

First-Order Mortality Basis for Life Annuities

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Mortality improvements pose a challenge for the life annuity business. For the management of such portfolios, it is important to forecast future mortality rates. Standard models for mortality forecasting assume that the force of mortality at age x in calendar year t is of the form $\exp(\alpha_x + \beta_x \kappa_t)$, where the dynamics of the time index κ_t is described by a random walk with drift. Starting from such a best estimate of future mortality (called second-order mortality basis in actuarial science), the paper explains how to determine a conservative life table serving as first-order mortality basis. The idea is to replace the stochastic projected life table with a deterministic conservative one, and to assume mutual independence for the remaining life times. The paper then studies the distribution of the present value of the payments made to a closed group of annuitants. It turns out that De Pril–Panjer algorithm can be used for that purpose under first-order mortality basis. The connection with ruin probabilities is briefly discussed. An inequality between the distribution of the present value of future annuity payments under first-order and second-order mortality basis is provided, which allows to link value-at-risk computed under these two sets of assumptions. A numerical example performed on Belgian mortality statistics illustrates how the approach proposed in this paper can be implemented in practice.

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Introduction and motivation

Lee and Carter (1992) proposed a simple model for describing the secular change in mortality as a function of a single time index. The method describes the log of a time series of age-specific death rates as the sum of an age-specific component that is independent of time and another component that is the product of a time-varying parameter reflecting the general level of mortality, and an age-specific component that represents how rapidly or slowly mortality at each age varies when the general level of mortality changes. This model is fitted to historical data. The resulting estimate of the time-varying parameter

is then modeled and projected as a stochastic time series using standard Box–Jenkins methods. From this forecast of the general level of mortality, the actual age-specific rates are derived using the estimated age effects. Often a simple random walk with drift is used to project the time index to the future. The Lee–Carter model is relatively easy to understand, and easy to implement in the stochastic simulations used by actuaries. Moreover, it performed quite well since it has been proposed in 1992.

This paper discusses the technical basis for life annuities. By technical basis, actuaries mean a set of assumptions used to set premium rates in life insurance. Here, we consider first-order mortality basis, which makes conservative or “safe-side” assumptions about future mortality. Experience basis is called second-order basis by actuaries. Contrarily to first-order mortality basis, second-order mortality basis consists in the best estimate of future death rates applying to the insured population. Second-order mortality basis will be taken as the Lee–Carter stochastic forecast.

What is “safe-side” in respect of mortality depends on the nature of the risk insured. In the context of life annuities, first-order mortality basis are routinely obtained by decreasing second-order death rates or one-year death probabilities. The present paper aims to provide a method to design first-order mortality basis in the context of life annuities.

The paper is organized as follows. In the next section, we briefly present the log-bilinear model for mortality projection proposed by Lee and Carter (1992). The following section explains how to determine a conservative, first-order life table. The subsequent section studies the distribution of the present values of the payments made to a closed group of life annuitants. In the penultimate section, a case study based on Belgian mortality statistics is proposed. The final section concludes and discusses further issues.

The contribution of this paper is twofold. First, we supplement the work by Frostig *et al.* (2003) with a simple formula to compute the ruin probability for life annuities portfolios. In the context of life annuity contracts, this amounts to study of the distribution of the present values of the payments made to a closed group of life annuitants. We explain how the actual computations can be performed with the help of De Pril–Panjer recursive algorithms under first-order mortality basis. Switching from a stochastic projected life table (under second-order mortality basis) to a deterministic conservative one (under first-order mortality basis) and replacing correlated life times (under second-order mortality basis) with independent ones (under first-order mortality basis) greatly facilitate the computations. As a second contribution, the paper compares the distribution of the present value of the payments made to a closed group of life annuitants under first-order and second-order mortality basis. This helps to quantify the impact of adopting the conservative life table suggested in this paper, and of assuming independence between the remaining

life times. As a direct application, we get useful inequalities between the value-at-risk computed under the two sets of technical assumptions.

In what follows, many quantities used have a dependence on n , the number of contracts. This dependence is suppressed for notational convenience.

Log-bilinear model for mortality forecasting

Log-bilinear form for the forces of mortality

Under the Lee–Carter model, the (central) death rate applying to age x in calendar year t is assumed to be of the form

$$m_x(t|\boldsymbol{\kappa}) = \exp(\alpha_x + \beta_x \kappa_t) \quad (1)$$

where the parameters β_x and κ_t are subject to constraints ensuring model identification. Interpretation of the parameters involved in model (1) is quite simple. The value of $\exp \alpha_x$ is the general shape of the mortality schedule. The actual forces of mortality change according to an overall mortality index κ_t modulated by an age response β_x . The shape of the β_x profile tells which forces of mortality decline rapidly and which slowly over time in response of change in κ_t . The time factor κ_t is intrinsically viewed as a stochastic process and Box–Jenkins techniques are then used to model and forecast κ_t .

