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Viewpoints

A novel method for the network reliability in terms of capacitated-minimum-paths without knowing minimum-paths in advance (Yeh, 2005)

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In a recent article, Yeh presents a method for computing the multi-state two-terminal reliability in terms of capacitated-minimum paths (CMPs). It is claimed that the proposed algorithm has the following advantages: (1) it is just based on the special property of CMPs of which the max-flow in these paths are all equal to a given capacity (say $\boldsymbol{d}$) and can be used to search for all capacitated minimum-paths without knowing all minimum-paths in advance; (2) it is simple and more effective in finding CMP candidates than the existing methods and (3) the proposed method is easier to understand and implement. Unfortunately, the proposed algorithm is incorrect as can be shown by the following counterexample.

Referring to Figure 1 and Table 1, we first note that since capacities of all arcs are 0, 3 and 5, any system-state vector that can (resp. cannot) send 1 or 2 unit of demand from s to t is able (respectively unable) to send 3 units of demand from s to t. Thus the sets of all 1-mps, 2-mps and 3-mps are equivalent. Similarly, the set of all 4-mps equals the set of all 5-mps.

Case 1: $\boldsymbol{d} = 1$, 2, or 4.
Since no summation of arc capacities equals $\boldsymbol{d}$ (Equation (8) in Theorem 2), Yeh's algorithm returns an empty set of $\boldsymbol{d}$-mp when $\boldsymbol{d}$ is 1, 2 or 4.

Case 2: $\boldsymbol{d} = 5$
To derive all 5-mps, a summation of arc capacities being equal to 5 (Equation (8) in Theorem 2) restricts arcs to take on capacity values 0 or 5. It is incorrect, since combination of two disjoint 3-mps from s to t (for example, system-state vector $(x_{s1}, x_{1t}, x_{12}, x_{s2}, x_{2t}) = (3, 3, 0, 3, 3)$) is also a 5-mp but is not searched by Yeh.

Consequently, Yeh's algorithm is unavailable for searching all $\boldsymbol{d}$-mps when $\boldsymbol{d}$ has values 1, 2 and 4, and is incorrect when $\boldsymbol{d} = 5$.

The mistake is due to the statement of advantage 1 'it is just based on the special property of CMPs of which the max-flow in these paths are all equal to a given capacity (say $\boldsymbol{d}$) and can be used to search for all capacitated minimum-paths without knowing all minimum-paths in advance' which is the fundamental of Theorem 2. In case 1 ($\boldsymbol{d} = 1$, 2 or 4) of the counterexample, it yields an empty set of 1-mp, 2-mp or 4-mp. In case 2 ($\boldsymbol{d} = 5$) of the counterexample, it yields an incorrect set of all 5-mps. It is also note here that the system reliability for level 3 in Example 2 is 0.611415, value 0.64515 in the article is not correct.

Since Yeh's algorithm is incorrect, there is no existing algorithm that can search for all $\boldsymbol{d}$-mps without knowing all binary-state minimal paths in advance. As a result, the NP-hard multi-state two-terminal reliability will be solved in terms of three NP-hard problems, say searching all binary-state minimal paths, searching all multi-state minimal paths ($\boldsymbol{d}$-mps), and applying the inclusion-exclusion method to compute the reliability value. The effort of finding an efficient algorithm that can search all $\boldsymbol{d}$-mps without locating all binary-state minimal paths in advance is worthwhile.

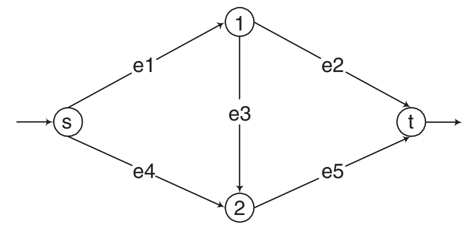

Figure 1 A limited-flow network.

Table 1 Capacities and capacity probabilities of arcs of Figure 1

Arc	*Capacities*			*Probabilities*		
e_1	0	3	5	0.05	0.10	0.85
e_2	0	3	5	0.05	0.10	0.85
e_3	0	3	5	0.05	0.10	0.85
e_4	0	3	5	0.05	0.10	0.85
e_5	0	3	5	0.05	0.10	0.85

References

Yeh W-C (2005). A novel method for the network reliability in terms of capacitated-minimum-paths without knowing minimum-paths in advance. *J Opl Res Soc* **56**: 1235–1240.

