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Hybrid subgames and copycat games in a pulsing model of advertising competition

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Two recent papers,[6,7] introduced the game of pulsing competition (PC) in advertising together with its related subgames of alternating pulsing competition (APC) and matching pulsing competition (MPC) for a duopoly. Following a game theoretic approach in conjunction with a continuous Lanchester model, the above authors basically concluded that when at least one of the response functions is convex, generalising monopolistic advertising pulsation results to a competitive setting might not be adequate. This paper expands the scope of the PC game by incorporating in its structure for the first time in the literature, two versions of a hybrid pulsing competition (HPC) subgame. The article compares the payoffs of the four alternative subgames and provides an analytical solution of a special case of the PC game. In addition, the article also introduces for the first time a variant of the PC game designated by 'the copycat advertising game' and shows analytically that for such a game the policy of constant advertising spending over time is optimal for both firms irrespective of the shape of their advertising response functions. The paper illustrates at its end how to solve numerically the expanded PC game in its general form using linear programming and how to derive a solution for a copycat advertising game.

Keywords: advertising; game theory; marketing

Introduction

Since the early part of the 1970s, the issue of whether a uniform advertising policy (even-spending over time) is superior (inferior) to an equivalent advertising policy of pulsation that costs the same has attracted the attention of several researchers. Considering continuous advertising response models, Sasieni[1] in his path-breaking work indicated that for a concave or linear advertising response function, a policy of uniform spending is superior in the long run, to a cyclic (pulsation) policy that costs the same. Mahajan and Muller[2] and later Sasieni[3] showed that for advertising response functions that contain a convex portion, a pulsation policy is superior to an equivalent even policy of advertising spending. This same argument is the basis for resource allocation models like Lodish's CALLPLAN.[4] A summary of the advertising pulsation literature for monopolistic markets is found in Mesak and Darrat.[5]

Following a game theoretic approach for a duopoly, Mesak and Calloway[6] found that Uniform Advertising Policy (UAP) is optimal for a firm having a concave or linear response function competing against another having a concave or a linear response function, and therefore generalising the above monopoly results for such functions to a duopoly. When at least one of the response functions is convex, the above study demonstrated that generalising monopolistic results might not be adequate. For such situations, the optimal policy of the firm may involve the Advertising Pulsing/Maintenance Policy (APMP) replacing the monopolistic Advertising Pulsing Policy (APP) optimal when the response function is convex, or the monopolistic UAP optimal when the response function is concave or linear. In a sequel to the above study, Mesak and Calloway[7] estimated empirically the continuous Lanchester model with considerable success.

Research in advertising pulsing competition is sparse. Notable examples include Park and Hahn,[8] Villas-Boas,[9] and Mesak and Callaway.[6,7] The general theme of the modeling framework used in these studies appears to be static. All studies assume equal periods for the high and low advertising pulses together with equal cycle lengths for all rivals. Average undiscounted profits over the infinite planning horizon is mostly chosen as the performance measure used to evaluate alternative advertising pulsation policies. In addition, Lanchester models are often employed to represent sales or market share response to the advertising effort of competing firms. Furthermore, the reviewed studies consider only two competitive situations: Matching Pulsing Competition and Alternating Pulsing Competition.

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In this paper, the authors extend the earlier works of Mesak and Calloway[6,7] in several directions. Firstly, we introduce for the first time in the literature the 'hybrid' advertising strategy for a firm composed of different APC, MPC, and HPC subgames, as a feasible response to a pulsation policy of its rival. We show that if a firm chooses to engage in a 'Hybrid Pulsing Competition (HPC)' with its competitor, its average sales, in the long run, could be superior to those acquired under Alternating Pulsing Competition (APC) or under Matching Pulsing Competition (MPC) considered in Mesak and Calloway.[6] Furthermore, the paper introduces and analyses for the first time a 'copycat' game in advertising competition. In addition, we show how to solve the game of Pulsing Competition (PC), using linear programming (LP). The paper also demonstrates that irrespective of the shape of the advertising response function, a policy of constant advertising over time is optimal for the two firms competing in a copycat advertising game.

As in Mesak and Calloway,[6,7] a game theoretic approach that employs a Lanchester continuous model is followed (see Sasieni[3] for a justification of using continuous advertising response models and Rao[10] for a discussion of their advantages over the discrete counterparts). Similar to previous studies in advertising pulsing competition, the current research follows a static modeling framework according to which each competitor has to choose among three alternative policies: (a) Uniform Advertising Policy (UAP), in which the firm advertises at some constant level throughout; (b) Advertising Pulsing/Maintenance Policy (APMP), in which the firm alternates between a high level of advertising and a lower level, usually a maintenance level; and (c) Advertising Pulsing Policy (APP), in which the firm alternates between high and zero levels of advertising.

In competitive situations, Wells and Chinsky[11] suggest as a result of their laboratory study that messages are most effective when they are delivered in bursts. Also, Ackoff and Emshoff[12] together with Rao and Miller[13] indicate as a result of their field experiments that an Advertising Pulsing/Maintenance Policy could be superior to the even policy. These studies and others, justify that the relatively narrow pulsation policy set considered above is not only of theoretical interest but also of practical relevance.