Calibration

The main statistical tool of Lee and Carter (1992) is least-squares estimation via singular value decomposition of the matrix of the log age-specific observed forces of mortality. To account for the higher variability at older ages, Brouhns *et al.* (2002a, b) and Renshaw and Haberman (2003) replaced ordinary least-squares regression with Poisson regression for the death counts. Alternative approaches include Negative Binomial regression as in Delwarde *et al.* (2007), Binomial regression as in Cossette *et al.* (2007), as well as penalized estimation methods as in Delwarde *et al.* (2007) and Bayesian methods as in Czado *et al.* (2005).

Booth *et al.* (2002) designed procedures for selection of an optimal calibration period. An *ad hoc* procedure for selecting the optimal fitting period has been suggested in Denuit and Goderniaux (2005). The restriction of the optimal fitting period favors the random walk with drift model for the κ_t 's. It also corresponds to a conservative approach, since the decline in the κ_t 's usually tends to fasten after the 1970s (where the optimal fitting period starts in most cases). In the numerical illustrations of the penultimate section, we will nevertheless select the appropriate ARIMA model on the basis of standard goodness-of-fit criteria.

Projecting the time index

To forecast, Lee and Carter (1992) assume that the α_x 's and β_x 's remain constant over time and forecast future values of κ_t using a standard univariate time series model. If the κ_t 's obey to a random walk with drift model, as it is the case in the majority of applications after having selected the optimal fitting period, then

$$\kappa_t = \kappa_{t-1} + \theta + \xi_t \text{ with iid } \xi_t \sim \mathcal{N}or(0, \sigma^2), \tag{2}$$

where θ is known as the drift parameter and $\mathcal{N}or(0, \sigma^2)$ stands for the Normal distribution with mean 0 and variance σ^2 . We will retain the model (2) throughout this paper (and justify it in the numerical illustrations). Note that since the κ_t 's obey to the dynamics of Eq. (2), the death rates are not constant but develop over time following a stochastic process.

We will assume in the remainder of this paper that the values $\kappa_1, \dots, \kappa_{t_0}$ are known but that the κ_{t_0+k} 's, $k=1, 2, \dots$, are unknown and have to be projected from the random walk with drift model (2). To forecast the time index at time $t_0 + k$ with all data available up to t_0 , we use

$$\kappa_{t_0+k} = \kappa_{t_0} + k\theta + \sum_{j=1}^k \xi_{t_0+j}.$$

The point estimate of the stochastic forecast is thus $\mathbb{E}[\kappa_{t_0+k} | \kappa_1, \dots, \kappa_{t_0}] = \kappa_{t_0} + k\theta$ which follows a straight line as a function of the forecast horizon k , with slope θ . The conditional variance of the forecast is $\mathbb{V}[\kappa_{t_0+k} | \kappa_1, \dots, \kappa_{t_0}] = k\sigma^2$. Therefore, the conditional standard errors for the forecast increase with the square root of the distance to the forecast horizon k . Now, the covariance structure of the κ_{t_0+k} 's is given by $\mathbb{C}ov[\kappa_{t_0+k_1}, \kappa_{t_0+k_2}] = \sigma^2 \min\{k_1, k_2\}$.

Let us denote the ultimate age of the life table as ω . Precisely, ω is such that $p_\omega(t)=0$ for every year t . The random vector $(\kappa_{t_0+1}, \dots, \kappa_{t_0+\omega-x_0})$ governing the survival of the cohort aged x_0 in year t_0 is Multivariate Normal with mean vector $\mathbf{m} = (\kappa_{t_0} + \theta, \dots, \kappa_{t_0+\omega-x_0} + (\omega - x_0)\theta)^\top$ and variance-covariance matrix

$$\Sigma = \begin{pmatrix} \sigma^2 & \sigma^2 & \dots & \sigma^2 \\ \sigma^2 & 2\sigma^2 & \dots & 2\sigma^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2 & 2\sigma^2 & \dots & (\omega - x_0)\sigma^2 \end{pmatrix}.$$

