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A Theory of Insurance Premiums*

by Karl Borch**

1. Introduction

1.1. The buyer of an insurance contract buys security, and the seller accepts a risk. The premium charged by the seller must give him adequate compensation for the risk bearing service he provides, and of course be acceptable to the buyer.

It is useful to see an insurance contract as a contingent claim. The buyer pays a premium in advance, and will get a random amount in return – as settlement of the claims he can make under the contract. Formally the transaction is of the same type as the purchase of a share in a risky business. The price of such shares is presumably determined by supply and demand in the market, and it is natural to assume that insurance premiums are determined in the same way. Economic theory has taken a long time to develop satisfactory models for the pricing of contingent claims. The breakthrough came just over thirty years ago, with the work of Arrow [1953]. In the following three decades a number of models have been developed, i.a. the so-called "Capital Asset Price Model" (CAPM) due to Sharpe [1964], Lintner [1965] and Mossin [1966], which has found extensive applications in practice.

1.2. Actuaries have of course been busy computing insurance premiums for more than a century. The more recent achievement in economics and finance do not seem to have had much influence on the work of actuaries. A recent survey volume of the theory of insurance premiums by Goovaerts et al. [1983] contains no references to highly relevant results in economic theory. One of the few exceptions is Bühlmann [1980] and [1984], which contain no references to actuarial literature.

One explanation of why actuaries have ignored economics may be that they consider CAPM – a one-period model depending only on expectation and variance – as too primitive for their purposes. Instead of adapting CAPM and similar models, they have continued the development of the "actuarial theory of risk", which places the focus on a class of stochastic process in continuous time. There is much pretty mathematics in this theory, but one inevitably feels that most contacts with economic reality have been lost. It is tempting to borrow a

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** Professor at the Norwegian School of Economics and Business Administration, Bergen, Norway.

term from Mac Lane [1983], and tell actuaries that they tend to take "too much of a Hungarian view of mathematics – that the science consists not in good answers, but in hard questions". One purpose of this paper is to induce actuaries to provide some answers.

1.3. The next section gives a short presentation of a model of an insurance market, and shows that a full generalization of CAPM is fairly simple. Section 3 gives some examples, and indicates how the model can be applied in practice. Section 4 reviews a model due to De Finetti, which makes it possible, in theory to determine the utility functions used in section 2. Section 5 presents a modified, and possibly more realistic version of De Finetti's model. The last section contains some comments on the realism of the models, and possible applications.

2. A model of an insurance market

2.1. The model presented in this section is essentially a special case of Arrow's model from 1953.

It is convenient to interpret the model as a reinsurance market, in which n insurance companies trade among themselves. We shall take as given:

(i) The risk attitude of company r, represented by the Bernoulli utility function $u_r(\cdot)$, with the properties $u_r' > 0$ and $u_r'' < 0$.

(ii) The initial portfolio of company r, represented by the stochastic variable x_r. $r = 1,2,..., n$

In the market these n companies exchange parts of their initial portfolio among themselves. As a result of these exchanges company r obtains a final portfolio, represented by the stochastic variable, y_r. $r = 1,2,..., n$.

If the companies cannot trade with outsiders, we have

(1) $$\sum_{r=1}^{n} y_r = \sum_{r=1}^{n} x_r = x$$

where x is the sum of the stochastic variables representing the initial portfolios.

At this stage it is necessary to make an assumption of *homogeneous beliefs*, i.e. that all companies hold the same opinion on the joint density $f(x_1, ..., x_n)$, and hence on $f(x)$. The assumption appears reasonable for a reinsurance market, where trade is supposed to take place under conditions of *uberrimae fidei*, and no information is hidden.

2.2. Any set of exchanges which satisfies (1) is feasible. The subset of Pareto optimal exchanges is determined by (1) and the conditions

(2) $$k_r u_r'(y_r) = k_s u_s'(y_s) \qquad r,s=1,2,...,n$$

where k_r and k_s are arbitrary positive constants. The result has been proved explicitely by Borch [1962], but is really contained in earlier work by Arrow.

