomorphisms of the contact metric structure.

In this setting, the Lorentz equation (2.7) can be written as

$$\nabla_{\dot\gamma}\dot\gamma = q\varphi\dot\gamma, \tag{3.3}$$

where $\gamma : I \subseteq \mathbb{R} \to M^{2n+1}$ is a smooth curve parametrized by arc length. The solutions of (3.3) are called *normal magnetic curves* or *trajectories* for $F_q$.

In what follows we classify normal magnetic curves associated to the contact magnetic field $F_q$ on a Sasakian manifold.

**Theorem 3.1.** *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian manifold and consider the contact magnetic field $F_q$ for $q \neq 0$, on $M^{2n+1}$. Then $\gamma$ is a normal magnetic curve associated to $F_q$ in $M^{2n+1}$ if and only if $\gamma$ belongs to the following list:*

*a) geodesics obtained as integral curves of $\xi$;*
*b) non-geodesic $\varphi$-circles of curvature $\kappa_1 = \sqrt{q^2-1}$ for $|q| > 1$, and of constant contact angle $\theta = \arccos \frac{1}{q}$ ;*
*c) Legendre $\varphi$-curves in $M^{2n+1}$ with curvatures $\kappa_1 = |q|$ and $\kappa_2 = 1$, i.e., 1-dimensional integral submanifolds of the contact distribution;*
*d) $\varphi$-helices of order 3 with axis $\xi$, having curvatures $\kappa_1 = |q|\sin\theta$ and $\kappa_2 = |q\cos\theta - 1|$, where $\theta \neq \frac{\pi}{2}$ is the constant contact angle.*

**Proof.** If the magnetic curve $\gamma$ is a geodesic, we have $\varphi\dot\gamma = 0$, and then $\dot\gamma$ is parallel to $\xi$. Since both $\dot\gamma$ and $\xi$ are unitary, then $\dot\gamma = \pm\xi$, and hence $\gamma$ is an integral curve of $\xi$.

From now on we suppose that $\gamma$ is a non-geodesic Frenet curve of osculating order $r > 1$. We can compute the following

$$0 = g(q\varphi T, \xi) = g(\nabla_T T, \xi) = \frac{d}{ds}g(T, \xi) - g(T, \nabla_T \xi).$$

Using relation (2.3) we obtain that $\frac{d}{ds}g(T,\xi)$ vanishes. Therefore, the curve is slant [16], that is the angle $\theta \in (0, \pi)$ between $T$ and $\xi$ is constant. See also [14, Lemma 3.1]. As a consequence, we have

$$\eta(T) = \cos\theta. \tag{3.4}$$

On the other hand, the Lorentz equation and the first Frenet formula lead to the equality

$$\kappa_1 \nu_1 = q\varphi T. \tag{3.5}$$

It follows that

$$\kappa_1 = |q| \sin\theta. \tag{3.6}$$

Plugging (3.6) into (3.5), we see that

$$\varphi T = \mathrm{sgn}(q) \sin\theta \, \nu_1, \tag{3.7}$$

where $\mathrm{sgn}(a)$ denotes the signature of the real number $a$. If $\kappa_2 = 0$, then the magnetic curve $\gamma$ is a Frenet curve of osculating order 2, and since $\kappa_1$ is constant, $\gamma$ is a circle. From (3.7) we get $\eta(\nu_1) = 0$, and taking the covariant derivative with respect to $T$ we obtain $\sin\theta(1 - q\cos\theta) = 0$. Since $\gamma$ is not a geodesic, we should have $\cos\theta = \frac{1}{q}$. It follows that for $|q| > 1$, the curve $\gamma$ is a non-geodesic circle of curvature $\kappa_1 = \sqrt{q^2 - 1}$, proving item b) of the theorem.

Next, taking the covariant derivative with respect to $T$ in (3.7), and using (2.2), the Lorentz equation (3.3) and the second Frenet formula, we obtain

$$\eta(T)T - g(T,T)\xi + q\varphi^2 T = \sin\theta(-q\sin\theta T + \mathrm{sgn}(q)\kappa_2 \nu_2),$$

and hence

$$(q\cos\theta - 1)(\xi - \cos\theta T) = \mathrm{sgn}(q) \sin\theta \, \kappa_2 \nu_2, \tag{3.8}$$

yielding

$$\kappa_2 = |q\cos\theta - 1|.$$

Moreover, from (3.8) we can express $\xi$ in terms of the Frenet frame of $\gamma$:

$$\xi = \cos\theta T + \varepsilon \; \mathrm{sgn}(q) \sin\theta \, \nu_2, \tag{3.9}$$

where $\varepsilon = \mathrm{sgn}(q\cos\theta - 1)$. Applying $\varphi$ we obtain that

$$\varphi \nu_2 = -\varepsilon \cos\theta \, \nu_1.$$

From (3.4), (3.7), and (3.9), it follows that

$$\varphi \nu_1 = -\mathrm{sgn}(q) \sin\theta T + \varepsilon \cos\theta \, \nu_2. \tag{3.10}$$

As a consequence, we have $\eta(\nu_2) = \varepsilon \; \mathrm{sgn}(q) \sin\theta$.

The case $\theta = \frac{\pi}{2}$ is rather special. More precisely, we have $\nu_2 = -\mathrm{sgn}(q)\xi$ and the first two curvatures are $\kappa_1 = |q|$ and $\kappa_2 = 1$. See also [11, Proposition 8.2, page 133]. Then, we compute

$$\nabla_T \nu_2 = -\mathrm{sgn}(q)\varphi T = -\nu_1.$$

From the Frenet formulas we conclude that $\kappa_3 = 0$. Item c) of Theorem 3.1 is obtained.

Let now $\theta \neq \frac{\pi}{2}$. Taking the covariant derivative with respect to $T$ in (3.10), straightforward computations yield $\nabla_T \nu_2 = -\varepsilon(q\cos\theta - 1)\nu_1$, and hence $\kappa_3 = 0$.

Thus, the osculating order of the non-geodesic magnetic curves corresponding to the Lorentz force $\phi_q$ on the Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is equal to 3, concluding the proof of item d) of Theorem 3.1.

To prove the converse part of Theorem 3.1, we proceed as follows.