First-order mortality basis

Requirement

Now, let us determine a conservative life table as follows. Let us consider the cohort reaching age x_0 (typically, retirement age) in year t_0 . For this cohort, we determine the first-order death rates $m_{x_0+k}^{[1]}$, $k=1, 2, \dots$, in order to satisfy

$$\Pr[\exp(\alpha_{x_0+k} + \beta_{x_0+k}\kappa_{t_0+k}) \leq m_{x_0+k}^{[1]} \text{ for some } k = 1, 2, \dots] \leq \varepsilon_{mort}$$

for some probability level ε_{mort} small enough. In order to find the $m_{x_0+k}^{[1]}$'s, we express them as a percentage π of a set of reference forces of mortality $m_{x_0+k}^{ref}$, that is, $m_{x_0+k}^{[1]} = \pi m_{x_0+k}^{ref}$. Then, the value of π comes from the constraint

$$\Pr \left[\kappa_{t_0+k} \geq \frac{\ln(\pi m_{x_0+k}^{ref}) - \alpha_{x_0+k}}{\beta_{x_0+k}} \text{ for all } k = 1, 2, \dots \right] = 1 - \varepsilon_{mort}.$$

Note that the reduction of death rates by a constant factor π is in line with the proportional hazard transform approach to measure risk that has been proposed by Wang (1995).

Reference life table

The set of the $m_{x_0+k}^{ref}$'s can be the latest available population life table, for instance. Here, we take for the $m_{x_0+k}^{ref}$'s the exponential of the point estimates of the κ_{t_0+k} 's, that is,

$$m_{x_0+k}^{ref} = \exp(\alpha_{x_0+k} + \beta_{x_0+k}(\kappa_{t_0} + k\theta)).$$

The $m_{x_0+k}^{ref}$'s thus correspond to the deterministic projected life table produced by the Lee–Carter approach to mortality forecasting. In order to fix the value of π , we then require that

$$\Pr[\exp(\alpha_{x_0+k} + \beta_{x_0+k}\kappa_{t_0+k}) \geq \pi \exp(\alpha_{x_0+k} + \beta_{x_0+k}(\kappa_{t_0} + k\theta)) \text{ for all } k = 1, 2, \dots] \geq 1 - \varepsilon_{mort}$$

$$\Leftrightarrow \Pr[\beta_{x_0+k}(\kappa_{t_0+k} - (\kappa_{t_0} + k\theta)) \geq \ln \pi \text{ for all } k = 1, 2, \dots] \geq 1 - \varepsilon_{mort}.$$

The value of $\ln \pi$ can then be determined as a quantile of the random vector

$$(\beta_{x_0+1}(\kappa_{t_0+1} - (\kappa_{t_0} + \theta)), \dots, \beta_{\omega}(\kappa_{t_0+\omega-x_0} - (\kappa_{t_0} + (\omega - x_0)\theta)))^T$$

that is multivariate Normal with $\mathbf{0}$ mean and variance–covariance matrix

$$\tilde{\Sigma} = \begin{pmatrix} \sigma^2\beta_{x_0+1}^2 & \sigma^2\beta_{x_0+1}\beta_{x_0+2} & \cdots & \sigma^2\beta_{x_0+1}\beta_\omega \\ \sigma^2\beta_{x_0+1}\beta_{x_0+2} & 2\sigma^2\beta_{x_0+2}^2 & \cdots & 2\sigma^2\beta_{x_0+2}\beta_\omega \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2\beta_{x_0+1}\beta_\omega & 2\sigma^2\beta_{x_0+2}\beta_\omega & \cdots & (\omega - x_0)\sigma^2\beta_\omega^2 \end{pmatrix}.$$

Present value of life annuity benefits

First-order and second-order mortality basis

Let us consider a portfolio of n life annuity contracts. The policyholders have respective lifetimes T_1, T_2, \dots, T_n . We assume that all the annuity contracts pay one monetary unit at the end of each year, as long as the annuitant survives.

Let us consider the random variable Z_k representing the present value of the payments made to the annuitants up to time k , that is,

$$Z_k = \sum_{i=1}^n a_{\min\{T_i, k\}},$$

where $a_{\bar{k}|} = \sum_{j=1}^k v(0, j)$ denotes the present value of a sequence of unit cash-flows due at times $1, \dots, k$ discounted with the help of the $v(0, j)$'s. This definition is easily extended to non-integer k 's by first rounding it to its integer part. Taking $k = \omega - x_0$ gives the random variable $Z = Z_{\omega - x_0}$ representing the present value of all the payments made to the group of n annuitants aged x_0 in year t_0 .