It is easy to see that (1) and (2) will determine the Pareto optimal exchanges as n functions $y_1(x), \ldots, y_n(x)$ of the stochastic variable x. These functions will contain arbitrary constants $k_1, \ldots, k_n$ as parameters, and $y_r(x)$ must be interpreted as payoff to company r, if total payoff is x.

It is easy to see that (2) can be written in the form

$$(3) \qquad k_r u_r'(y_r(x)) = u'(x)$$

where the function $u'(x)$ in a sense represents aggregate marginal utility in the market. It will of course depend on the parameters $k_1, \ldots, k_n$.

To facilitate the interpretation of $u'(x)$ we differentiate (3) and obtain

$$k_r u_r''(y_r(x)) y_r'(x) = u''(x)$$

Division by (3) gives

$$\frac{u_r''(y_r(x)) y_r'(x)}{u'(y_r(x))} = \frac{u''(x)}{u'(x)}$$

which we shall write

$$\frac{y_r'(x)}{R(x)} = \frac{1}{R_r(y_r(x))}$$

where $R = -u''/u'$ stands for absolute risk aversion.

From (1) it follows that $\Sigma\, y_r'(x) = 1$, and hence that

$$\frac{1}{R(x)} = \sum_{r=1}^{n} \frac{1}{R_r(y_r)}$$

The inverse of absolute risk aversion is some times called "risk tolerance". Thus we obtain the attractive result that in a Pareto optimum the risk tolerance of the market as a whole is equal to the sum of the risk tolerances of the participants. If one participant is risk neutral, his risk tolerance will be infinite, and hence that of the market. This corresponds to the obvious, that a Pareto optimum implies that all risk should be carried by the risk neutral participants.

The result has also been found by Bühlmann [1980] for the special case of exponential utility functions.

2.3. To determine the competitive equilibrium of the market in the usual way, we need two assumptions:

(i) The market is complete, in the sense that it will assign a unique value $P\{y\}$ to an arbitrary final portfolio, described by a stochastic variable y.

(ii) The value operator $P\{\cdot\}$ is a linear functional of $y(x)$.

From the Riesz representation theorem it then follows that there exists a function $G(x)$ such that

$$P\{y\} = \int_{-\infty}^{+\infty} y(x) dG(x)$$

Any actuary worth his salt believes that he can compute the premium for any risk with known stochastic properties, so to him the first assumption will be trivial. Some economists, i.a. Hirshleifer [1970] seem to doubt that markets in the real world are complete, and the study of "incomplete markets" has become fashionable. These studies are however not relevant in the present context, since no restrictions are placed on the exchange arrangements which the companies are allowed to make.

If a derivative of $G(x)$ exists, we can without loss of generality write $G'(x) = V'(x)f(x)$, so that the formula above takes the form:

$$(4) \qquad P\{y\} = \int_{-\infty}^{+\infty} y(x)V'(x)f(x)dx = E\{y(x)V'(x)\}$$

In (4) y represents a Pareto optimal portfolio, and hence it is a function $y(x)$ of the variable x. A non-optimal initial portfolio is described by the variable x_r, which is stochastically dependent on x. Analogy with (4) suggests that we can write

$$(5) \qquad P\{x_r\} = E\{x_rV'(x)\} = \int_{-\infty}^{+\infty} x_rV'(x)f(x,x_r)dxdx_r$$

Clearly (4) is a special case of (5).

2.4. To find the competitive equilibrium in the conventional way, we note that the problem of company r is:

$$max\ E\{u_r(y_r)\} \qquad r=1,2,\dots,n$$

subject to

$$(6) \qquad P\{y_r\} = P\{x_r\} \qquad r=1,2,\dots,n$$

Conditions (6) say simply that the market value of a portfolio must remain constant if all exchange transactions are settled at market prices.

The problem is equivalent to

$$max \int_{-\infty}^{+\infty} \{u_r(y_r(x)) + \lambda_r(E\{x_rV'(x)\} - y_r(x)V'(x))\} f(x)dx$$

which can be seen as a problem in the calculus of variation. The Euler equation is

$$(7) \qquad u_r'(y_r(x)) = \lambda_r V'(x)$$

and from (3) it follows that this can be written as

$$u'(x) = \lambda_r k_r V'(x)$$

There is no loss of generality if we take $\lambda_r = k_r^{-1}$ and write $V'(x) = u'(x)$. It then follows that the assumption that the function $G(x)$ is differentiable is equivalent to assuming that the aggregate utility function is differentiable.