The next section of this paper describes the pulsing competitive game (PC) and its subgames. The third section develops and compares alternative analytical expressions for the payoff of the focal firm under alternating, matching, and hybrid competitive situations, and also solves analytically a special case of the PC game and introduces and analyses a copycat game in advertising competition. In the fourth section, linear programming is used to provide a numerical solution for the PC game formulated in the second and third sections. A numerical example for the copycat advertising game discussed in the third section is also illustrated in the fourth section. Finally, the fifth section summarises and concludes the study. Derivation of key formulas and proofs of all results are available from the authors upon request.

The pulsing competitive game and its subgames

Assume there are two firms competing in a single market where advertising is the major element in the marketing effort of both firms. Each of the firms manages only one brand of an established frequently purchased product. We identify the following competitive situations in a duopoly.

Firm 1 is said to be engaged in an Alternating Pulsing Competition (APC) with its rival if it cycles low-high (L-H) pulsing while firm 2 cycles high-low (H-L) pulsing or cycles H-L pulsing while firm 2 cycles L-H pulsing.

On the other hand, firm 1 is said to be engaged in a Matching Pulsing Competition (MPC) with its rival if both firms cycle the same type of pulsing. Finally, firm 1 is said to be engaged in a Hybrid Pulsing Competition (HPC) with its rival if both firms cycle similarly part of the time and cycle differently for another part of the time in a repetitive fashion. There are two versions of HPC. Figure 1 illustrates examples of the competitive situations discussed above.

We assume that in each cycle of length 4τ, the high and low advertising periods are of the same length 2τ. As illustrated in Figure 1, HPC (versions H1 or H2) can be conceived as some mixture of the APC and MPC competitive situations. A simple procedure for distinguishing between the two versions will be discussed shortly. In this paper, as in Mesak and Calloway,[6] the chosen measure of performance for each firm is the average undiscounted profit over the infinite planning horizon. Park and Hahn[8] argue in favor of such performance measure as it has the advantage of being independent of arbitrary initial conditions in addition that its weakness is substantially mitigated because of the periodic nature of the considered advertising policies over time. In the game of pulsing competition (PC), each firm aims at determining the advertising level in each period (and therefore one of the alternative policies UAP, APMP, or APP) as well as the advertising timing relative to the competition (and therefore one or perhaps some combination of the alternative competitive situations APC, MPC, or HPC) in order to maximise its performance measure. Figure 2 introduces schematically the expanded PC game together with its related subgames.

Figure 2 asserts that the PC game is composed of 16 subgames (4 APCs, 4 MPCs, and 8 HPCs). The PC game is seen to be of dimension 4 blocks $\times 4$ blocks such that each subgame of the four alternative subgames appears only once in its block row and once in its block column. Two versions of the HPC subgames are identified. They are the Hybrid (version H1) and the Hybrid (version H2). Figure 3 helps in distinguishing between the two versions H1 and H2.