Let us now explicitly distinguish the two sets of technical assumptions. Henceforth, we denote as $P_1[E]$ the probability of the event E taken under the first-order mortality basis, and as $P_2[E]$ the probability of the event E taken under the second-order mortality basis. We assume that under P_1 the lifetimes T_1, \dots, T_n are independent with death rates $m_{x_0+k}^{[1]}$, $k=1, 2, \dots$. Recall that under P_2 the lifetimes are conditionally independent (given the κ_{t_0+k} 's) and have common death rates $m_{x_0+k}(t_0 + k|\kappa)$ given by (1). The probability measure \Pr used in the preceding section corresponds to P_2 . This section aims to derive the distribution function of Z_k under P_1 and then to relate it to the distribution function of Z_k under P_2 .

Distribution of Z_j under first-order mortality basis

Computing the distribution function of Z_j thus amounts to computing the distribution function of the sum of the $a_{\min\{T_i, j\}}$'s. Under second-order

mortality basis, we must account for the dependence induced among the T_i 's by the unknown life table (i.e., by the κ random vector). See Denuit and Frostig (2007a, b) for more details about the correlation structure between the T_i 's in the Lee–Carter setting. However, this dependence disappears under P_1 .

Let us now work under the first-order mortality basis, that is, we assume from now on that the T_i 's are independent and identically distributed, and all conform to the life table defined by the forces of mortality $m_{x_0+k}^{[1]}$. Let us denote as $q_{x_0+k}^{[1]}$ the corresponding one-year death probabilities given by

$$q_{x_0+k}^{[1]} = 1 - \exp(-\pi \exp(\alpha_{x_0+k} + \beta_{x_0+k}(\kappa_{t_0} + k\theta)))$$

for $k = 0, 1, \dots$

In this case, computing the distribution function of Z_j thus amounts to computing the distribution function of a sum of n independent and identically distributed random variables $a_{\overline{\min\{T_{1,j}\}}}, \dots, a_{\overline{\min\{T_{n,j}\}}}$. Clearly, $a_{\overline{\min\{T_{i,j}\}}}$ is valued in $\{0, a_{\overline{1}}, \dots, a_{\overline{j}}\}$ and has probability distribution

$$P_1[a_{\overline{\min\{T_{i,j}\}}} = 0] = P_1[T_i < 1] = q_{x_0}^{[1]}$$

$$P_1[a_{\overline{\min\{T_{i,j}\}}} = a_{\overline{\ell}}] = P_1[\ell \leq T_i < \ell + 1]$$

$$= p_{x_0}^{[1]} \dots p_{x_0+\ell-1}^{[1]} q_{x_0+\ell}^{[1]} \text{ for } \ell = 1, \dots, j - 1.$$

$$P_1[a_{\overline{\min\{T_{i,j}\}}} = a_{\overline{j}}] = P_1[T_i \geq j]$$

$$= p_{x_0}^{[1]} \dots p_{x_0+j-1}^{[1]}$$

Let X_i be $a_{\overline{\min\{T_{i,j}\}}}$ that has been appropriately discretized. Here, we keep the original probability mass at the origin, and round the other values in the support of $a_{\overline{\min\{T_{i,j}\}}}$ to the least upper integer (after having selected an appropriate monetary unit). The probability mass function p_X of the X_i 's has support $\{0, 1, \dots, \lceil a_{\overline{j}} \rceil\}$, with $p_X(0) > 0$ (since the probability mass $q_{x_0}^{[1]}$ of $a_{\overline{\min\{T_{i,j}\}}}$ at the origin is kept unchanged). De Pril (1985) developed a simple recursion giving the n -fold convolution of p_X directly in terms of p_X . This substantially reduces the number of required operations. Specifically, the probability mass function of the sum $S = \sum_{i=1}^n X_i$ can be computed from the following recursive formula:

$$p_S(s) = \frac{1}{p_X(0)} \sum_{\eta=1}^s \left(\frac{n+1}{s} \eta - 1 \right) p_X(\eta) p_S(s - \eta),$$

$$s = 1, 2, \dots,$$

starting from $p_S(0) = (p_X(0))^n$. This recurrence relation is a particular case of Panjer recursion formula in the compound Binomial case. It is known to be

numerically unstable so that particular care is needed when performing the computations. Backward and forward computations are often needed to reach a given numerical accuracy. For more details about these issues, we refer the reader to Panjer and Wang (1993).

Link between first-order and second-order mortality bases

As mentioned earlier, the distribution of the Z_k 's under second-order mortality basis is more difficult to obtain because of the correlation arising between the T_i 's (all of them being influenced by the κ_{t_0+k} 's). The next result gives an inequality between the distribution of the Z_k 's under P_1 and P_2 .