We can now sum up the results:

Conditions (1) and (3) determine the form of the functions $y_r(x)$.

The equations (6) determine the parameters $k_1, ..., k_n$ in the functions $y_r(x)$.

For the market value of an arbitrary portfolio we find

$$(8) \qquad P\{x_r\} = E\{x_r u'(x)\}.$$

2.5. We have so far discussed only market values of portfolios. To obtain market premiums for insurance contracts, we note that the initial portfolio of company r consists of assets R_r, and of liabilities under the insurance contracts held by the company. Let the non-negative stochastic variable z_r represent claim payments under the contracts, and write

$$x_r = R_r - z_r$$

If for the sake of simplicity we assume that the assets are risk-free, we have

$$(9) \qquad P\{x_r\} = R_r - P\{z_r\}$$

Formula (9) says simply that the market value of the company's portfolio is equal to risk-free assets, less the market premium for insurance of the liabilities. The formula makes it easy to translate results expressed in terms of portfolio values into insurance premiums.

3. Examples and applications

3.1. As a simple example we shall take

$$u_r'(x) = sgn(c_r - x)\,|(c_r - x)^\alpha| \qquad r = 1,...,n$$

The condition $u_r' > 0$ is satisfied only when $x < c_r$, so the analysis is valid only in this domain. If α is an odd integer we can take $u_r'(x) = (c_r - x)^\alpha$. We shall use this simpler notation, and fall back on the complete notation only when it is necessary to avoid misunderstanding.

In the example (3) takes the form

$$(3') \qquad k_r(c_r - y_r)^\alpha = u'(x)$$

which can be written

$$c_r - y_r = (k_r^{-1} u'(x))^{\frac{1}{\alpha}}$$

Summing these equations over all r, and using (1), we obtain

$$\sum_{r=1}^{n} c_r - x = (u'(x))^{\frac{1}{\alpha}} \sum_{r=1}^{n} k_r^{-\frac{1}{\alpha}}$$

Here we write

$$\sum_{r=1}^{n} c_r = c \qquad \text{and} \qquad \sum_{r=1}^{n} k_r^{-\frac{1}{\alpha}} = K^{-\frac{1}{\alpha}}$$

so that the equation takes the form

$$u'(x) = K(c-x)^{\alpha}$$

From (3′) we then find

$$k_r(c_r-y_r(x))^{\alpha} = K(c-x)^{\alpha}$$

and the explicit expression for $y_r(x)$

$$y_r(x) = c_r - (c-x)(Kk_r^{-1})^{\frac{1}{\alpha}} = c_r - (c-x)h_r$$

where $h_r = (Kk_r^{-1})^{\frac{1}{\alpha}}$, and $\Sigma\, h_r = 1$.

The equations (6) then take the form

$$(6') \qquad KE\{(c_r-(c-x)h_r)(c-x)^{\alpha}\} = KE\{x_r(c-x)^{\alpha}\}$$

and the parameters are determined by the equations

$$h_r = \frac{E\{(c_r-x_r)(c-x)\}^{\alpha}}{E\{(c-x)^{\alpha+1}\}} \qquad r=1,2,\ldots,n$$

3.2. The right-hand side of (6′) gives the market value of an arbitrary portfolio, so we can write

$$(10) \qquad P\{x_r\} = KE\{x_r(c-x)^{\alpha}\}$$

The special case $\alpha = 1$ has led to models which are widely used in practical financial analysis, so it may be useful to discuss this case in some detail.

For $\alpha = 1$ the equation (10) takes the form:

$$P\{x_r\} = KcE\{x_r\} - KE\{xx_r\} = K(c-E\{x\})E\{x_r\} - K\,cov\; xx_r$$

If we here take $x_r = x$, we obtain in the same way

$$P\{x\} = K(c-E\{x\})E\{x\} - K\;var\; x$$

Combining these two equations, we obtain

$$(11) \qquad P\{x_r\} = KAE\{x_r\} + (P\{x\} - KAE\{x\})\,\frac{cov\; xx_r}{var\; x}$$

where $c - E\{x\} = A$.