We start with a $\varphi$-helix $\gamma$ with axis $\xi$ of order 3 with $\kappa_1 = |q|\sin\theta$ and $\kappa_2 = |q\cos\theta - 1|$, where $\theta$ denotes the constant contact angle between $\gamma$ and $\xi$, i.e.,

$$\cos\theta = g(T, \xi),$$

and we show that $\gamma$ satisfies the Lorentz equation (3.3). Successive covariant derivatives with respect to $T$ in the previous expression yield $g(\nu_1, \xi) = 0$. Denoting $\mathscr{V}(s) := \mathrm{span}\{T(s), \nu_1(s), \nu_2(s)\}$, which is a 3-dimensional $\varphi$-invariant space, it follows that $\xi(s) \in \mathscr{V}(s)$ for all $s$. Thus, we may decompose $\xi$ as

$$\xi = \cos\theta T + \rho \nu_2, \tag{3.11}$$

where $\rho = g(\nu_2, \xi)$ is a real constant such that $\rho^2 = \sin^2\theta$, since $\xi$ is unitary. Taking the covariant derivative with respect to $T$ in (3.11) and using the Frenet formulas, we get

$$\varphi T = (\kappa_1 \cos\theta - \rho\kappa_2)\,\nu_1. \tag{3.12}$$

Inserting now the expressions of $\kappa_1$ and $\kappa_2$ into (3.12) and using the fact that $g(\varphi T, \varphi T) = \sin^2\theta$, we immediately obtain $\rho = \varepsilon\mathrm{sgn}(q)\sin\theta$, and hence (3.12) becomes $\varphi T = \mathrm{sgn}(q)\sin\theta\nu_1$. From the first Frenet formula one has

$$\nabla_T T = |q|\sin\theta\nu_1 = \mathrm{sgn}(q)q\sin\theta\nu_1 = q\varphi T,$$

showing that the Lorentz equation (3.3) is fulfilled by $\gamma$ and the converse of item d) is proved.

In particular, when $\theta = \frac{\pi}{2}$ we have $\kappa_1 = |q|$, and $\varphi T = \mathrm{sgn}(q)\nu_1$. Again the first Frenet formula yields that the Legendre curves from c) satisfy the Lorentz equation (3.3).

Furthermore, when $\kappa_2$ vanishes identically, namely $\cos\theta = \frac{1}{q}$, we have $\kappa_1 = \sqrt{q^2 - 1}$, and (3.12) yields $\varphi T = \frac{\sqrt{q^2-1}}{q}\nu_1$. Consequently, $\nabla_T T = \sqrt{q^2 - 1}\nu_1 = q\varphi T$, and thus the Lorentz equation is fulfilled also by the circles from item b).

Concerning a), notice that an integral curve $\gamma$ of $\xi$ is a geodesic, and satisfies $\varphi\dot{\gamma} = 0$. □

**Remark 3.1.** For an arbitrary $\varphi$-helix of order 3 in a Sasakian manifold, not all $\varphi$-torsions are constant, hence the $\varphi$-helix is not necessary a magnetic curve. Yet, if the contact angle is constant, or equivalently $\tau_{02} = 0$, then the three $\varphi$-torsions are constant. Consequently,

$$\kappa_1 \tau_{12} - \kappa_2 \tau_{01} + g(\nu_2, \xi) = 0 \text{ and } \tau_{01}^2 + \tau_{12}^2 = 1.$$

It follows that the $\varphi$-helix is a magnetic curve with the strength $q = -\frac{\kappa_1}{\tau_{01}}$ and the contact angle is given by $\cos\theta = \frac{\tau_{01}\tau_{12}}{\kappa_2\tau_{01} - \kappa_1\tau_{12}}$. In particular, if $\tau_{12}$ vanishes, then the magnetic curve becomes the Legendre $\varphi$-curve stated at item c) of Theorem 3.1.

Form the proof of Theorem 3.1 we may infer two more, interesting, results.

**Proposition 3.1.** *Let $\gamma$ be a non-geodesic Legendre $\varphi$-curve of order 3 in a Sasakian manifold. Then $\kappa_2 = 1$ and $\nu_2 = \pm\xi$.*

This statement generalizes [11, Proposition 8.2, page 133].

**Theorem 3.2.** *Let $\gamma$ be a $\varphi$-helix of order $r \leq 3$ on a Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$. Then we have:*

1. *If $\cos\theta = \pm 1$, then $\gamma$ is an integral curve of $\xi$, hence it is a normal magnetic curve for $F_q$ with an arbitrary $q$.*
2. *If $\cos\theta = 0$ and $\kappa_1 \neq 0$ (that is $\gamma$ is a non geodesic Legendre curve), then $\gamma$ is a magnetic curve for $F_{\pm\kappa_1}$.*
3. *If $\cos\theta = \varepsilon / \sqrt{\kappa_1^2 + 1}$, then $\gamma$ is a magnetic curve for $F_{\varepsilon\sqrt{\kappa_1^2+1}}$, where $\varepsilon = sgn(\tau_{01})$. In such case, $\gamma$ is a $\varphi$-circle, that is $\kappa_2 = 0$.*
4. *If $\cos\theta = (\varepsilon \pm \kappa_2) / \sqrt{\kappa_1^2 + (\varepsilon \pm \kappa_2)^2}$, then $\gamma$ is a magnetic curve for $F_{\varepsilon\sqrt{\kappa_1^2+(\varepsilon\pm\kappa_2)^2}}$, where $\varepsilon = sgn(\tau_{01})$ and the sign $\pm$ corresponds to the sign of $\eta(\nu_2)$.*
5. *Except the above cases, $\gamma$ is not a magnetic curve for contact magnetic fields.*

Moreover, we can state the following.