Property 1 For any time horizon k , we have

$$P_2[Z_k \leq z] \geq (1 - \varepsilon_{mort})P_1[Z_k \leq z]$$

for any $z \geq 0$.

Proof Let us define

$$A = \{\kappa | m_{x_0+k}(t_0 + k | \kappa) \geq m_{x_0+k}^{[1]} \text{ for all } k = 1, 2, \dots\}.$$

For any event E , let $\mathbb{1}[E]$ be the indicator of E , that is, $\mathbb{1}[E] = 1$ if E is realized, and 0 otherwise. Then, denoting as \mathbb{E}_i the mathematical expectation taken under P_i , for $i=1, 2$,

$$\begin{aligned} P_2[Z_k \leq z] &= \mathbb{E}_2[\mathbb{1}[Z_k \leq z] \mathbb{1}[A]] + \mathbb{E}_2[\mathbb{1}[Z_k \leq z] \mathbb{1}[\bar{A}]] \\ &\geq \mathbb{E}_2[\mathbb{1}[Z_k \leq z] \mathbb{1}[A]] \\ &\geq P_1[Z_k \leq z] P_2[A] \\ &\geq P_1[Z_k \leq z] (1 - \varepsilon_{mort}), \end{aligned}$$

which is the announced result. \square

This result is very useful for risk management. If you take for z the $(1-\varepsilon)$ th quantile of Z_k under P_1 then the probability that Z_k is less than z under P_2 is at least equal to $(1-\varepsilon_{mort})(1-\varepsilon)$. Taking $\varepsilon_{mort}=\varepsilon=1$ percent for example, shows that the 99th percentile of Z_k under P_1 is at least equal to the 98.01th percentile of Z_k under P_2 .

Remark 2 Property 1 can be extended to the expectation of any non-increasing function of the T_i 's. It is indeed easy to see that considering any non-negative non-increasing function $g : \mathbb{R}^n \rightarrow \mathbb{R}^+$, we have

$$\begin{aligned} \mathbb{E}_2[g(T_1, \dots, T_n)] &\geq \mathbb{E}_2[g(T_1, \dots, T_n) \mathbb{1}[A]] \\ &\geq \mathbb{E}_1[g(T_1, \dots, T_n)] (1 - \varepsilon_{mort}). \end{aligned}$$

Ruin probability

Let U_t be the surplus of the insurance company at time t . Starting from $U_0=u$, the surplus obeys to the dynamics

$$U_t = U_{t-1}(1 + r_t) - L_t, \quad t = 1, 2, \dots, \quad (3)$$

where $L_0=n$ by convention, where L_t denotes the (random) number of survivors at time t among the initial n annuitants, and where r_t is the (deterministic) interest rate earned on the reserve during period t (from $t-1$ to t).

Let us denote the ruin probability at horizon k computed under first-order basis ($i=1$) and under second-order basis ($i=2$) as

$$\begin{aligned} \psi_k^{[i]}(u) &= 1 - \phi_k^{[i]}(u) \\ &= P_i[U_t < 0 \text{ for some } t = 1, \dots, k | U_0 = u] \\ &= 1 - P_i[U_1 \geq 0, U_2 \geq 0, \dots, U_k \geq 0 | U_0 = u] \end{aligned}$$

where $\phi_k^{[i]}$ is the non-ruin probability. We thus work conditionally on the initial reserve u , for a given initial number n of annuitants all aged x_0 at issuance (in calendar year t_0).

For any horizon k , the non-ruin probability can be expressed as

$$\phi_k^{[i]}(u) = P_i[Z_1 \leq u, Z_2 \leq u, \dots, Z_k \leq u].$$

Coming back to (3), we see that $U_t < 0 \Rightarrow U_{t+j} < 0$ for any $j=1, 2, \dots$, which in turn implies that the non-ruin probability can be expressed as

$$\phi_k^{[i]}(u) = P_i[Z_k \leq u]. \quad (4)$$

Computing the non-ruin probability at horizon k then amounts to evaluate the distribution function of Z_k .

Let u be the initial capital such that the ultimate ruin probability $\psi_{\omega-x_0}^{[1]}(u)$ is at most ε_{solv} for a portfolio of n annuitants. Note that the number of annuities comprised in the portfolio is an element of the computation. Specifically, we consider the random variable $Z_{\omega-x_0}$, and we determine its $1-\varepsilon_{solv}$ quantile under P_1 .