It is worth noting that the arbitrary constant K can be interpreted as a risk-free rate of interest in the market.

Formula (11) is of course CAPM, which has found widespread applications in practice, and is discussed in virtually every textbook in finance – in spite of its obvious shortcomings.

We derived (11) by assuming that marginal utilities were linear, i.e. that all utility functions were quadratic. This assumption seems to be too strong for most economists. The equivalent assumption, implied by (11), that values are determined by the two first moments of a stochastic variable, seems however to be acceptable. A multivariate normal distribution is completely described by means and covariances. If the variables $z_1, z_2, \ldots, z_m$ are normally distributed, the variable $z = t_1 z_1 + \ldots, t_m z_m$ defining a portfolio, is also normally distributed. This seems to be the current justification for the practical application of CAPM.

The normal is the only distribution with finite variance which has the "stability" property sketched above. This means that strictly speaking CAPM can assign a value only to portfolio defined by normally distributed variables, and that may be the explanation of the interest in "incomplete" markets.

3.3. Actuaries seem to consider CAPM as too primitive for their purposes. They may find some support for this view in a statement by Cramér [1930] who wrote: "... in many cases the approximation obtained by using the normal function is not sufficiently good to justify the conclusions that have been drawn in this way". The statement has led to a number of hard questions which must delight any actuary who takes the "Hungarian" view. In their efforts to answer Cramér's questions these actuaries have made contributions of some importance to the development of probability theory. It is however doubtful if their work has had any noticeable effect on insurance practice.

Economists working on finance seem to have taken the opposite attitude. They have found an easy answer in CAPM, which may not be a particularly good one, and put it to application. One can discuss how useful this model really has been in practice. It is however certain that the applications of CAPM has led to deeper insight into the functioning of financial markets.

3.4. Let us now return to the more general version of CAPM given by (10). From (9) it follows that we have

$$P\{z_r\} = R_r - P\{x_r\} = R_r - KE\{x_r(c-x)^{\alpha}\}$$

Here we substitute

$$x_r = R_r - z_r \quad \text{and} \quad x = R - z$$

and find the following expression for the premium

$$P\{z_r\} = R_r\{1 - KE\{(c-R+z)^{\alpha}\} + KE\{z_r(c-R+z)^{\alpha}\}$$

where $R = \Sigma\ R_r$ and $z = \Sigma\ z_r$.

For the degenerate case $z_r \equiv 0$, consistency requires that $P\{z_r\} = 0$, so that we must have

$$K^{-1} = E\{(c-R+z)^{\alpha}\}$$

Hence the premium formula takes the form

$$P\{z_r\} = KE\{z_r(c-R+z)^{\alpha}\} = \frac{E\{z_r(c-R+z)^{\alpha}\}}{E\{(c-R+z)^{\alpha}\}}$$

or written full

$$(12) \qquad P\{z_r\} = K\int_0^\infty\int_0^\infty z_r(c-R+z)^{\alpha} f(z,z_r)dzdz_r$$

Formula (12) shows how the premium for a risk which can lead to losses represented by the stochastic variable z_r, depends on:

(i) The stochastic properties of the risk itself.

(ii) The stochastic relationship between the particular risk and claims in the market as a whole, described by the joint density $f(z, z_r)$.

(iii) The attitude to risk in the market as a whole, represented by c.

(iv) The total assets of all insurance companies in the market, R.

A realistic theory of insurance premiums must of course take all these four elements into account. This is however rarely done in actuarial risk theory. The recent book by Goovaerts et.al. covers only the first of the four elements.

One could consider as a fifth element the interest earned on the premium before claims are paid. This is regularly done in financial theory, but is rather trivial in the present context. One has just to multiply the premium by the appropriate discount factor.