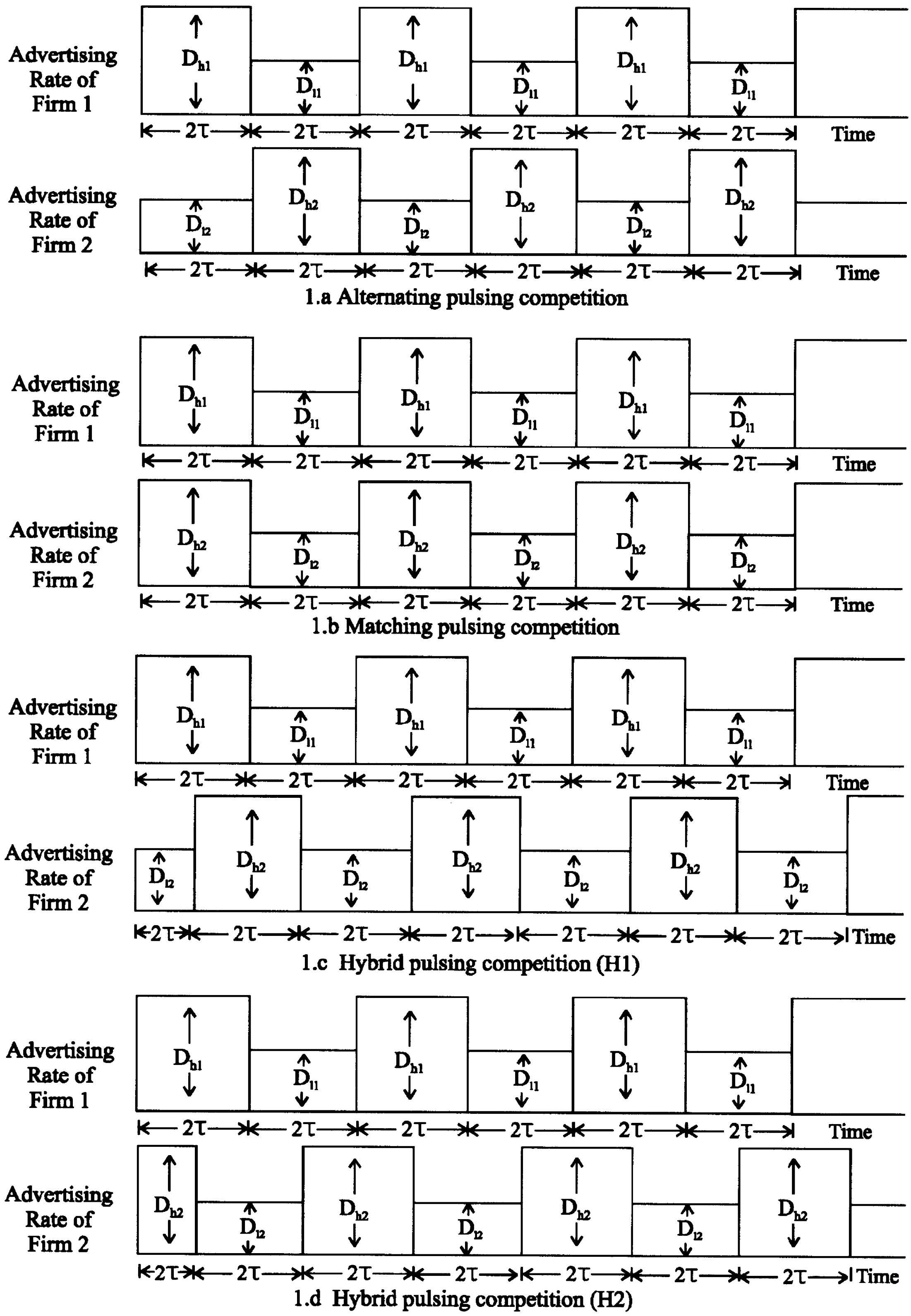

Figure 1 Competitive advertising pulsation in a duopoly.

Firm 2

Firm 1	(H, H, L, L)	(L, L, H, H)	(L, H, H, L)	(H, L, L, H)
(H, H, L, L)	Matching	Alternating	Hybrid (H1)	Hybrid (H2)
(L, L, H, H)	Alternating	Matching	Hybrid (H2)	Hybrid (H1)
(L, H, H, L)	Hybrid (H2)	Hybrid (H1)	Matching	Alternating
(H, L, L, H)	Hybrid (H1)	Hybrid (H2)	Alternating	Matching

Figure 2 Alternative subgames of the PC game.

Regarding Figure 3a in which firm 1 cycles according to the sequence H, H, L, L and firm 2 cycles according to the sequence L, H, H, L or the sequence H, L, L, H related to the first block row in Figure 2, the versions Hybrid (H1) and Hybrid (H2) are identified as such because for H1 firm 1 leads in time firm 2 in clockwise cycling, whereas for H2 firm 1 lags in time behind firm 2 in clockwise cycling. A similar procedure has also been employed in identifying the H1 and H2 versions of the HPC subgames in conjunction with Figures 3b, 3c, and 3d related to the second, third and fourth block rows in Figure 2. We show in the next section that the payoff to firm 1 (or firm 2) under subgame H1 is different from that under subgame H2.

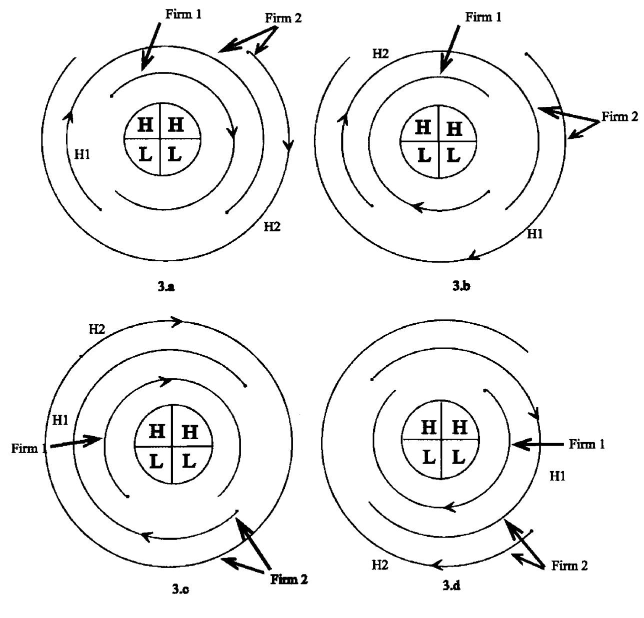

Figure 3 Identification of H1 and H2 versions of HPC subgame.