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Reply to Jane

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My method proposed in the published paper entitled 'A Novel Method for the Network Reliability in Terms of Capacitated-Minimum-Paths without Knowing Minimum-Paths in Advance' (*J Opl Res Soc*, 2005, Vol. 56, pp 1235–1240) can be implemented to search for all CMPs in the network of which the state (ie the capacity level) of each arc, for example arc e, is $0, 1, 2, \ldots, W(e)$.

If the arc state is defined as in the special cases discussed in Jane's comment, two modifications are needed for the proposed method to find all CMPs as follows:

1. The state variable x_i (in the definition of X of Section 2 and in Equation (9) of Theorem 2) needs to change from the original definition '$x_i = 0, 1, \ldots, W(e_i)$' to the new definition '$x_i \in \{c_{ik}|$ where c_{ik} is the kth state of $e_i \in E\}$'.
2. Change Step 1 in the proposed algorithm of Section 4: *Step* 1 (Implement Theorem 2): Construct and use the implicit algorithm to solve all of the feasible solutions (CMP candidates), say $p_1, p_2, \ldots, p_\pi$, to Equations (7)–(9). If there is any feasible solution, go to Step 2. If no feasible solution exists and d is less than the max-flow in the network, let $d = d + 1$ and go to Step 1. Otherwise, halt.

After the above two simple modifications, all CMPs can be found using my method without needing any MPs in advance.

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Comment on Berman and Huang (2004): Minisum collection depots location problem reduces to the *p*-median problem

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Berman and Huang (2004) introduce the minisum collection depots location problem with multiple facilities on a network. In this note, we show that the problem is equivalent to the p-median problem. Let $G = (N, L)$ be a graph where the node set N denotes n customers and the edge set L represents the links among them. $X = \{x_1, x_2, \ldots, x_m\}$ is the set of m collection depots on G, not necessarily on the nodes only. ω_i is the weight of customer i and $d(u, v)$ is the shortest distance between any points u and v on G. Given the locations of customers and collection depots, the problem is to locate p facilities to minimize the total (weighted) routing distances, where each route is devoted to a single customer and involves three stops for servicing. A route originates from a starting facility, moves to a customer for picking up the material at its location, delivers the material to a collection depot, and returns to the starting facility.

Berman and Huang prove that an optimal set of facility locations of the problem belongs to an extended node set $N' = N \cup X$. They propose the following mixed integer programming model to formulate the problem. Let

$$z_{ijk} = \begin{cases} 1 & \text{if node } i \text{ is assigned to the facility at } j \text{ and} \\ & \quad \text{depot } k \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi_j = \begin{cases} 1 & \text{if there is a facility located at } j \in N' \\ 0 & \text{otherwise} \end{cases}$$

Model P

$$min \sum_{i \in N} \sum_{j \in N'} \sum_{k \in X} \omega_i [d(j, i) + d(i, k) + d(k, j)] z_{ijk}$$

st

$$\sum_{j \in N'} \pi_j = p \tag{1}$$

$$\sum_{j \in N'} \sum_{k \in X} z_{ijk} = 1, \quad i \in N \tag{2}$$

$$z_{ijk} \leqslant \pi_j, \quad i \in N, \; j \in N', \; k \in X \tag{3}$$

$$z_{ijk}, \pi_j \in \{0, 1\}, \quad i \in N, \; j \in N', \; k \in X \tag{4}$$

Model P locates p facilities and assigns each customer i to a pair of facility j and depot k. Note that, however, if customer i is to be assigned to facility j, the best collection depot assignment k^* for the (i, j) pair can be selected in advance by considering the shortest links among them, that is, $k^* = \text{argmin}\{d(i, k) + d(k, j)\}$. Let $t(i, j)$ be the length of single customer route if customer i is assigned to facility j so that $t(i, j) = d(j, i) + \min_{k \in X}\{d(i, k) + d(k, j)\}$. Given all $t(i, j)$ values, the problem can be remodelled as follows:

$$q_{ij} = \begin{cases} 1 & \text{if node } i \text{ is assigned to the facility at } j \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi_j = \begin{cases} 1 & \text{if there is a facility located at } j \in N' \\ 0 & \text{otherwise} \end{cases}$$

Model P-median

$$min \sum_{i \in N} \sum_{j \in N'} \omega_i t(i, j) q_{ij}$$

st

$$\sum_{j \in N'} \pi_j = p \tag{1$'$}$$

$$\sum_{j \in N'} q_{ij} = 1, \quad i \in N \tag{2$'$}$$

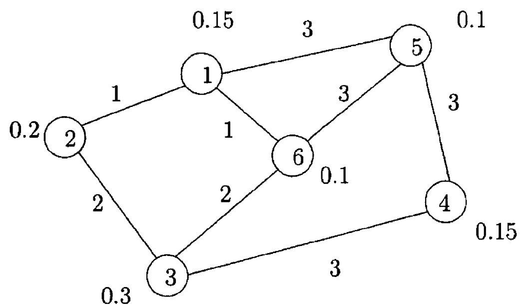

Figure 1 Network of the second example in Berman and Huang (2004).