3.5. As another simple example we shall take

$$u_r'(x) = exp\{-\frac{x}{\alpha_r}\} \qquad r = 1,2, ..., n$$

Equation (3) then takes the form

$$k_r\, exp\{-y_r\alpha_r^{-1}\} = u'(x)$$

Taking logarithms on both sides, we obtain

$$log\, k_r - y_r\alpha_r^{-1} = log\, u'(x)$$

Multiplication by α_r and summation over all r gives

$$\Sigma\,\alpha_r log\, k_r - x = log\, u'(x)\Sigma\,\alpha_r$$

Writing

$$\sum_{s=1}^{n} \alpha_s \log k_s = K \quad \text{and} \quad \sum_{s=1}^{n} \alpha_s = A$$

we obtain

$$u'(x) = exp\{-\frac{K-x}{A}\}$$

and

$$y_r(x) = \frac{\alpha_r x}{A} + \alpha_r \log k_r - \frac{\alpha_r K}{A}$$

We can then proceed as in the first example. From (8) we determine the value of an arbitrary portfolio in the market, and hence also the market premium for an arbitrary insurance contract.

3.6. The two preceding examples are fairly simple because the "sharing rules" are linear, i.e.

$$y_r(x) = q_r x + c_r$$

where

$$\Sigma q_r = 1 \quad \text{and} \quad \Sigma c_r = 0$$

The sharing rules have this form only if all utility functions belong to one of the following two classes:

(i) $u_r'(x) = (x+c_r)^{\alpha}$, or

(ii) $u_r'(x) = e^{-\alpha_r x}$

The result is well known, and has been proved by many authors, i.a. by Borch [1968].

If the sharing rules are linear, it is possible for the companies to reach a Pareto optimum through proportional reinsurance arrangements. Other forms of reinsurance are widely used, so there should be a need for studying utility function which do not lead to linear sharing rules. The problem is difficult, and it is surprising that it has only occassionally been taken up by those who cherish "hard questions" for their own sake. We shall not discuss the problem in any detail, but it may be useful to study a simple example.

Consider 2 persons with the utility functions

$$u_1(x) = 1-e^{-x}, \text{ and } u_2(x) = log(c+x).$$

Condition (2) then takes the form

$$k_1 e^{-y_1} = \frac{k_2}{c+y_2}$$

As $y_1+y_2 = x$, we can write this as

$$(c+y_2)e^{-(x-y_2)} = k$$

Clearly we must have $y_2 > -c$. Hence under a Pareto optimal rule of loss-sharing, person 2 can under no circumstances pay more than c, so that the rule will correspond approximately to a stop loss insurance arrangement.

3.7. Let us now return to the general case (3), and consider a risk described by the stochastic variable z.

From (8) it follows that the insurance premium for this risk is

(13) $$P\{z\} = \int_0^\infty \int_0^\infty z\, u'(x) f(x,z)dxdz$$

Here we have written x for total claims in the market, and we have not brought the assets explicitly into the utility function.

It is convenient to write (13) in the form:

$$P\{z\} = \int_0^\infty u'(x) \left\{ \int_0^\infty z f(x,z)dz \right\} dx$$

Let us now write $z = z_1+z_2$, where $z_1 = min(z,D)$ and $z_2 = max(0,z-D)$.

For the two components we find the following premiums

$$P\{z_1\} = \int_0^\infty u'(x) \left\{ \int_0^D z f(x,z)dz + D \int_D^\infty f(x,z)dz \right\} dx$$

$$P\{z_2\} = \int_0^\infty u'(x) \left\{ \int_D^\infty (z-D) f(x,z)dz \right\} dx$$

These two formulae can be given a number of different interpretations, i.a.:

$P\{z_1\}$ can be seen as the reduction in the premium offered to a buyer, if he will accept a deductible D.

$P\{z_2\}$ can be seen as the reinsurance premium for a stop loss contract, under which the reinsurer pays all claims in excess of the limit D.

We now write $P\{z_2\} = P\{D\}$, and differentiate twice with respect to D. This gives:

$$P'(D) = -\int_0^\infty u'(x) \left\{ \int_D^\infty f(x,z)dz \right\} dx$$

(14) $$P''(D) = \int_0^\infty u'(x) f(x,D)dx$$

If a reinsurer is willing to quote premiums for a number of stop loss contracts, with different limits, the left-hand side of (14) can be estimated for an arbitrary D. If the joint density $f(x,z)$ is known, at least approximately it may be possible to obtain an estimate of the kernel function $u'(x)$. With this estimate, we can then compute the premium for an arbitrary insurance contract in the market.

