Journal of the Operational Research Society (2012) 63, 1018–1020 0160-5682/12

www.palgrave-journals.com/jors/

Corrigendum

Scheduling jobs with position-dependent processing times

A Janiak[1] and MY Kovalyov[2]

[1]Wroclaw University of Technology, Wroclaw, Poland; and [2]United Institute of Informatics Problems, National Academy of Sciences of Belarus, Minsk, Belarus

Bachman and Janiak provided a sketch of the proof that the problem $1|r_i,p_i(v)=a_i/v|C_{\max}$ is NP-hard in the strong sense. However, they did not show how to avoid using harmonic numbers whose encoding is not pseudo-polynomial, which makes the proof incomplete. In this corrigendum, we provide a new complete proof.

Journal of the Operational Research Society (2012) **63,** 1018–1020. doi:10.1057/jors.2012.18
Published online 25 April 2012

Keywords: sequencing; single machine; learning effect; computational complexity

Correction to: A Bachman and A Janiak (2004) Scheduling jobs with position-dependent processing times, *Journal of the Operational Research Society* **55**: 257–264. doi:10.1057/palgrave.jors.2601689

In the problem $1|r_i,p_i(v)=a_i/v|C_{\max}$, there are n jobs to be scheduled for processing on a single machine to minimize the makespan, $C_{\max}$. Each job i is associated with a given release date r_i before which its processing cannot start, and a position-dependent processing time $p_i(v)=a_i/v$, where a_i is a given coefficient and v is the position of job i in the processing sequence. This dependence of processing times is called *learning effect* in the scheduling literature.

Our note is concerned with the proof of NP-hardness in the strong sense of the above problem. Similarly to Bachman and Janiak (2004), it is based on a reduction from the NP-hard in the strong sense problem 3-PARTITION, which can be defined as follows.

3-PARTITION: Given $3m+1$ positive integer numbers $x_1,\ldots,x_{3m}$ and B such that $\sum_{j=1}^{3m}x_j=mB$ and $B/4<x_j<B/2$ for $j=1,\ldots,3m$, is there a partition of the set $\{1,\ldots,3m\}$ into m disjoint subsets X_l such that $\Sigma_{j\in X_l}x_j=B$ for $l=1,\ldots,m$?

Dr Radoslaw Rudek pointed out to us that a sketch of the proof given in Bachman and Janiak (2004) is incorrect due to the usage of *harmonic numbers* $H_l=\sum_{k=1}^{l}(1)/(k)$, $l=1,\ldots,m$, in the calculation of the job release dates. Due to this, the release dates cannot be expressed as irreducible fractions of the type A_l/B_l such that A_l and B_l are pseudo-polynomially bounded in the length of 3-PARTITION, which means that Bachman and Janiak (2004) proves only NP-hardness in the ordinary sense of the problem $1|r_i,p_i(v)=a_i/v|C_{\max}$. We appreciate Dr Rudek's note. Below we demonstrate how to avoid the trouble with the harmonic numbers.

Theorem 1 *The problem* $1|r_i,p_i(v)=a_i/v|C_{\max}$ *is NP-hard in the strong sense.*

Proof We use the following reduction from 3-PARTITION. There are $3m$ *partition jobs* and m groups of *enforcer jobs*. For a partition job i we have $r_i=0$ and $a_i=W^2x_i$, where W is a sufficiently large number to be determined later. All the enforcer jobs have $a_i=1$. There are $g_1=(W)/(m)$ enforcer jobs of Group 1 and $g_l=(W)/((m-l+1)(m-l+2))$ enforcer jobs of Group l, $2\leqslant l\leqslant m$, where $[\cdot]$ is the rounding up operator. Jobs of Group l have the same release date r_l. There is a threshold y on the $C_{\max}$ value. The values of the release dates and y will be determined later such that $r_1=0$ and the interval $[r_l,r_{l+1}]$ can accommodate all the enforcer jobs of Group l and some part of partition jobs for $l=1,\ldots,m$, where $r_{m+1}=y$. Note that the release date sequence is increasing.

Consider an arbitrary optimal schedule. Because of the learning effect, the total processing time of all jobs is minimized if the enforcer jobs (with small values a_i) are scheduled at the earliest possible times. As the enforcer jobs differ only by their release dates, they can always be interchanged to be sequenced in the non-decreasing order of their release dates. Therefore, assume without loss of generality that the groups of the enforcer jobs are scheduled in the order $1,2,\ldots,m$, and that jobs of Group l are scheduled contiguously and start no earlier than at the time r_l, $l=1,\ldots,m$. Jobs of Group 1 start at time $r_1=0$.