Comparison of subgames' payoffs

As in Mesak and Calloway,[6] we consider the situation of two firms competing in a single market of a constant size m where competitive behavior in the market is assumed to be governed by a continuous Lanchester model proposed in Little.[14]

Lanchester model

Let S_1 = sales rate of firm 1 (\$ /unit time), S_2 = sales rate of firm 2 (\$ /unit time), u_1 = advertising rate of firm 1 (\$ / unit time), and u_2 = advertising rate of firm 2 (\$ /unit time). The change in the sales rates (dS_1/dt) and (dS_2/dt) are expressed by two general first order linear differential equations:

$$(dS_1/dt) = -(dS_2/dt) = f_1(u_1)S_2 - f_2(u_2)S_1, \qquad (1)$$

where $f_1(u_1)$ and $f_2(u_2)$ are the advertising response functions of firms 1 and 2 respectively. Each response function can be concave, linear, or convex. Expression (1) is a modification over Kimball's[15] earlier formulation that assumes each response function to be linear in advertising. Expression (1) implies that a firm gains sales proportional to its advertising response function and the sales of its rival. Moreover, the firm is losing sales proportional to its sales and the advertising response function of its rival.

The assumption of a constant market of size m is plausible for products in the maturity stage of their product life cycles (Gruca *et al*[16]; Erickson[17]) and implies for the Lanchester model that

$$S_1 + S_2 = m. \qquad (2)$$

The steady state sales rates r_1 and r_2 can be obtained through equating $(dS_1/dt) = (dS_2/dt) = 0$, then solving (1) and (2) simultaneously for S_1 and S_2 to obtain:

$$r_1(u_1, u_2) = m - r_2(u_1, u_2) = \frac{mf_1(u_1)}{f_1(u_1) + f_2(u_2)}. \qquad (3)$$

Using (2) and (3), it can be shown that for rectangular or uniform advertising policies, (1) takes the following form:

$$(dS_1/dt) = -(dS_2/dt) = (f_1(u_1) + f_2(u_2))[r_1(u_1, u_2) - S_1]. \qquad (4)$$

Sales response over time $S_1(t)$ and $S_2(t)$ are obtained through solving the differential equation in (4) for certain given initial conditions $S_1(t_0) = S_{1,0}$ and $S_2(t_0) = S_{2,0}$ at time $t = t_0$.

Sales response to advertising pulsation

Figure 4 illustrates a situation for two firms engaged in a Hybrid Pulsing Competition (HPC), version H1, according to Figure 3a.

When the three advertising policies UAP, APMP, and APP are specified to have the same undiscounted total advertising budget in a cycle of length 4τ, then for any given average rate D_j of advertising for firm j given by:

$$\frac{1}{4\tau}\int_0^{4\tau} U_j(t)dt = D_j, \qquad j = 1, 2,$$

where $U_j(t)$ is the uniform advertising rate at time t, implies that

$$\frac{1}{2}(D_{hj} + D_{lj}) = D_j, \qquad j = 1, 2, \tag{5}$$

and D_{hj} and D_{lj} are the high and low advertising rates of each firm $j; j = 1, 2$. Since all advertising policies are assumed to cost the same for a given firm, then maximising the undiscounted profit in the long-run would be equivalent to maximising sales revenues in the long-run as such. It can be shown that the sales at the end of cycle n for firm 1, $S_{1,n+1}$, is related to the initial sales in the cycle, $S_{1,n}$, through the following expression:

$$S_{1,n+1} = KS_{1,n} + C, \tag{6}$$

where K and C are constants such that $0 < K < 1$ and $C > 0$. According to (6), and as in Mesak and Calloway,[6]