It is obviously impossible to determine the function $u'(x)$ from (3), i.e. from the preferences of all companies, and then use (8) to calculate the market premium for an arbitrary insurance contract.

It is however clear that $u'(x)$ plays an important part in determining market premiums. Since premiums are observable, we can turn the problem around, and estimate the function $u'(x)$ from observations. This approach is widely used in economics. Demand curves and underlying utility functions cannot be observed, but must be estimated from observed behaviour. This approach is some times referred to as the "principle of revealed preference".

4. A dynamic approach to utility

4.1. In the preceding sections we have studied simple one-period decision problems, and assumed there was a utility function behind the decision. In practice it may often be difficult to specify this utility function – for good reasons. The utility a company assigns to the profits earned in one period will presumably depend on how that profit can be applied in future periods. This suggests that the problem should be studied in a dynamic or multi-period framework. If the company has an overall, long-term objective, this must contain the utility functions which govern the decisions in each period. This overall objective may often be simpler to specify than the Bernoulli utility function.

One of the first to study this problem in an insurance context was De Finetti [1957] in one of his pioneering papers. As a long-term objective he suggested that an insurance company should seek to maximize the expected present value of its dividend payments. In the following decades this problem has been discussed by several other authors, i.a. by Borch [1967] and [1969], Bühlmann [1970], and Gerber [1979].

De Finetti made his observation in a critical study of the now obsolete "collective risk theory". This theory placed the focus on the probability that an insurance company shall remain solvent forever, provided that it does not change its operating procedures. This probability of achieving eternal life will inevitably be zero, unless the company allows its reserves to grow without limit. De Finetti pointed out that an insurance company could not really be interested in building up unlimited reserves, and argued in fact that the marginal utility of accumulated profit *(vincite utili)* must be decreasing. He illustrates his argument with a very simple example, based on a two-point distribution. In the following we shall consider a simple, slightly more general example.

4.2. Consider an insurance company which in each successive operating period underwrites identical portfolios, and take the following elements as given:

S = the company's initial capital

P = the premium received by the company at the beginning of each operating period

$f(x)$ = the probability density of claims paid by the company in each operating period.

If S_t is the company's capital at the end of period t, and x_{t+1} the claims paid by the company during period $t+1$, the company's capital at the end of period $t+1$ will be:

$$S_{t+1} = S_t + P - x_{t+1}$$

Assume now that the company operates under the following conditions:

(i) If $S_t < 0$, the company is insolvent, or "ruined", and cannot operate in any of the following periods.

(ii) If $S_t > Z$, the company pays a dividend $s_t = S_t - Z$. It is natural to assume that Z is chosen by the management, because accumulated profits beyond Z has lower utility than a paid out dividend.

The dividend payments $s_1, s_2, ..., s_t ...$ is a sequence of stochastic variables, and we shall write $V(S,Z)$, or when no misunderstanding is possible $V(S)$ for the expected discounted sum of the dividend payments which the company makes before the inevitable ruin, i.e.

$$V(S) = \sum_{t=1}^{\infty} v^t E\{s_t\}$$

where v is a discount factor. We shall assume that the company seeks the value of Z which maximizes $V(S,Z)$.

4.3. To study the function $V(S)$, we first note that for $S < Z < S + P$ it must satisfy the integral equation

$$(15) \qquad V(S) = v \int_Z^{S+P} \{x-Z+V(Z)\} f(S+P-x)dx + v \int_0^Z V(x) f(S+P-x)dx$$

The equation says that if claims x are less than $S+P-Z$, a dividend $S+P-Z-x$ will be paid at the end of the period, and the company will begin the next period with a capital Z.

If $S+P-Z \leqslant x \leqslant S+P$ the company will pay no dividend, and begin the next period with a capital $S+P-x$. If $S+P < x$, the company is ruined, and cannot operate in any of the following periods.

In the interval stated (15) is an integral equation of Fredholm's type, and it is known that is has a unique continuous solution. For $S < Z - P$, no dividend can be paid at the first period, and some modifications are necessary.

On the whole (15) is difficult to handle, and unless one holds a strong "Hungarian" view on mathematics, a general solution appears uninteresting. The optimal dividend policy is given by the value of Z, which maximizes $V(S,Z)$. The existence of an optimal policy poses some hard questions, which we shall not take up.