**Proposition 3.2.** *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian manifold. There exist no non-geodesic circles as magnetic curves corresponding to the contact magnetic field $F_q$ for $0 < |q| \leq 1$, on $M^{2n+1}$.*

Now, let us emphasize that the study of magnetic curves in 3-dimensional Sasakian manifolds was done by Cabrerizo et al. in [15], obtaining the next result:

**Theorem A [15].** *Let $(M^3, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The normal flowlines $\gamma(t)$ of the contact magnetic field $F_\xi = dv_g(\xi, \_, \_)$ are helices of axis $\xi$ with constant curvature $\kappa_0 = \sin\theta_0$ and constant torsion $\tau_0 = 1 + \cos\theta_0$, where $\theta_0$ is the (constant) contact angle.*

This theorem is consistent with Theorem 3.1 for $n = 1$, taking into account the following remark. In our notation,

$$F_\xi = dv_g(\xi, \_, \_) = -3(\eta \wedge d\eta)(\xi, \_, \_) = -d\eta = -\Omega,$$

which shows that $F_\xi$ coincides with $F_{-1}$. (We use the *alt* convention for the exterior product $\wedge$ of differential forms.) Thus, the curvature $\kappa_0$ corresponds to $\kappa_1 = \sin\theta$, and the torsion $\tau_0$ corresponds to $\kappa_2 = 1 + \cos\theta$ in item d) of our theorem. Notice that Theorem A was formulated by the same authors in [14] as well, but with different sign conventions.

In the sequel, we describe the magnetic curves in simply connected Sasakian space forms. Let us recall a standard construction of Sasakian space forms (see e.g. [11]). Let $(M,\varphi,\xi,\eta,g)$ be a Sasakian manifold. Take a positive constant $a$ and define a new Sasakian structure $(\varphi,\hat{\xi},\hat{\eta},\hat{g})$ on $M$ by

$$\hat{\xi} := \frac{1}{a}\xi,\ \ \hat{\eta} := a\eta,\ \ \hat{g} := ag + a(a-1)\eta\otimes\eta. \tag{3.13}$$

This structure is called a *D-homothetic deformation* of $(\varphi,\xi,\eta,g)$. In particular, if $M(c)$ is a Sasakian space form of constant $\varphi$-sectional curvature $c$, then by the $D$-homothetic deformation (3.13) we obtain a Sasakian space form $M(\hat{c})$ of constant $\varphi$-sectional curvature $\hat{c} = \frac{c+3}{a} - 3$. For every value of $c$, there exists a Sasakian space form, as follows: the (non-standard) sphere $\mathbb{S}^{2n+1}$ if $c > -3$, the Heisenberg space $\mathbb{R}^{2n+1}(-3)$, if $c = -3$, and $B^{2n}\times\mathbb{R}$ when $c < -3$. Here $B^{2n}\subset\mathbb{C}^n$ is the unit ball in the complex Euclidean $n$-space. See also [11, Theorem 7.15]. Note that the case $c > -3$ includes the unit sphere $\mathbb{S}^{2n+1}(1)$. We emphasize the following result.

**Proposition 3.3.** *Let $\mathbb{S}^{2n+1}(1)$ be a unit sphere equipped with the canonical Sasakian structure $(\varphi,\xi,\eta,g)$. Then $\mathbb{S}^{2n+1}$ equipped with a D-homothetic deformation $(\varphi,\hat{\xi},\hat{\eta},\hat{g})$ is a Sasakian space form of constant $\varphi$-sectional curvature $c = \frac{4}{a} - 3$. Conversely, let $\mathscr{M}^{2n+1}(c)$ be a simply connected Sasakian space form of constant $\varphi$-sectional curvature $c > -3$. Then $\mathscr{M}^{2n+1}(c)$ is isomorphic to $\mathbb{S}^{2n+1}$ equipped with a D-homothetic deformed Sasakian structure.*

We show now that the study of trajectories associated to contact magnetic fields on Sasakian space forms with $c > -3$ reduces to their study on $\mathbb{S}^{2n+1}(1)$.

Let $\hat{\nabla}$ be the Levi-Civita connection of $\hat{g}$. Then $\hat{\nabla}$ is related to the Levi-Civita connection $\nabla$ of $g$ by

$$\hat{\nabla}_X Y = \nabla_X Y + (a-1)\big(\eta(X)\varphi Y + \eta(Y)\varphi X\big).$$

The corresponding fundamental 2-form $\hat{\Omega}$ is given by $\hat{\Omega} = a\Omega$.

Let $\gamma(s)$ be a magnetic trajectory parametrized by arc length in $\mathbb{S}^{2n+1}(1)$:

$$\nabla_{\dot\gamma}\dot\gamma = q\varphi\dot\gamma.$$

We study the $D$-homothetic image of $\gamma$. We know that $\gamma$ has constant contact angle $\theta$. Since

$$\hat{g}(\dot\gamma,\dot\gamma) = a + a(a-1)\eta(\dot\gamma)^2 = a(\sin^2\theta + a\cos^2\theta),$$

the arc length parameter $\hat{s}$ of $\gamma$ with respect to $\hat{g}$ is

$$\hat{s} = ms,\ \text{when}\ m = \sqrt{a(\sin^2\theta + a\cos^2\theta)},$$

and hence $\hat{\nabla}_{\gamma'}\gamma' = \frac{1}{m^2}\,\hat{\nabla}_{\dot\gamma}\dot\gamma$, where $\gamma'$ denotes the derivative of $\gamma$ with respect to $\hat{s}$. It follows that

$$\hat{\nabla}_{\gamma'}\gamma' = \hat{q}\varphi\gamma',\ \text{where}\ \hat{q} = \frac{1}{m}\big(q + 2(a-1)\cos\theta\big).$$

This formula shows that $\gamma(\hat{s})$ is a magnetic trajectory for the contact magnetic field $\hat{q}\,\hat{\Omega}$ in $(\mathbb{S}^{2n+1},\varphi,\hat{\xi},\hat{\eta},\hat{g})$. The contact angle $\hat{\theta}$ of $\gamma(\hat{s})$ in $(M,\varphi,\hat{\xi},\hat{\eta},\hat{g})$ is given by $\cos\hat{\theta} = \frac{a}{m}\cos\theta$.

Conversely, let $\gamma(\hat{s})$ be a normal magnetic trajectory in $(\mathbb{S}^{2n+1}, \varphi, \hat{\xi}, \hat{\eta}, \hat{g})$ with respect to $\hat{q}\,\hat{\Omega}$ for some constant $\hat{q}$, namely, $\gamma(\hat{s})$ satisfies

$$\hat{\nabla}_{\gamma'}\gamma' = \hat{q}\,\varphi\gamma'.$$

Then we have $\nabla_{\dot{\gamma}}\dot{\gamma} = q\,\varphi\dot{\gamma}$, with arc length parameter $s = \hat{m}\hat{s}$ and strength $q$, where

$$\hat{m} = \frac{1}{a}\sqrt{a\sin^2\hat{\theta} + \cos^2\hat{\theta}}, \quad q = \frac{1}{\hat{m}}\left(\hat{q} - \frac{2(a-1)}{a}\cos\hat{\theta}\right).$$

At this point, we describe the magnetic trajectories on the three models of Sasakian space forms, that is $\mathbb{S}^{2n+1}(1)$, $\mathbb{R}^{2n+1}(-3)$ and $B^{2n}\times\mathbb{R}$, respectively.