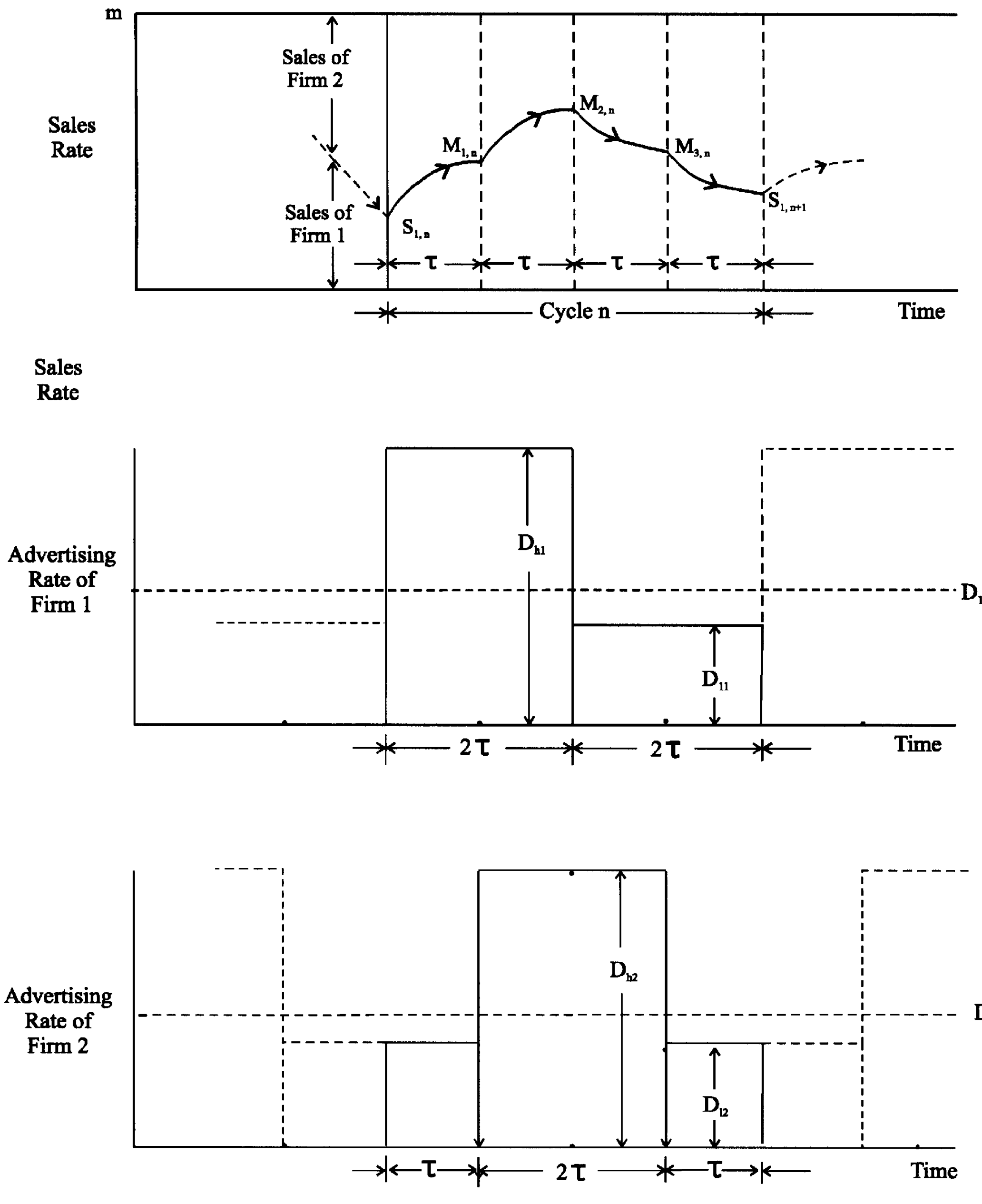

Figure 4 Sales response to advertising pulsation.

sales of both firms arrive at a quasi-steady state in which each cycle starts and ends with the same level of sales for each firm independent of the initial sales values. This finding is quite general and is applicable for all the other competitive subgames. This finding may also be conceived as a generalisation of a similar result by Sasieni[3] from a monopoly to a duopoly. Here, as in Sasieni[3] the term 'quasi-steady state' is used in realisation of the fact that sales over the steady state cycle are not constant. The above finding, based on (6), gives rise to the following result, the proof of which is found in Mesak and Calloway.[6]

Result 1 For an infinite planning horizon, the long-run mean sales rate of a firm is independent of the initial level of sales rate, and is equal to the mean sales rate over the steady state cycle.

The above result is useful in providing the long-run average sales revenues of firm 1 per unit time for an infinite planning horizon associated with the HPC (version H1), HPC (version H_2), APC, and MPC subgames, designated by R_{1h1}, R_{1h2}, R_{1a}, and R_{1m} respectively. The result implies that in order to derive the above stated measure of performance, one only needs to analyze sales response to advertising over the finite period of the steady state cycle rather than doing the same over an infinite period of time in which sales configurations are dissimilar for different cycles. For notational convenience, we denote the advertising response $f_1(D_{h1}) = f_{h1}$, and $f_1(D_{l1}) = f_{l1}$ for firm 1. The terms f_{h2} and f_{l2} for firm 2 are designated in a similar manner. Regarding expression (3), we further designate $r_1(D_{h1}, D_{h2})$ by $r_1(h, h), r_1(D_{h1}, D_{l2})$ by $r_1(h, l), r_1(D_{l1}, D_{h2})$ by $r_1(l, h)$, and $r_1(D_{l1}, D_{l2})$ by $r_1(l, l)$. We are now in a position to introduce the following expressions for R_{1h1}, R_{1h2}, R_{1a}, and R_{1m} for firm 1.