4.4. The main idea behind De Finetti's paper can be presented as follows:

Assume an insurance company with equity capital S is offered a premium Q, if it will underwrite a one-period insurance contract described by the claim density $g(x)$. If the company accepts the contract, expected present value of future dividend payments will be

$$U(S) = \int_0^{S+Q} V(S+Q-x)g(x)dx \tag{16}$$

Hence, given the company's long-term objective, the contract will be acceptable if and only if $U(S) \geqslant V(S)$. This means that $V(S)$ serves as the utility function which determines the company's decision, and that in (8) $u'(x)$ can be replaced by $V'(x)$. This assumes of course that the company receives the offer after it made its reinsurance arrangements, or that it takes the reinsurance possibilities into account when deciding about the particular offer.

De Finetti did not elaborate his suggestion, but went instead on to discuss other functions of an arbitrary upper limit to accumulated profits, which could serve as alternative objectives. His paper was generally ignored for a decade, even in actuarial circles. The influential paper on the subject is by Shubik and Thompson [1959], who independently developed a model which is virtually identical to that of De Finetti, but not interpreted in terms of insurance.

4.5. De Finetti's model implies that an insurance company is basically risk-neutral. This may at first sight seem surprising, but it makes some sense if one considers the alternatives. It seems artificial to assume that the board of a company will consider different utility functions, and possibly take a vote, to pick one which adequately represents the company's attitude to risk. It seems more natural to assume that a board will settle for some simpler rule, such as that of maximizing expected present value of dividend payment. With this rule the company will behave as if it were risk-averse in its underwriting and reinsurance transactions.

At this stage we can see dimly the outline of an attractive unified theory of insurance:

(i) Premiums and reinsurance arrangements are determined by the methods sketched in Section 2.

(ii) The utility functions which play an important role in Section 2, can be determined by the methods indicated in this Section.

A number of hard questions must be answered before a complete theory can take form. Before taking up the challenge, it may however be useful to pause, and ask if it really is necessary to answer these questions.

5. A modification of the De Finetti's model

5.1. De Finetti's innovation consisted in removing the ruin probability as part of the objective of the insurance company. Instead he put the emphasis on the dividend payments, and placed an upper limit on the reserves which the company would accumulate. His model presented actuaries with a number of new hard questions, and seems to have given the collective risk theory a *coup de grâce.*

It is natural also to consider a lower limit on the equity capital which the company will hold at the beginning of a new operating period. This means that the company will have to obtain new equity capital after an unfavorable underwriting period. If capital markets function efficiently, this should be possible. If the insurance company is owned by a holding company, one must expect that the owner will see that the subsidiary enters each period with an optimal equity capital, i.e. that the owner will risk the optimal amount of capital in the insurance business. This leads us to consider an insurance company which initially holds an equity capital Z, and has adopted the following policy:

If claims in a period amounts to x

(i) The company pays out a dividend $max(P-x,0)$

(ii) An amount $min(x-P, Z)$ is paid into the company as new equity capital.

This policy implies that if the company is solvent at the end of a period, it will enter the next period with an equity capital Z.

5.2. Under the policy outlined expected present value of the dividend payment at the end of the first period is

$$W_1(Z) = v \int_0^{P+Z} (P-x)f(x)dx$$

The probability that the company shall be solvent at the end of the period, and hence be able to operate in the second period is

$$Pr(x \leqslant P+Z) = F(P+Z)$$

It then follows that the expected present value of all payments is

$$W(Z) = v \int_0^{P+Z} (P-x)f(x)dx \sum_{n=0}^{\infty} [vF(P+Z)]^n$$

or

$$(17) \qquad W(Z) = \frac{v \int_0^{P+Z} (P-x)f(x)dx}{1-vF(P+Z)} = \frac{v \left\{ \int_0^{P+Z} F(x)dx-ZF(P)Z) \right\}}{1-vF(P+Z)}$$

5.3. The problem of the company is now to determine the value of Z which maximizes (17), i.e. to find the optimal amount of capital which the owners should put at risk in their company.