The ruin probability under the second-order mortality basis is difficult to evaluate in an analytical way, but we can provide the following easy-to-compute lower bound. Specifically, considering (4), Property 1 allows us to write

$$\phi_k^{[2]}(u) \geq (1 - \varepsilon_{mort})\phi_k^{[1]}(u) \geq (1 - \varepsilon_{mort})(1 - \varepsilon_{solv}).$$

For instance, taking $\varepsilon_{mort} = \varepsilon_{solv} = 1$ percent gives a ruin probability of at most 1.99 percent.

Numerical illustration

Context

This section discusses a practical example. We consider individuals buying a life annuity at the age of 65 in year 2005. We can think of people getting retired and desiring to convert their pension fund benefits into periodic payments. In this section, we consider Belgian males aged 65 in year 2005. We take in all the computations $\omega=100$.

We use the 2005 zero coupon yield curve published by Eurostat (for more details, we refer the interested reader to the website <http://epp.eurostat.ec.eu.int>) to obtain the discount factors used for the numerical illustration. Specifically, the r_t 's correspond to the forward rates deduced from the prices of Euro zero coupon bonds. The Eurostat curve shows the structure of interest rates for maturities of one year up to 30 years. The bonds are identified as government issues to guarantee their quality. They are thus appropriate to use by annuity providers (being non-defaultable and having the same currency as the insurer's liabilities).

Mortality projection

We apply the Poisson modeling to the general population data of Belgium. The ages considered here range from 65 to 100, and the observation period is 1950–2002. All the data are available from the Human Mortality Database (maintained by the University of California, Berkeley, U.S.A., and the Max Planck Institute for Demographic Research, Germany), <http://www.mortality.org> (data downloaded in October 2006).

The higher variability observed at advanced ages rules out the least-squares approach to estimating the Lee–Carter parameters α_x , β_x and κ_t (since it assumes a constant variance across ages) unless the mortality surface is previously closed as explained in Denuit and Goderniaux (2005). Here, we resort to the Poisson maximum likelihood approach to estimating the α_x 's, β_x 's and κ_t 's, allowing for heteroskedasticity.

Applying the methodology proposed by Booth *et al.* (2002) to Belgian males yields an optimal calibration period starting in 1974. The *ad hoc* method of Denuit and Goderniaux (2005) suggests that we could start the fitting period in 1970, since the adjustment coefficient peaks around 1970. Therefore, we focus on the period 1970–2002 (Figure 1).

Let us now model the time index κ_t . As in the Lee–Carter methodology the time factor κ_t is intrinsically viewed as a stochastic process. Box–Jenkins techniques are therefore used to estimate and forecast κ_t within an ARIMA times series model. First, the κ_t 's are differenced, to remove the downward linear trend. Considering the first differences of the time index, the

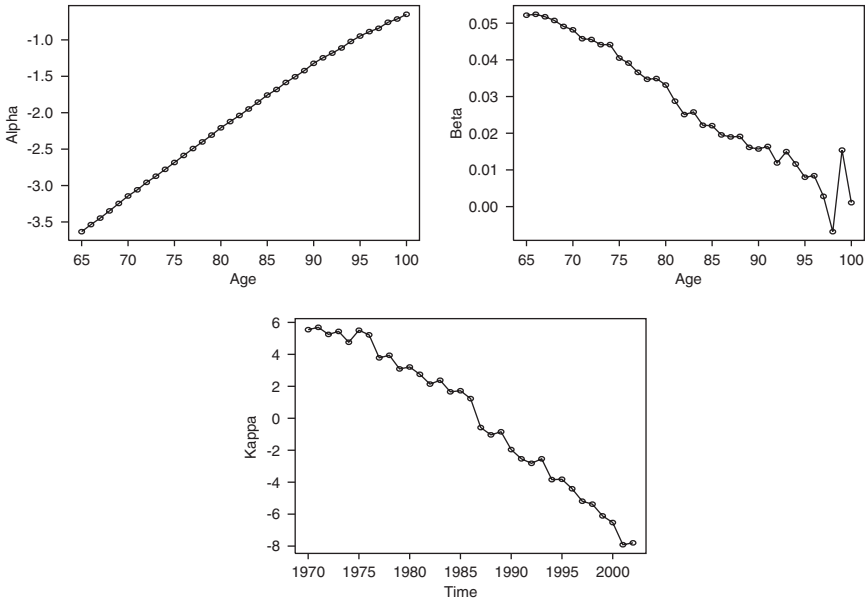


Figure 1. Estimated α_x 's, β_x 's and κ_t 's obtained for the optimal fitting period 1970–2002.