### 3.1. *Magnetic curves in* $\mathbb{S}^{2n+1}(1)$

The $(2n+1)$-dimensional sphere $\mathbb{S}^{2n+1} = \{p \in \mathbb{R}^{2n+2} \ : \ \langle p, p\rangle = 1\}$, where $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^{2n+2}$, carries a natural Sasakian structure induced from the canonical complex structure of $\mathbb{R}^{2n+2}$. If $\mathbb{R}^{2n+2}$ is identified with $\mathbb{C}^{n+1}$, let $J$ denote the multiplication with the imaginary unit $i = \sqrt{-1}$, on $\mathbb{R}^{2n+2}$. At any point $p \in \mathbb{S}^{2n+1}$, the unit outward normal to the sphere coincides with the position vector $p$. Let $\xi = Jp$ be the characteristic vector field. If $X$ is tangent to $\mathbb{S}^{2n+1}$, then $JX$ fails in general to be tangent and we decompose it into the tangent and the normal parts, respectively: $JX = \varphi X - \eta(X)p$. We have defined an almost contact structure on $\mathbb{S}^{2n+1}$. Together with the induced metric, a Sasakian structure is obtained.

In what follows, we recall a result proved by Erbacher [21] about the reduction of the codimension of an isometric immersion.

Let $\psi : M^m \to \widetilde{M}^{m+k}(c)$ be an isometric immersion of a connected $m$-dimensional Riemannian manifold $M$ into an $(m+k)$-dimensional Riemannian manifold $\widetilde{M}$ of constant sectional curvature $c$.

The first normal space $N_1(p)$ is defined to be the orthogonal complement of $\{\zeta \in T_p^{\perp}M : A_\zeta = 0\}$ in $T_p^{\perp}M$. Here $A_\zeta$ denotes the shape operator corresponding to a normal vector $\zeta$ at $p$. It is known, see [5], that $N_1(p)$ can also be expressed as

$$N_1(p) = \mathrm{span}\{\sigma(X,Y) : X, Y \in T_pM\}. \tag{3.14}$$

Here $\sigma$ denotes the second fundamental form of $M$ in $\widetilde{M}^{m+k}(c)$. In order to define the second normal space $N_2(p)$, let us consider the space

$$L(p) = \mathrm{span}\{(\nabla^{\perp}_X\sigma(Y,Z))_p : X, Y, Z \in T(M)\}.$$

Define

$$N_2(p) := \begin{cases} L(p)\cap N_1^{\perp}(p), & \text{if } L(p)\cap N_1^{\perp}(p) \neq \{0\}, \\ N_1^{\perp}(p), & \text{otherwise.}\end{cases}$$

The orthogonal complement $N_1^{\perp}(p)$ is taken in $T_p^{\perp}M$. One may continue the procedure in order to define the third normal space $N_3(p)$, and so on. Obviously, there exists the last normal space, see [21]. We recall the following result which will be used later.

**Theorem B [21].** *If $N \supset N_1$ has constant dimension $l$ and $N$ is invariant under parallel translations with respect to the normal connection, then there exists a totally geodesic submanifold $N^{m+l}$ of $\widetilde{M}^{m+k}(c)$ such that $\psi(M^m) \subset N^{m+l}$.*

Let us consider now $m = 1$. It is not difficult to prove the following proposition.

**Proposition 3.4.** *Let $\gamma$ be a Frenet curve of osculating order r on a Riemannian space form $\widetilde{M}(c)$. Then, there exist r normal spaces mutually orthogonal such that the normal bundle of $\gamma$ decomposes as*

$$T_s^{\perp}\gamma = N_1(s) \oplus N_2(s) \oplus \cdots \oplus N_{r-1}(s) \oplus N_r(s),$$

*where s is the arc length parameter of $\gamma$, $N_i(s) = \mathrm{span}\{\nu_i\}$ for $i = 1,\ldots,r-1$, and $N_r(s) = (N_1(s) \oplus \cdots \oplus N_{r-1}(s))^{\perp}$.*

As a consequence of Theorem 3.1 and Proposition 3.4 we have the following result.

**Corollary 3.1.** *Let $\gamma$ be a normal magnetic curve on the sphere $\mathbb{S}^{2n+1}(1)$, endowed with the standard Sasakian structure, corresponding to the contact magnetic field $F_q$. Then, the normal bundle of $\gamma$ splits as*

$$T_s^{\perp}\gamma = N_1(s) \oplus N_2(s) \oplus N_3(s),$$

*where s is the arc length parameter of $\gamma$, $N_1(s) = \mathrm{span}\{\nu_1\}$, $N_2(s) = \mathrm{span}\{\nu_2\}$, and $N_3(s) = (N_1(s) \oplus N_2(s))^{\perp}$.*

Now, we may describe the geometry of the normal magnetic curves on the Sasakian sphere $\mathbb{S}^{2n+1}(1)$.

**Theorem 3.3.** *Let $\gamma$ be a normal magnetic curve on the Sasakian sphere $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$, corresponding to the contact magnetic field $F_q$. Then $\gamma$ is a normal magnetic curve on a 3-dimensional sphere $\mathbb{S}^3(1)$, embedded as a Sasakian totally geodesic submanifold in $\mathbb{S}^{2n+1}(1)$.*

**Proof.** Set $N(s) = N_1(s) \oplus N_2(s) = \mathrm{span}\{\nu_1, \nu_2\}$. Since $N(s) \supset N_1(s)$ is invariant under parallel translations with respect to the normal connection $\nabla^{\perp}$, we obtain, applying Erbacher's reduction theorem (Theorem B), that there exists a 3-dimensional totally geodesic submanifold of $\mathbb{S}^{2n+1}(1)$ which contains the magnetic curve $\gamma$. Hence, $\gamma$ lies on a unit 3-dimensional sphere $\mathbb{S}^3(1)$.