The first order condition for a maximum, $W'(Z) = 0$, takes the form

$$(18) \qquad Z \left\{ 1-vF(P+Z) \right\} = v \int_0^{P+Z} (P-x)f(x)dx$$

From (18) Z can be determined, and comparison with (17) shows that for the optimal Z we have

$$Z = W(Z)$$

This simple condition expresses the obvious, that the optimal capital to put into a venture is equal to the expected present value of the return.

In the discount factor $v = (1+i)^{-1}$ which occurs in (17) and (18), i must be interpreted as the return on competing investments. This invites some conventional comparative static analysis. Differentiation of (18) gives

$$\frac{dZ}{dv}\left\{1-vF(P+F)\right\} = \int_0^{P+Z} F(x)dx > 0$$

This again expresses the obvious. If return on competing investments decreases, i.e. if v increases, more capital will flow into the insurance industry. In the opposite case insurance companies may be unable to attract new equity capital.

Clearly $F(P+Z)$ is the probability that an insurance company shall be able to pay its claims. In most countries government regulations require that this probability shall be close to unity. In practice this means that an insurance company is only allowed to operate if it satisfies a solvency conditions of the form $F(P+Z) \geqslant \alpha$. If there is a general increase in returns on investments, it is evident that insurance companies must increase their premiums, in order to attract the capital necessary to satisfy the solvency condition. This may be a useful result. It shows that the government cannot decree both high solvency and "reasonable" premiums in a free capital market. The result should be obvious, but does not always seem to be so.

6. The models and the real world

6.1. The model presented in Section 2 was interpreted as a reinsurance market, and we determined the equilibrium premiums. One must expect that these premiums, at least in the longer run, will have some influence on the premiums in direct insurance.

Insurance markets are far from perfect, and a company may well – for some time – be able to charge premiums above those determined by the model. This company can however earn a risk-free profit if it reinsures its whole portfolio in the market, so the situation cannot be stable in the long run. One must also expect that the customers of this company eventually will discover that they can buy their insurance cheaper elsewhere.

Similarly an insurance company may charge lower premiums than those determined by the model. This company will probably find that reinsurance costs more than it is worth, and carry most of the risk itself. With good luck this may go on for quite some time, and with ample reserves the company will be able to satisfy the solvency conditions laid down by the government. When luck runs out, as it eventually must, the company may well find itself insolvent, when its liabilities are valued at market prices.

6.2. In Section 2 it was proved that a Pareto optimum could be reached only if every insurer participated in every risk in the market. This does not happen in practice, and it is natural to seek the explanation in transaction costs. This is however not entirely satisfactory, since it should not be prohibitively expensive to carry out the calculations necessary to divide a risk in a Pareto optimal way. It seems more likely that some elements of moral hazard enter. It is always stressed that reinsurance is carried out under conditions of the "utmost good faith". In practice this means that a reinsurer accepts the ceding company's report on the stochastic properties of the risk covered, without a costly checking of the statistics. The possibility that the ceding company cheats is always there, so the reinsurer has the choice of either expenses, or of being exposed to moral hazard. It is therefore natural that reinsurers should try to cover a medium sized risk within a fairly small group, and this may lead to some segmentation of the market.

One way to model this would be to assume that there are cost c involved in checking the portfolio offered by a would-be ceding company. If n companies participate in an exchanges arrangement, each checking the other, total costs will be $n(n-1)c$. With increasing n there will clearly be a point where increasing costs will offset the gain by wider diversification of the risk.

6.3. In Section 4 we have in a sense banished utility and risk aversion from the supply side of the insurance market. There are however some reasons for assuming that insurance companies should be risk-neutral. Insurance companies are intermediaries between those who want to buy insurance, and those who are willing to risk their money in the underwriting of insurance contracts. If these two parties are risk-averse, a set of Pareto optimal arrangements exists. The parties may however be unable to reach a Pareto optimum, if the transactions have to be carried out through an intermediary that imposes its own risk aversion. The problem has been discussed in economic literature, i.a. by Malinvaud [1972]. We shall not discuss it further here, since a complete discussion will require detailed specification of the institutional framework.

On the demand side risk aversion remains essential, since a buyer of insurance is by definition risk-averse. When he is faced by a market which can quote a premium for any kind of insurance contract, he can compute or program his way to the contract which is optimal according to his preferences.

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