$$
\begin{aligned}
(4\tau)R_{1hl} = {} & [r_1(h,l) + r_1(h,h) + r_1(l,h) + r_1(l,l)\tau] \\
& + \frac{r_1(h,l)(1-e^{-\alpha_{hl}\tau})}{(1-e^{-\alpha\tau})}\Bigg[-\frac{(1-e^{-(\alpha_{hh}+\alpha_{lh}+\alpha_{ll})\tau})}{\alpha_{hl}} \\
& + \frac{(1-e^{-\alpha_{hh}\tau})}{\alpha_{hh}} + \frac{(1-e^{-\alpha_{lh}\tau})e^{-\alpha_{hh}\tau}}{\alpha_{lh}} \\
& + \frac{(1-e^{-\alpha_{ll}\tau})e^{-(\alpha_{lh}+\alpha_{hh})\tau}}{\alpha_{ll}}\Bigg] \\
& + \frac{r_1(h,h)(1-e^{-\alpha_{hh}\tau})}{(1-e^{-\alpha\tau})}\Bigg[\frac{(1-e^{-\alpha_{hl}\tau})e^{-(\alpha_{lh}+\alpha_{ll})\tau}}{\alpha_{hl}} \\
& - \frac{(1-e^{-(\alpha_{hl}+\alpha_{lh}+\alpha_{ll})\tau})}{\alpha_{hh}} \\
& + \frac{(1-e^{-\alpha_{lh}\tau})}{\alpha_{lh}} + \frac{(1-e^{-\alpha_{ll}\tau})e^{-\alpha_{lh}\tau}}{\alpha_{ll}}\Bigg] \\
& + \frac{r_1(l,h)(1-e^{-\alpha_{lh}\tau})}{(1-e^{-\alpha\tau})}\Bigg[\frac{(1-e^{-\alpha_{hl}\tau})e^{-\alpha_{ll}\tau}}{\alpha_{hl}} \\
& + \frac{(1-e^{-\alpha_{hh}\tau})e^{-(\alpha_{hl}+\alpha_{ll})\tau}}{\alpha_{hh}} \\
& - \frac{(1-e^{-(\alpha_{hh}+\alpha_{hl}+\alpha_{ll})\tau})}{\alpha_{lh}} + \frac{(1-e^{-\alpha_{ll}\tau})}{\alpha_{ll}}\Bigg] \\
& + \frac{r_1(l,l)(1-e^{-\alpha_{ll}\tau})}{(1-e^{-\alpha\tau})}\Bigg[\frac{(1-e^{-\alpha_{hl}\tau})}{\alpha_{hl}} \\
& + \frac{(1-e^{-\alpha_{hh}\tau})e^{-\alpha_{hl}\tau}}{\alpha_{hh}} \\
& + \frac{(1-e^{-\alpha_{lh}\tau})e^{-(\alpha_{hh}+\alpha_{hl})\tau}}{\alpha_{lh}} - \frac{(1-e^{-(\alpha_{hh}+\alpha_{hl}+\alpha_{lh})\tau})}{\alpha_{ll}}\Bigg],
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
(4\tau)R_{1h2} = {} & [r_1(h,h) + r_1(h,l) + r_1(l,l) + r_1(l,h)]\tau \\
& + \frac{r_1(h,h)(1-e^{-\alpha_{hh}\tau})}{(1-e^{-\alpha\tau})}\Bigg[-\frac{(1-e^{-(\alpha_{hl}+\alpha_{ll}+\alpha_{lh})\tau})}{\alpha_{hh}} \\
& + \frac{(1-e^{-\alpha_{hl}\tau})}{\alpha_{hl}} \\
& + \frac{(1-e^{-\alpha_{ll}\tau})e^{-\alpha_{hl}\tau}}{\alpha_{ll}} + \frac{(1-e^{-\alpha_{lh}\tau})e^{-(\alpha_{ll}+\alpha_{hl})\tau}}{\alpha_{lh}}\Bigg] \\
& + \frac{r_1(h,l)(1-e^{-\alpha_{hl}\tau})}{(1-e^{-\alpha\tau})}\Bigg[\frac{(1-e^{-\alpha_{hl}\tau})e^{-(\alpha_{ll}+\alpha_{lh})\tau}}{\alpha_{hh}} \\
& - \frac{(1-e^{-(\alpha_{hh}+\alpha_{ll}+\alpha_{lh})\tau})}{\alpha_{hl}} \\
& + \frac{(1-e^{-\alpha_{ll}\tau})}{\alpha_{ll}} + \frac{(1-e^{-\alpha_{lh}\tau})e^{-\alpha_{ll}\tau}}{\alpha_{lh}}\Bigg] \\
& + \frac{r_1(l,l)(1-e^{-\alpha_{ll}\tau})}{(1-e^{-\alpha\tau})}\Bigg[\frac{(1-e^{-\alpha_{hh}\tau})e^{-\alpha_{lh}\tau}}{\alpha_{hh}} \\
& + \frac{(1-e^{-\alpha_{hl}\tau})e^{-(\alpha_{hh}+\alpha_{lh})\tau}}{\alpha_{hl}} \\
& - \frac{(1-e^{-(\alpha_{hl}+\alpha_{hh}+\alpha_{lh})\tau})}{\alpha_{ll}} + \frac{(1-e^{-\alpha_{lh}\tau})}{\alpha_{lh}}\Bigg] \\
& + \frac{r_1(l,h)(1-e^{-\alpha_{lh}\tau})}{(1-e^{-\alpha\tau})}\Bigg[\frac{(1-e^{-\alpha_{hh}\tau})}{\alpha_{hh}} \\
& + \frac{(1-e^{-\alpha_{hl}\tau})e^{-\alpha_{hh}\tau}}{\alpha_{hl}} \\
& + \frac{(1-e^{-\alpha_{ll}\tau})e^{-(\alpha_{hl}+\alpha_{hh})\tau}}{\alpha_{ll}} - \frac{(1-e^{-(\alpha_{hl}+\alpha_{hh}+\alpha_{ll})\tau})}{\alpha_{lh}}\Bigg],
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
(4\tau)R_{1a} = {} & 2[r_1(h,l) + r_1(l,h)]\tau \\
& + \frac{[r_1(h,l) - r_1(l,h)](1-e^{-2\alpha_{hl}\tau})(1-e^{-2\alpha_{lh}\tau})}{(1-e^{-2(\alpha_{hl}+\alpha_{lh})\tau})} \\
& \times \left(\frac{1}{\alpha_{lh}} - \frac{1}{\alpha_{hl}}\right),
\end{aligned} \tag{9}
$$

and

$$(4\tau)R_{1m} = 2[r_1(h,h) + r_1(l,l)\tau + \frac{[r_1(h,h) - r_1(l,l)](1 - e^{-2\alpha_{hh}\tau})(1 - e^{-2\alpha_{ll}\tau})}{(1 - e^{-2(\alpha_{hh}+\alpha_{ll})\tau})} \times \left(\frac{1}{\alpha_{ll}} - \frac{1}{\alpha_{hh}}\right), \quad (10)$$