Moreover, from (3.9) we see that $\xi \in \mathrm{span}\{T, \nu_2\}$, hence $\xi$ is tangent to $\mathbb{S}^3(1)$ along $\gamma$. Yet, it can be proved that $\xi$ is tangent to $\mathbb{S}^3(1)$ at any point. Consequently, the Sasakian structure on $\mathbb{S}^3(1)$ is that induced from the Sasakian structure on $\mathbb{S}^{2n+1}(1)$. See also [27]. □

**Remark 3.2.** The study of the normal magnetic curves corresponding to the contact magnetic field $F_q$ in $\mathbb{S}^{2n+1}(1)$ reduces to their study in $\mathbb{S}^3(1)$.

From Theorem 3.3 and [15, Corollary 6.1] we conclude:

**Theorem 3.4.** *The normal magnetic curves on $\mathbb{S}^{2n+1}(1)$, corresponding to the contact magnetic field $F_q$, are the geodesics of the Hopf tori in $\mathbb{S}^3(1)$, constructed over geodesic circles in $\mathbb{S}^2(\frac{1}{2})$, where $\mathbb{S}^3(1)$ is a Sasakian totally geodesic sphere embedded in $\mathbb{S}^{2n+1}(1)$.*

### 3.2. *Magnetic curves in* $\mathbb{R}^{2n+1}(-3)$

In this subsection, we explicitly find the magnetic curves in the Sasakian space form $\mathbb{R}^{2n+1}(-3)$ generated by the contact magnetic field $F_q = q\Omega$ defined by (3.1).

Let us recall the Sasakian structure on $\mathbb{R}^{2n+1}$. Denote by $(x^i, y^i, z)$, where $i = 1, \ldots, n$, the canonical coordinates on $\mathbb{R}^{2n+1}$. The contact structure is determined by the 1-form $\eta = dz - 2\sum_{i=1}^{n} y^i dx^i$. The contact distribution is spanned by $X_i = \dfrac{\partial}{\partial y^i}$ and $Y_i = \dfrac{\partial}{\partial x^i} + 2y^i \dfrac{\partial}{\partial z}$, while the characteristic vector field is $\xi = \dfrac{\partial}{\partial z}$. The Riemannian metric $g = \eta \otimes \eta + \sum_{i=1}^{n} \left((dx^i)^2 + (dy^i)^2\right)$ gives a contact metric structure on $\mathbb{R}^{2n+1}$, meaning that $d\eta = \Omega$, where the fundamental 2-form $\Omega$ is defined by (2.1). The resulting contact metric manifold is a Sasakian space form of constant $\varphi$-sectional curvature $-3$ and denoted by $\mathbb{R}^{2n+1}(-3)$.

The vector fields $X_i$, $Y_i$, where $i = 1, \ldots, n$, and $\xi$ form a $\varphi$-basis, i.e.,

$$\varphi X_i = -Y_i, \ \varphi Y_i = X_i, \ \varphi \xi = 0.$$

Moreover, the Lie brackets are given by

$$[X_i, Y_i] = 2\xi, \ \ [X_i, \xi] = 0, \ \ [Y_i, \xi] = 0.$$

With these hypotheses, we may formulate the following two results for later use.

**Lemma 3.1.** *Let $c_1^i, c_2^i$, where $i = 1, \ldots, n$, and $\theta \neq 0, \pi$ be real constants such that*

$$\sum_{i=1}^{n} \left((c_1^i)^2 + (c_2^i)^2\right) = \sin^2\theta.$$

*We define two orthonormal vector fields in* $\operatorname{Ker}\eta$ *by setting*

$$\mu_1 = \frac{1}{\sin\theta}\left(c_1^i X_i + c_2^i Y_i\right), \ \ \mu_2 = \varphi\mu_1 = \frac{1}{\sin\theta}\left(c_2^i X_i - c_1^i Y_i\right).$$

*Then,* $W = \operatorname{span}\{\mu_1, \mu_2, \xi\}$ *is an integrable distribution in* $\mathbb{R}^{2n+1}$.

Since $W$ is integrable, there exists $M_0^3$ an integral submanifold of $W$ in $\mathbb{R}^{2n+1}(-3)$. We can state the following result.

**Proposition 3.5.** *The submanifold $M_0^3$ is totally geodesic in $\mathbb{R}^{2n+1}(-3) = (\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ and it carries a Sasakian structure induced from $\mathbb{R}^{2n+1}(-3)$.*

Let now $\gamma : I \subseteq \mathbb{R} \to (\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ be a magnetic curve associated to the contact magnetic field $F_q$, that is the Lorentz equation $\nabla_{\dot\gamma}\dot\gamma = q\varphi\dot\gamma$ is satisfied. We formulate first a classification result consisting in the explicit parametrizations of such magnetic curves.

**Theorem 3.5.** *Let $\gamma : I \subseteq \mathbb{R} \to (\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ be a smooth curve parametrized by arc length s, and let $F_q$ be the contact magnetic field. Then $\gamma$ is a normal magnetic curve in $\left(\mathbb{R}^{2n+1}(-3), g, F_q\right)$ if and only if:*

*a) $\gamma$ is a helix with axis $\xi$ parametrized as*

$$\gamma(s)=\Big(x_0+\tfrac{1}{\lambda}(c_2\sin(\lambda s)+c_1\cos(\lambda s)),\ y_0+\tfrac{1}{\lambda}(c_1\sin(\lambda s)-c_2\cos(\lambda s)),\\ z_0+\big(\cos\theta-\tfrac{1}{\lambda}\sin^2\theta\big)s+\tfrac{2}{\lambda}\langle y_0,c_2\rangle\sin(\lambda s)+\tfrac{2}{\lambda}\langle y_0,c_1\rangle\cos(\lambda s)\\ +\tfrac{1}{2\lambda^2}(|c_1|^2-|c_2|^2)\sin(2\lambda s)-\tfrac{1}{\lambda^2}\langle c_1,c_2\rangle\cos(2\lambda s)\Big),$$

*where $\lambda=q-2\cos\theta\neq 0$, $\theta$ denotes the constant contact angle, and $c_1,c_2\in\mathbb{R}^n$ satisfy $|c_1|^2+|c_2|^2=\sin^2\theta$;*