Denote by δ_l the total processing time of the enforcer jobs of Group l in the considered optimal schedule. Taking into account $g_l \leqslant W$ for $l=1,\ldots,m$, we obtain

$$\delta_l \leqslant H_W = \sum_{j=1}^{W} \frac{1}{j} < \ln W + \gamma + 1 < \ln W + 2, \quad l = 1,\ldots,m,$$

where H_W is a *harmonic number* and γ is the *Euler-Mascheroni* constant. In contrast to Bachman and Janiak (2004), we will use an integer evaluation of the harmonic numbers rather than their exact values. Below we will show that this evaluation is sufficient for the proof.

Denote by P_l the set of partition jobs scheduled between the enforcer jobs of Groups l and $l+1$. Denote by v_i the position of the partition job i. For $i \in P_l$ we have

$$v_i = \sum_{j=1}^{l} g_j + \xi_i,$$

where ξ_i is the number of partition jobs scheduled before job i including this job. By using the definitions of the g_j values and evaluating the rounding up operators, we obtain

$$v_i = \frac{W}{m} + \sum_{j=2}^{l} \frac{W}{(m-j+1)(m-j+2)} + \chi_i + \xi_i = \frac{W}{m-l+1} + \psi_i,$$

where $0 \leqslant \chi_i \leqslant l$ is the value coming from discarding the rounding up operators, $\psi_i = \chi_i + \xi_i$, and $\sum_{j=2}^{l}(W)/((m-j+1)(m-j+2)) := 0$ if $l=1$. It is easy to notice that $1 \leqslant \xi_i \leqslant \psi_i \leqslant 3m + l \leqslant 4m$, as $1 \leqslant \xi_i \leqslant 3m$.

Calculate the total processing time of the partition jobs of the set P_l and enforcer jobs of Group l, denoted as T_l.

$$\begin{aligned} T_l &= \sum_{i \in P_l} p_i(v_i) + \delta_l = \sum_{i \in P_l} \frac{a_i}{v_i} + \delta_l \\ &= \sum_{i \in P_l} \frac{W^2 x_i}{\frac{W}{m-l+1} + \psi_i} + \delta_l \\ &= W(m-l+1) \sum_{i \in P_l} x_i + \delta_l - (m-l+1)^2 \\ &\quad \times \sum_{i \in P_l} \frac{\psi_i x_i}{1 + \frac{\psi_i(m-l+1)}{W}}, \quad l = 1,\ldots,m. \end{aligned} \tag{1}$$

The transition between the two lines in Equation (1) is justified by the following chain of equalities.

$$\begin{aligned} &W(m-l+1) - \frac{(m-l+1)^2 \psi_i}{1 + \frac{\psi_i(m-l+1)}{W}} \\ &= \frac{W(m-l+1)[W + (m-l+1)\psi_i] - (m-l+1)^2 \psi_i W}{W + \psi_i(m-l+1)} \\ &= \frac{(m-l+1)W^2}{W + \psi_i(m-l+1)} = \frac{W^2}{\frac{W}{m-l+1} + \psi_i}. \end{aligned}$$

Observe that for any schedule feasible with respect to the release dates, $C_{\max} \leqslant y$ *if and only if*

$$\sum_{j=l}^{m} T_j \leqslant y - r_l, \quad l = m,\ m-1,\ldots,1. \tag{2}$$

We choose $r_1,\ldots,r_m$ and $y = r_{m+1}$ such that $r_1 = 0$ and $r_l - r_{l-1} = B + WB(m-l+2)$, $l = 2,\ldots,m+1$. Hence, $y = mB + WB(m+1)m/2$ and $y - r_l = (m-l+1)B + WB \sum_{j=1}^{m-l+1} j$, $l = 1,\ldots,m$.