$\alpha_{hl} = f_{h1} + f_{l2}$, $\alpha_{hh} = f_{h1} + f_{h2}$, $\alpha_{lh} = f_{l1} + f_{h2}$, $\alpha_{ll} = f_{l1} + f_{l2}$, $\alpha = \alpha_{hl} + \alpha_{hh} + \alpha_{lh} + \alpha_{ll}$, and from (3), $r_1(h,\ l) = mf_{h1}/\alpha_{hl}$, $r_1(h,\ h) = mf_{h1}/\alpha_{hh}$, $r_1(l,\ h) = mf_{l1}/\alpha_{lh}$, and $r_1(l,\ l) = mf_{l1}/\alpha_{ll}$.

Based on expressions (7) through (10), we are in a position to introduce the following self-explanatory results for the focal firm, firm 1.

Result 2 R_1 (Match) $\geqslant R_1$ (Alter) if $f_{h1} + f_{l1} \geqslant f_{h2} + f_{l2}$.

Result 3 R_1(Hybrid 2) $\geqslant R_1$(Hybrid1).

Result 2 asserts that in response to the pulsation policy of firm 2, firm 1 should prefer a matching (alternating) pulsation policy over the alternating (matching) counterpart if its *attraction power,* $f_{h1} + f_{l1}$, is larger (smaller) than the attraction power of its competitor. When the attraction power of both firms are equal, however, firm 1 would be indifferent between the two pulsation policies. Result 3, on the other hand asserts that firm 1 would consistently prefer to engage with its rival in a HPC (version H2) subgame rather than a HPC (version H1) subgame irrespective of the relative attraction power of both parties. It is interesting to mention that the equality signs in results 2 and 3 would only hold if firm 2 employs a Uniform Advertising Policy [see result (iii) in Mesak and Calloway.[6]

Problem formulation

Designating the policy parameter for firm 1 by x and that for firm 2 by y, both $0 \leqslant x, y \leqslant 1$, such that $D_{l1} = xD_1$ and $D_{l2} = yD_2$, then the different advertising policies that could be employed by firms 1 and 2 can be characterised as follows: UAP is characterised by x (or y) $= 1$. APMP is characterized by $0 < x$ (or y) < 1 and APP is characterised by x (or y) $= 0$.

At this point, the PC game considered in this article may be formulated as follows. For exogenously arbitrarily determined values of the quantities D_1, D_2 and τ in a market of a fixed size m, what is the optimal value x^* chosen by firm 1 and the optimal value y^* chosen by firm 2 together with the optimal sequence of pulsation for each (H,H,L,L), (L,L,H,H), (L,H,H,L), or (H,L,L,H) provided that the payoff to firm 1 is represented by (7), (8), (9), or (10) for every policy pair (x, y) such that $0 \leqslant x, y \leqslant 1$? This problem formulation is within the context of a special class of infinite two person, zero-sum games known in game theory as *Games on the Square.* Though we deal with a constant-sum game, such a game can be converted to a zero-sum game by subtracting the quantity $m/2$ from the payoff R_1 (usually called the *Kernel*). Early treatments of such games for specific situations are found in McKinsey[18] and Karlin.[19]

As in Mesak and Calloway[6] assuming that each of the two firms strive to make its worst possible outcome as good as possible (Von Neumann-Morgenstern behavior), the *Maxmin–Minimax* solution concept would be appropriate in analysing such a competitive situation (more details regarding that solution concept may be found in Taha).[20] Therefore, the above mentioned solution concept will be adopted throughout.

The situation for which the subgames MPC, APC, HPC(H1), and HPC(H2) have saddle points at the same corner can be dealt with analytically. Mesak and Calloway[7] found in their empirical study that such a situation is practically relevant for the MPC and APC subgames analyzed in Mesak and Calloway[6] study (see also Figure 6). Defining the game G by the (4 × 4) matrix the entries of which represent the values of the subgames composed of the PC game shown in Figure 2, then the following important result is obtained as a direct consequence of a theorem by Sherman.[21]

Result 4 In a game of pulsing competition for which the related subgames posses saddle points at the same corner:

(i) The value of the PC game is equal to the value of game *G*.

(ii) The optimal strategy for a given firm (pure or mixed) is the same as that derived from game *G*, distributed over the policies associated with the saddle points in the PC game.