Table 1 The values of Lagrangean multipliers, upper bounds and lower bounds

	Iterations									
	0	1	2	3	4	5	...	21	22	23
λ_1	0	0.300	0.900	0.300	1.020	0.060		0.331	0.683	0.683
λ_2	0	0.400	1.000	0.400	1.120	0.160		0.331	0.683	0.683
λ_3	0	1.200	1.200	1.800	1.800	1.800	...	1.531	1.531	1.531
λ_4	0	0.900	0.900	1.500	0.780	1.740		1.583	1.231	1.231
λ_5	0	0.600	1.200	0.600	1.320	0.360		0.331	0.683	0.683
λ_6	0	0.200	0.800	0.200	0.920	0.000		0.331	0.683	0.400
UB	$+\infty$	2.7	2.7	2.7	2.7	2.7	...	2.4	2.1	2.1
LB	0	1.5	0.9	0.9	0.3	−0.26	...	0.64	1.82	2.1

$$q_{ij} \leqslant \pi_j, \quad i \in N, \quad j \in N' \qquad (3')$$

$$q_{ij}, \pi_j \in \{0, 1\}, \quad i \in N, \quad j \in N' \qquad (4')$$

The *Model P-median* is the well-known formulation of the p-median problem and represents the minisum collection depots location problem with p facilities on an extended network. While there are $O(n^2m+m^2n)$ variables and constraints in the *Model P*, the *Model P-median* has $O(n^2 + mn)$ variables and constraints.

We note that *Model P-median* is stronger than *Model P* in terms of the solution quality of the linear programming relaxations of the two models because our variable redefinition ensures that each customer and facility pair uses a single collection depot. Consider the example of Berman and Huang given in Figure 1 to illustrate this property. There are six customers and four collection depots located at nodes 1, 2, 3 and 4. The problem is to locate two facilities on the network. The length of links and weight of customers are given next to links and nodes in the figure.

The optimal solution to the example problem is to locate two facilities at nodes 1 and 3 with a solution value of 2.1. All customers are assigned to the facility at 1 except that the customers 3 and 4 are assigned to the facility at 3. The optimal linear relaxation solution value is 1.9 for *Model P*, where $\pi_1 = \pi_4 = \frac{1}{2}$, $\pi_3 = 1$, $z_{111} = z_{112} = \frac{1}{2}$, $z_{211} = z_{212} = \frac{1}{2}$, $z_{333} = 1$, $z_{433} = z_{444} = \frac{1}{2}$, $z_{511} = z_{544} = \frac{1}{2}$, and $z_{611} = z_{612} = \frac{1}{2}$. In this solution there are several fractional valued variables, for example, the customers 1, 2 and 6 are assigned to the facility at 1, but use the two collection depots at 1 and 2. The linear relaxation of the *Model P-median*, however, yields the optimal (integer valued) solution of the original problem.

Berman and Huang suggest a Lagrangean relaxation-based branch and bound algorithm to solve the minisum collection depots location problem, using *Model P*. They illustrate that they need to apply the branch and bound algorithm to find an optimal solution to the above example problem. Similar to their approach (which they call the Lagrangean dual solution procedure as *procedure-RELAX*), we solve the example problem by relaxing constraint 2′ of the *Model P-median* in the Lagrangean manner. We use the same conditions for the Lagrangean dual solution procedure and set the initial upper bound as $+\infty$. It takes only 23 iterations of *procedure-RELAX* to prove the optimality of the solution to the example problem and no branching is needed. The values of Lagrangean multipliers (λ_i's), upper bounds (UB) and lower bounds (LB) in the selected sample iterations are given in Table 1.

This suggests that the *Model P-median* also yields strong Lagrangean bounds than the Berman and Huang's model to solve the minisum collection depots location problem.

References

Berman O and Huang R (2004). Minisum collection depots location problem with multiple facilities on a network. *J Opl Res Soc* **55**: 769–779.

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Reply to Özpeynirci and Süral

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I read the comment and concur with the authors that their model p-median is stronger than model P as it results in $(m-1)(n^2+mn)$ less variables and constraints. Moreover, there has been a large body of research on the p-median problem. Therefore, I do not believe that the second part of the comment regarding the solution quality and the Lagrangean bounds is essential.

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