autocorrelation functions and partial autocorrelation functions (which both tail off) clearly suggests that an ARIMA(0,1,0) process is appropriate. Running a Shapiro–Wilk test yields a P -value of 0.2358, which indicates that the residuals seem to be approximately Normal. The corresponding Jarque–Bera statistics is 0.4734, which confirms that there is no significant departure from Normality. The random walk with drift model outperforms its competitor on the basis of standard information criteria. The ARIMA(0,1,0) estimated parameters for the period 1970–2002 are $\hat{\theta} = -0.4169175$ and $\hat{\sigma}^2 = 0.3333644$. The projected κ_t 's are then obtained from last $\hat{\kappa}_{2002}$ by adding a linear trend with slope $\hat{\theta}$.

First-order mortality basis

We are now in a position to determine the conservative life table. The criterion is that the random future life table should not produce smaller death rates than the conservative life table with probability at least 99 percent (corresponding to $\varepsilon_{mort} = 11$ percent). This gives $\pi = 0.9320154$. This value of π has been found using the qmvnorm function of the R package mvtnorm.

This gives a pure life annuity premium amount of 11.62939, to be compared with 11.37335 obtained with the projected life table. The increase in the premium amount is rather moderate. This can be explained by the Gaussian modeling for the time index together with the effect of discounting. The distribution function of the present values $a_{\overline{T_i}|}$ of the life annuity payments for policyholder i is depicted in Figure 2 (left panel) where T_i obeys to the first-order life table. The right panel of Figure 2 describes the distribution function obtained after discretization.

Ruin probabilities

The distribution of Z_{35} can easily be computed as explained in the previous section. Figure 3 gives the probability mass function and the distribution function for $n=10, 20$ and 30 . Imposing a non-ruin probability of 99 percent (i.e., $\epsilon_{solv}=0.01$) under P_1 requires an initial capital $u(10, 1\%, 1\%)=152$, $u(20, 1\%, 1\%)=287$, and $u(30, 1\%, 1\%)=417.9$, respectively, under P_1 . These values correspond to ruin probabilities of at most 2 percent under P_2 . Increasing the number of policies clearly reduces the amount of capital per policy, as expected.

Let us now augment this preliminary set of results to indicate how u depends on ϵ_{mort} and ϵ_{solv} under P_1 . To this end, we have computed the values displayed in the next table.

	$\epsilon_{mort} = 0.5\%$ $\pi = 92.50\%$	$\epsilon_{mort} = 1\%$ $\pi = 93.20\%$	$\epsilon_{mort} = 2\%$ $\pi = 93.97\%$
$\epsilon_{solv} = 0.5\%$			
$n = 10$	15.50	15.50	15.50
$n = 20$	14.60	14.55	14.55
$n = 30$	14.17	14.13	14.10
$\epsilon_{solv} = 1\%$			
$n = 10$	15.20	15.20	15.20
$n = 20$	14.35	14.35	14.30
$n = 30$	13.97	13.93	13.93
$\epsilon_{solv} = 2\%$			
$n = 10$	14.90	14.90	14.90
$n = 20$	14.15	14.10	14.05
$n = 30$	13.77	13.73	13.70

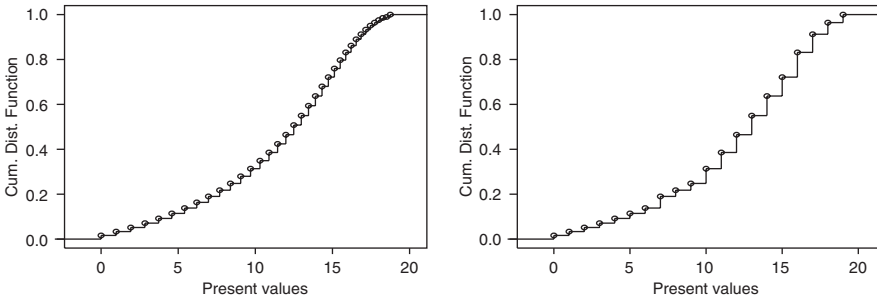


Figure 2. Distribution function of the present values of the annuity payments $a_{\overline{T}|}$ (on the left) and its discretized version (on the right).

These values correspond to the capital needed to have a ruin probability of at most ε_{solv} under P_1 (when P_1 corresponds to ε_{mort}), divided by the number n of policies. As expected for fixed ε_{solv} , these values decrease as ε_{mort} increases. Similarly, for fixed ε_{mort} , these values decrease as ε_{solv} increases. Some values coincide because of the discrete nature of the underlying distribution. Note that combining $\varepsilon_{mort}=0.5$ percent with $\varepsilon_{solv}=0.5$ percent gives a ruin probability of at most 1 percent under P_2 .