By substituting Equation (1) into Equation (2), we obtain that for the considered optimal schedule $C_{\max} \leqslant y$ *if and only if*

$$\begin{aligned} \sum_{j=l}^{m} T_j &= W \sum_{j=l}^{m} \left((m-j+1) \sum_{i \in P_j} x_i \right) \\ &\quad + \sum_{j=l}^{m} \left(\delta_j - (m-j+1)^2 \sum_{i \in P_j} \frac{\psi_i x_i}{1 + \frac{\psi_i(m-j+1)}{W}} \right) \\ &\leqslant y - r_l = (m-l+1)B + WB \sum_{j=1}^{m-l+1} j, \\ &l = m,\ m-1,\ldots,1. \end{aligned} \tag{3}$$

Rearrange Equation (3) so that

$$\sum_{j=l}^{m} \left((m-j+1) \sum_{i \in P_j} x_i \right) \leqslant B \sum_{j=1}^{m-l+1} j + D_l, \quad l = m,\ m-1,\ldots,1, \tag{4}$$

where

$$D_l = \frac{(m-l+1)B + \sum_{j=l}^{m} \left((m-j+1)^2 \sum_{i \in P_j} \frac{\psi_i x_i}{1 + \frac{\psi_i(m-j+1)}{W}} - \delta_j \right)}{W}, \quad l = 1,\ldots,m.$$

We would like to have $0 \leqslant D_l < 1$, $l = 1,\ldots,m$. Consider the following chain of relations:

$$\begin{aligned} &(m-j+1)^2 \sum_{i \in P_j} \frac{\psi_i x_i}{1 + \frac{\psi_i(m-j+1)}{W}} \\ &\leqslant m^2 \sum_{i \in P_j} (\psi_i x_i) \leqslant 4m^3 \sum_{i \in P_j} x_i \leqslant 12 m^4 B. \end{aligned}$$

Then $D_l \leqslant ((m-l+1)B - \sum_{j=l}^{m} \delta_j + 12m^5 B)/(W)$. We set $W = 12m^5 B + mB + 1$ and assume without loss of generality that $\ln W + 2 \leqslant B$, which implies $\delta_j \leqslant B$ for all j. Here we use the integer evaluation of the harmonic numbers. Then $0 \leqslant (m-l+1)B - \sum_{j=l}^{m} \delta_j \leqslant mB$ and $0 \leqslant D_l < 1$, $l = 1, \ldots, m$.

Due to the integrality of the left-hand side and the first summand of the right-hand side in Equation (4), relations (4) are *equivalent* to

$$\sum_{j=1}^{m-l+1} j \sum_{i \in P_{m-j+1}} x_i \leqslant B \sum_{j=1}^{m-l+1} j, \quad l = m, m-1, \ldots, 1.$$

By making index substitution $h := m-l+1$, the latter relations can be expressed as:

$$\sum_{j=1}^{h} j \left(B - \sum_{i \in P_{m-j+1}} x_i \right) \geqslant 0, \quad h = 1, \ldots, m. \qquad (5)$$

Suppose that at least one of the inequalities in Equation (5) is strict. Then by forming a linear combination of the m inequalities with positive coefficients $1/h - 1/(h+1)$ for $h = 1, \ldots, m-1$, and $1/m$ for $h = m$, we obtain a strictly positive value

$$\sum_{h=1}^{m-1} \left(\frac{1}{h} - \frac{1}{h+1} \right) \sum_{j=1}^{h} j \left(B - \sum_{i \in P_{m-j+1}} x_i \right) + \frac{1}{m} \sum_{j=1}^{m} j \left(B - \sum_{i \in P_{m-j+1}} x_i \right) > 0.$$

The latter expression can be rearranged as follows:

$$\sum_{j=1}^{m} j \left(B - \sum_{i \in P_{m-j+1}} x_i \right) \left(\sum_{h=j}^{m-1} \left(\frac{1}{h} - \frac{1}{h+1} \right) + \frac{1}{m} \right) = mB - \sum_{j=1}^{m} \sum_{i \in P_{m-j+1}} x_i > 0,$$

which is a contradiction because $\sum_{j=1}^{m} \sum_{i \in P_{m-j+1}} x_i = mB$ by the definition of 3-PARTITION. Hence, the left-hand side of each inequality in Equation (5) is equal to 0, which implies that 3-PARTITION has a solution *if and only if* for the considered optimal schedule $C_{\max} \leqslant y$.

An evident observation that our reduction is pseudo-polynomial in the length of 3-PARTITION completes the proof. □

Reference

Bachman A and Janiak A (2004). Scheduling jobs with position-dependent processing times. *Journal of the Operational Research Society* **55**(3): 257–264.