*b) $\gamma$ is parametrized as*

$$\gamma(s)=\Big(x_0+\beta_0 s,\ y_0+\alpha_0 s,\ z_0+(2\langle\beta_0,y_0\rangle+\cos\theta)s+\langle\alpha_0,\beta_0\rangle s^2\Big),$$

*where $\alpha_0,\beta_0\in\mathbb{R}^n$ satisfy $|\alpha_0|^2+|\beta_0|^2=\sin^2\theta$.*

*Here $x_0,y_0,z_0\in\mathbb{R}^n$ are integration constants and $\langle\cdot,\cdot\rangle$ denotes the inner product induced from the metric $g$.*

**Proof.** Let us begin with parametrizing the curve $\gamma$ by

$$\gamma(s)=(x^i(s),y^i(s),z(s)),\ \text{where } i=1,\dots,n. \tag{3.15}$$

Then the velocity $\dot\gamma$ may be decomposed in the $\varphi$-basis $\{X_i,Y_i,\xi\}$ as

$$\dot\gamma(s)=\alpha^i(s)X_i+\beta^i(s)Y_i+c(s)\xi,\ \text{where } i=1,\dots,n, \tag{3.16}$$

and where $\alpha^i(s)$, $\beta^i(s)$ and $c(s)$ are smooth functions to be determined. The Lorentz equation (3.3) is equivalent to the following ordinary differential equations (ODE) system:

$$\begin{cases}\dot\alpha^i(s)-(q-2c(s))\beta^i(s)=0,\\ \dot\beta^i(s)+(q-2c(s))\alpha^i(s)=0,\\ \dot c(s)=0.\end{cases} \tag{3.17}$$

From the third equation of (3.17) it follows that there exists a constant contact angle $\theta$ such that

$$c(s)=\cos\theta. \tag{3.18}$$

Since $\gamma$ is parametrized by arc length, we have $\sum_{i=1}^{n}\big(\alpha^i(s)^2+\beta^i(s)^2\big)=\sin^2\theta$. Combining the first two equations of (3.17) and using (3.18), we get the following ODE

$$\ddot\alpha^i(s)+(q-2\cos\theta)^2\alpha^i(s)=0.$$

Denoting $\lambda=q-2\cos\theta$, we obtain two sets of solutions for the previous ODE, corresponding to the cases where $\lambda\neq 0$ and $\lambda=0$, respectively.

First, when $\lambda \neq 0$ we get

$$\alpha^i(s) = c_1^i \cos(\lambda s) + c_2^i \sin(\lambda s), \quad \beta^i(s) = c_2^i \cos(\lambda s) - c_1^i \sin(\lambda s), \tag{3.19}$$

where $c_1^i, c_2^i$ are real constants such that $\sum_{i=1}^{n}((c_1^i)^2 + (c_2^i)^2) = \sin^2\theta$. Computing $\dot{\gamma}$ in (3.15) and combining it with (3.16) and (3.18), we see that the coordinate functions of $\gamma$ satisfy:

$$\dot{x}^i(s) = \beta^i(s), \quad \dot{y}^i(s) = \alpha^i(s), \quad \dot{z}(s) = 2\sum_{i=1}^{n}\beta^i(s)y^i(s) + \cos\theta.$$

Furthermore, denoting $c_1 = (c_1^i)_{i=1,\dots,n}$ and $c_2 = (c_2^i)_{i=1,\dots,n}$, we see that $\gamma$ is a helix with axis $\xi$ parametrized as in item a) of Theorem 3.5.

Second, if $\lambda = 0$, then

$$\alpha^i(s) = \alpha_0^i, \quad \beta^i(s) = \beta_0^i, \tag{3.20}$$

where $\alpha_0^i, \beta_0^i$ are real constants such that $\sum_{i=1}^{n}((\alpha_0^i)^2 + (\beta_0^i)^2) = \sin^2\theta$. Following a reasoning similar to the above, we obtain the parametrization of $\gamma$ from item b) of the theorem.

The converse follows by straightforward computations. □

Next, let us get more information about the geometry of these normal magnetic curves. Recall that the mapping

$$\pi : \mathbb{R}^{2n+1}(-3) \to \mathbb{E}^{2n}, (x^i, y^i, z) \mapsto (x^i, y^i) \text{ for } i = 1, \dots, n,$$

is a Riemannian submersion called the *Hopf fibration*. The inverse image $\pi^{-1}(\bar{\gamma})$ of the curve $\bar{\gamma}$ in $\mathbb{E}^{2n}$ is called a *Hopf cylinder*.

Notice that $\xi$ is a vertical vector field, i.e., $\pi_*\xi = 0$, and $X_i$, $Y_i$ are the horizontal lifts of $\frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial x^i}$, respectively; $\pi_* X_i = \frac{\partial}{\partial y^i}$, $\pi_* Y_i = \frac{\partial}{\partial x^i}$ for $i = 1, \dots, n$.

At this point, a second classification result may be formulated as follows.

**Theorem 3.6.** *The normal magnetic curves in $\mathbb{R}^{2n+1}(-3)$ generated by the contact magnetic field $F_q$ are given by*

- *a) helices on the constant mean curvature Hopf cylinders of radius $\frac{\sin\theta}{|q-2\cos\theta|}$, with $q \neq 2\cos\theta$, in the Heisenberg group* Heis$_3$ *which is totally geodesically embedded in $\mathbb{R}^{2n+1}$, carrying the induced Sasakian structure of $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$.*
- *b) geodesics in the minimal "planes" $(\bar{\gamma}) \times \mathbb{R}$, where $\bar{\gamma}$ is a straight line in $\mathbb{E}^2$.*

**Proof.** Let $\gamma(s)$ be a normal magnetic curve in $\mathbb{R}^{2n+1}(-3)$ corresponding to the contact magnetic field $F_q$, such that $\gamma(s) = \pi^{-1}(\bar{\gamma}(s))$, where $\bar{\gamma} : [0, \pi] \to \mathbb{E}^{2n}$.

Equation (3.16) may be written in one of the following two forms

- $\dot{\gamma}(s) = \sin\theta\cos(\lambda s)\mu_1 + \sin\theta\sin(\lambda s)\mu_2 + \cos\theta\,\xi$, when (3.19) is used,
- $\dot{\gamma}(s) = \sin\theta\,\mu_1 + \cos\theta\,\xi$, when (3.20) is used.