Conclusion

In this paper, first-order mortality bases for life annuity contracts, using a conservative life table corresponding to a high-longevity scenario (determined by the ε_{mort} probability level). Assuming that the remaining life times are mutually independent, the computation of ruin probabilities is then straightforward from Panjer algorithm. We prove that this approach allows us to control the ruin probability in the second-order mortality basis.

In this paper, we have disregarded the sampling errors in the Lee–Carter parameters α_x , β_x and κ_t , and in the ARIMA parameters. The two sources of uncertainty that have to be combined are the sampling fluctuation in the α_x , β_x , and κ_t parameters, and the forecast error in the κ_t parameters. To this end, Brouhns *et al.* (2002a, b) sampled directly from the approximate multivariate Normal distribution of the maximum likelihood estimators $\hat{\alpha}$, $\hat{\beta}$, $\hat{\kappa}$. Brouhns *et al.* (2005) sampled from the death counts (under a Poisson error structure). Specifically, the bootstrapped death counts are obtained by applying a Poisson noise to the observed numbers of deaths. The bootstrapping procedure can also be achieved in a number of alternative ways. We refer to Renshaw and Haberman (2008) for a detailed study.

For individual death rate forecasts, we know from Lee and Carter (1992, Appendix B) that confidence intervals based on κ_t alone are a reasonable

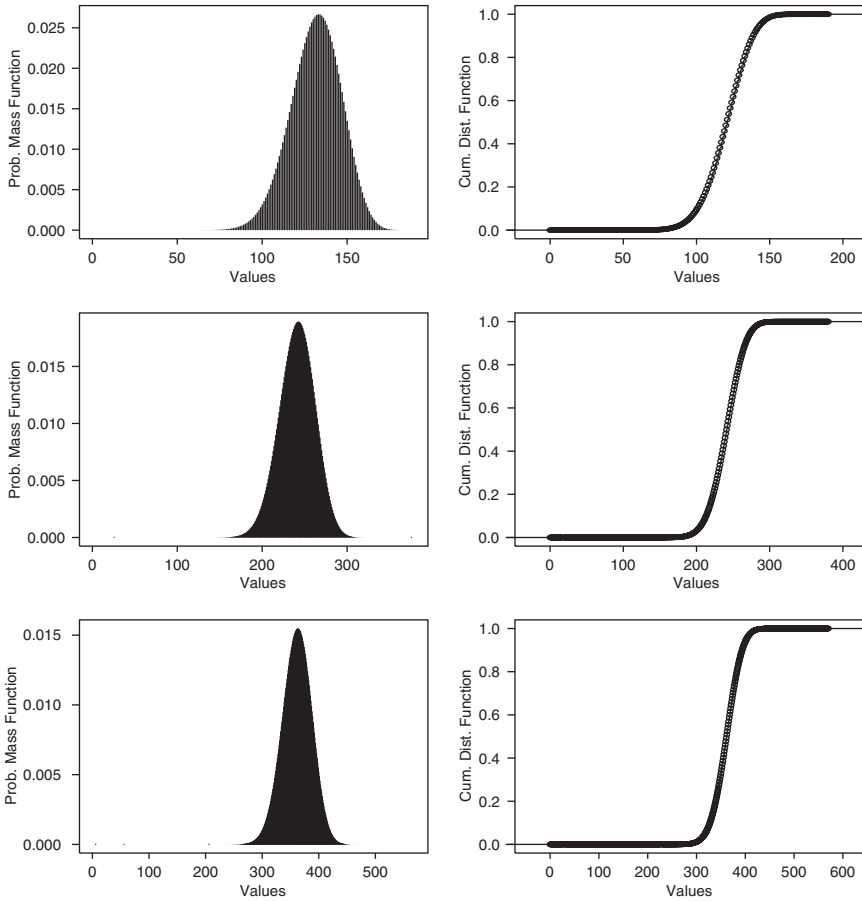


Figure 3. Probability mass functions (left panels) and corresponding distribution functions (right panels) of Z_{35} under P_1 for $n = 10$ (top), 20 (middle) and 30 (bottom).

approximation only for forecast horizons greater than 10–25 years. For long-run forecasts (35 years in the numerical illustration), the error in forecasting the mortality index clearly dominates the errors in fitting the mortality matrix.

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