Here $\lambda = q - 2\cos\theta$, while $\mu_1$, $\mu_2$ are defined as in Lemma 3.1.

Applying Proposition 3.5 we see that $\gamma$ lies in a certain 3-dimensional manifold $M_0^3$ totally geodesically embedded in $\mathbb{R}^{2n+1}(-3)$, as follows.

First, from item a) of Theorem 3.5 we see that

$$\bar{\gamma}(s) = (x(s), y(s)) = (x_0, y_0) + \sin\theta \left( \frac{1}{\lambda} \sin(\lambda s) V_1 - \frac{1}{\lambda} \cos(\lambda s) V_2 \right),$$

where $V_1 = \frac{1}{\sin\theta}(c_2, c_1)$ and $V_2 = \frac{1}{\sin\theta}(-c_1, c_2)$ are orthonormal vector fields. Thus $\bar{\gamma}$ is a circle of radius $\frac{\sin\theta}{|\lambda|}$ for $\theta \neq 0, \pi$, and hence $\gamma$ is a helix on the constant mean curvature Hopf cylinder of the same radius in $M_0^3 = \mathrm{Heis}_3$. Item a) of the theorem is proved.

Second, from item b) of Theorem 3.5, one has

$$\bar{\gamma}(s) = (x(s), y(s)) = (x_0, y_0) + (\beta_0, \alpha_0)s,$$

which is a straight line in $\mathbb{E}^2$. Furthermore, $\gamma$ is a geodesic in the minimal plane $(\bar{\gamma}) \times \mathbb{R}$ implicitly given by the equation $\beta_0 y - \alpha_0 x = \beta_0 y_0 - \alpha_0 x_0$. This proves item b) of the theorem. □

### 3.3. *Magnetic curves in* $B^{2n} \times \mathbb{R}$

Consider the complex unit ball $B^{2n} = \{z \in \mathbb{C}^n : |z|^2 < 1\}$ endowed with the Bergman metric $G_0$ of constant holomorphic sectional curvature $k = -1$. See, e.g., [12]. One can rescale this metric to $G$, in order to have constant holomorphic sectional curvature $k < 0$. Denote by $\Omega_B$ the corresponding Kähler form. Since $B^{2n}$ is simply connected, there exists a real analytic 1-form $\omega$ such that $\Omega_B = d\omega$.

Let $\pi : B^{2n} \times \mathbb{R} \to B^{2n}$ be the canonical projection and denote the coordinate on $\mathbb{R}$ by $t$. We construct an almost contact structure on $M^{2n+1} = B^{2n} \times \mathbb{R}$ by setting the 1-form $\eta = \pi^*\omega + dt$, the characteristic vector field $\xi = \frac{\partial}{\partial t}$ and let the $(1,1)$-tensor field $\varphi$ be the horizontal lift of the (almost) complex structure $J$ of $B^{2n}$ and zero in the vertical direction. Define the metric $g = \pi^*G + \eta \otimes \eta$. See, e.g., [11]. We have $\Omega = d\eta = \pi^*\Omega_B$, where $\Omega(\cdot,\cdot) = g(\varphi\ \cdot,\cdot)$ is the fundamental 2-form on $M^{2n+1}$ and $\Omega_B(\cdot,\cdot) = G(J\cdot,\cdot)$ is the Kähler 2-form on $B^{2n}$. Then $(M^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasakian manifold of constant $\varphi$-sectional curvature $c = k - 3$.

For $X \in T_z B^{2n}$, denote by $X^\uparrow$ its horizontal lift to $B^{2n} \times \mathbb{R}$, namely, $\pi_{*,(z,t)} X^\uparrow_{(z,t)} = X_z$ for all $z \in B^{2n}$ and $t \in \mathbb{R}$. Furthermore, we have $\eta(X^\uparrow) = 0$.

Note that the Lie brackets have the expressions

$$[X^\uparrow, Y^\uparrow] = [X,Y]^\uparrow - 2(\Omega_B(X,Y) \circ \pi)\xi, \quad [X^\uparrow, \xi] = 0,$$

and the Levi-Civita connection $\nabla$ on $B^{2n} \times \mathbb{R}$ is given by

$$\nabla_{X^\uparrow} Y^\uparrow = (\bar{\nabla}_X Y)^\uparrow - (\Omega_B(X,Y) \circ \pi)\xi, \quad \nabla_{X^\uparrow}\xi = \varphi X^\uparrow = (JX)^\uparrow,$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on $B^{2n}$.

We prove the following classification result.

**Theorem 3.7.** *Let $\gamma : I \subseteq \mathbb{R} \to B^{2n} \times \mathbb{R}$, be a smooth curve parametrized by arc length $s$ and let $F_q = q\Omega$ for $q \neq 0$ be the contact magnetic field. Then $\gamma$ is a normal magnetic curve associated to $F_q$ if and only if it belongs to the following list:*

*a) a geodesic obtained as an integral curve of $\xi$;*
*b) the horizontal lift of a magnetic trajectory in $\mathbb{H}^2(k)$ corresponding to the Kähler magnetic field $F = q\Omega_B$;*

*c) a helix in the 3-dimensional Sasakian space form $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ identified with $B^2 \times \mathbb{R}$ as totally geodesic submanifold in $B^{2n} \times \mathbb{R}$. Moreover, $\gamma$ is a geodesic on a vertical cylinder over a curve of constant curvature in $\mathbb{H}^2(k)$.*

*Here $\mathbb{H}^2(k)$ denotes the hyperbolic plane of constant curvature k.*

**Proof.** Item a) may be proved using a reasoning similar to that used for item a) of Theorem 3.1.

From now on we suppose that $\gamma$ is a non-geodesic smooth curve on $B^{2n} \times \mathbb{R}$, and hence $\theta \in (0,\pi)$. We have the following decomposition of $\dot\gamma$:

$$\dot\gamma = \dot{\bar\gamma}^{\uparrow} + \cos\theta\,\xi, \tag{3.21}$$

where $\bar\gamma$ denotes the projection of $\gamma$ on $B^{2n}$, namely $\bar\gamma = \pi\circ\gamma$. Using the previous expression, we compute

$$\nabla_{\dot\gamma}\dot\gamma = \left(\bar\nabla_{\dot{\bar\gamma}}\dot{\bar\gamma} + 2\cos\theta J\dot{\bar\gamma}\right)^{\uparrow}.$$

The Lorentz equation becomes

$$\left(\bar\nabla_{\dot{\bar\gamma}}\dot{\bar\gamma} + 2\cos\theta J\dot{\bar\gamma}\right)^{\uparrow} = \left(qJ\dot{\bar\gamma}\right)^{\uparrow}.$$

It follows that

$$\bar\nabla_{\dot{\bar\gamma}}\dot{\bar\gamma} = (q - 2\cos\theta)J\dot{\bar\gamma}. \tag{3.22}$$

Since $s$ is the arc length parameter of $\gamma$, we get $G(\dot{\bar\gamma},\dot{\bar\gamma}) = \sin^2\theta$. If $\theta \neq 0,\pi$, then $\bar s = s\sin\theta$ is the arc length parameter for $\bar\gamma$. Denoting by $'$ the derivative with respect to $\bar s$, one writes

$$\bar\nabla_{\bar\gamma'}\bar\gamma' = \frac{q-2\cos\theta}{\sin\theta}J\bar\gamma'.$$

Therefore, $\bar\gamma$ is a normal magnetic curve on $B^{2n}$ associated to the Kähler magnetic field $F_{\bar q} = \bar q\Omega_B$, where $\bar q = \frac{q-2\cos\theta}{\sin\theta}$. We can apply now [25, Theorem 4] to see that $\bar\gamma$ belongs to a totally geodesic complex submanifold of $B^{2n}$ of complex dimension 1. Hence, $\bar\gamma$ lies in a certain hyperbolic plane $\mathbb{C}H^1(k) \equiv \mathbb{H}^2(k)$ of constant negative curvature $k$. See also [3,7,8].

If $\theta = \frac{\pi}{2}$, then from (3.21) it follows that $\gamma$ is the horizontal lift of $\bar\gamma$, and (3.22) yields that $\bar\gamma$ is a magnetic trajectory in $\mathbb{H}^2(k)$ corresponding to the Kähler magnetic field $F = q\Omega_B$. Hence, item b) of the theorem is proved.

The geometry of magnetic flows on a Riemannian surface, in particular on the hyperbolic plane, is known from [30]. The behaviour of a magnetic trajectory in $\mathbb{H}^2$ depends on the strength, namely, when $|\bar q| > \sqrt{-k}$ it is closed, while for $|\bar q| \leq \sqrt{-k}$ it is open and unbounded, see [3].

In order to prove statement c), when the contact angle $\theta \neq \frac{\pi}{2}$, observe that $\gamma$ is a helix in a 3-dimensional Sasakian space form $B^2 \times \mathbb{R}$. More precisely, $\gamma$ is a curve on the cylinder $\pi^{-1}(\bar\gamma) \subset B^2 \times \mathbb{R}$. Since $\nabla_{\dot\gamma}\dot\gamma$ is orthogonal to both $\xi$ and $\dot{\bar\gamma}^{\uparrow}$ which generate the tangent plane of $\pi^{-1}(\bar\gamma)$, we conclude that $\gamma$ is a geodesic on the vertical cylinder over $\bar\gamma$.

Moreover, it is known that the universal covering of the Lie group $\mathrm{PSL}(2,\mathbb{R})$, endowed with the standard metric, can be identified with the universal covering of the unit tangent bundle $T_1\mathbb{H}^2(k)$ endowed with the Sasaki metric. This identification can be done since $\mathrm{PSL}(2,\mathbb{R})$ acts transitively on

$T_1\mathbb{H}^2(k)$ and the stabilizer of any element of $T_1\mathbb{H}^2(k)$ is trivial. Moreover, $B^2\left(\frac{2}{\sqrt{-k}}\right)\times\mathbb{R}$ endowed with the structure described above can be considered as model for $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. See, e.g., [19]. □

**Final remarks.** In a different approach, the Sasakian space forms may be realized as particular homogeneous real hypersurfaces in non-flat complex space forms $\mathbb{C}M^n(c)$. Adachi and Bao [6] showed that the fundamental 2-form of an orientable real hypersurface in a Kähler manifold is closed. They called a magnetic field on real hypersurfaces given by constant multiple of the fundamental 2-form a *Sasakian magnetic field* and they studied the magnetic trajectories on real hypersurfaces of type $A$ in $\mathbb{C}M^n(c)$ in [6], and of type $B$ in the hyperbolic complex space $\mathbb{C}H^n(c)$ in [7,8]. Observe that in general the real hypersurfaces in Kähler manifolds are neither contact, nor normal. More precisely, Adachi, Kameda and Maeda showed that the homogeneous real hypersurfaces in $\mathbb{C}M^n(c)$ of type $A_0$ and $A_1$ (for standard names of these types, see [28]) are Sasakian space forms of constant $\varphi$-sectional curvature $c+1$ [4, Corollary 1], while type $B$ consists of non-Sasakian contact metric hypersurfaces [4, Lemma 4, Proposition 3]. Furthermore, type $A_2$ hypersurfaces are the only quasi-Sasakian hypersurfaces that are non-homothetic to Sasakian manifolds (see [17]). Finally, notice that the classification of trajectories on types $A_0$ and $A_1$ hypersurfaces (which are Sasakian space forms) may be also retrieved as a particular case in our classification Theorem 3.1 in Sasakian manifolds.

We conclude this section recalling a result of Ikawa [24] on Sasakian-Kähler submersions. Let $\pi : M \to \bar{M}$ be a Sasakian-Kähler submersion, where $M$ is a $(2n+1)$-dimensional Sasakian manifold and $\bar{M}$ is a $(2n)$-dimensional Kähler manifold, and let $\gamma$ be a normal magnetic curve in $M$ corresponding to the contact magnetic field $F_q$. According to [24, Theorem 2.1], $\bar{\gamma}=\pi(\gamma)$ is also a normal magnetic curve in $\bar{M}$ corresponding to the Kähler magetic field $F_{\bar{q}}$, $\bar{q}=\frac{q-2\cos\theta}{\sin\theta}$. Hence, our results from Theorems 3.6 and 3.7 are consistent with [24].

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