To enhance the understanding of result 4 and appreciate its implications, let us consider an example. Assume that all the subgames of the PC game have saddle points at the corner (0, 0) and v_m, v_a, v_{h1}, and v_{h2} represent the values at the saddle points for firm 1 related to the subgames MPC, APC, HPC(H1), and HPC(H2) respectively. In reference to

		Firm 2			
		H-H-L-L	L-L-H-H	L-H-H-L	H-L-L-H
Firm 1	H-H-L-L	v_m	v_a	v_{h1}	v_{h2}
	L-L-H-H	v_a	v_m	v_{h2}	v_{h1}
	L-H-H-L	v_{h2}	v_{h1}	v_m	v_a
	H-L-L-H	v_{h1}	v_{h2}	v_a	v_m

Figure 5 Elements of game G.

Figure 2, game G in this case would be as shown in Figure 5.

It can be shown that an analysis of game G produces an optimal mixed strategy $(0.25, 0.25, 0.25, 0.25)$ for each firm. The value of game G would be given by $(v_m + v_a + v_{h1} + v_{h2})/4$. According to result 4, the PC game would have the same value for game G shown above and each of the two firms would use APP policies of sequences (H-H-L-L), (L-L-H-H), (L-H-H-L) and (H-L-L-H), 25% of the time.

For the situation in which the subgames have no saddle points at the same corner, numerical methods are called upon to arrive at a solution. This is done by first discretising the policy variables x and y and therefore converting the PC

Firm 2

Firm 1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	349.04	350.54	351.65	352.57	353.31	353.92	354.41	354.78	355.04	355.19	355.23
0.1	342.69	344.36	345.62	346.63	347.47	348.15	348.69	349.11	349.40	349.57	349.62
0.2	335.40	337.26	338.66	339.80	340.74	341.50	342.11	342.58	342.90	343.09	343.15
0.3	327.14	329.22	330.78	332.05	333.10	333.96	334.64	335.16	335.52	335.74	335.81
0.4	317.97	320.29	322.03	323.45	324.62	325.57	326.33	326.91	327.32	327.56	327.64
0.5	308.14	310.70	312.63	314.19	315.49	316.55	317.39	318.03	318.48	318.75	318.84
0.6	298.10	300.90	303.01	304.73	306.15	307.31	308.23	308.94	309.43	309.73	309.83
0.7	288.60	291.62	293.89	295.75	297.28	298.54	299.54	300.30	300.84	301.16	301.27
0.8	280.62	283.82	286.23	288.20	289.83	291.16	292.22	293.04	293.61	293.95	294.06
0.9	275.25	278.57	281.07	283.12	284.81	286.19	287.29	288.14	288.73	289.09	289.20
1.0	273.35	276.71	279.24	281.31	283.03	284.43	285.54	286.40	287.00	287.36	287.48

a. Payoff matrix of MPC subgame (R_{1m})

Firm 2

Firm 1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	348.92	350.43	351.56	352.48	353.24	353.86	354.36	354.74	355.01	355.18	355.23
0.1	342.59	344.27	345.53	346.56	347.41	348.10	348.65	349.08	349.37	349.55	349.62
0.2	335.31	337.19	338.59	339.74	340.69	341.46	342.08	342.55	342.88	343.08	343.15
0.3	327.07	329.15	330.72	332.00	333.06	333.92	334.61	335.14	335.51	335.73	335.81
0.4	317.92	320.24	321.98	323.40	324.58	325.54	326.30	326.89	327.31	327.55	327.64
0.5	308.10	310.66	312.59	314.16	315.46	316.52	317.37	318.02	318.47	318.75	318.84
0.6	298.07	300.87	302.98	304.70	306.12	307.29	308.22	308.93	309.43	309.73	309.83
0.7	288.57	291.60	293.87	295.73	297.27	298.52	299.53	300.29	300.83	301.16	301.27
0.8	280.60	283.81	286.22	288.19	289.82	291.15	292.22	293.03	293.61	293.95	294.06
0.9	275.24	278.56	281.06	283.11	284.80	286.18	287.29	288.13	288.73	289.08	289.20
1.0	273.35	276.71	279.24	281.31	283.03	284.43	285.54	286.40	287.00	287.36	287.48

b. Payoff matrix of APC subgame (R_{1a})

Firm 2

Firm 1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	347.61	349.23	350.48	351.53	352.42	353.18	353.81	354.33	354.74	355.04	355.24
0.1	341.29	343.08	344.46	345.62	346.59	347.42	348.10	348.66	349.10	349.42	349.62
0.2	334.03	336.01	337.54	338.81	339.88	340.79	341.53	342.14	342.61	342.95	343.15
0.3	325.81	328.01	329.69	331.09	332.27	333.26	334.08	334.74	335.24	335.60	335.81
0.4	316.72	319.14	321.00	322.54	323.83	324.91	325.80	326.51	327.05	327.43	327.64
0.5	306.99	309.65	311.68	313.36	314.77	315.94	316.90	317.67	318.24	318.63	318.84
0.6	297.10	299.98	302.18	304.00	305.51	306.78	307.80	308.62	309.22	309.62	309.83
0.7	287.78	290.87	293.22	295.15	296.77	298.11	299.19	300.04	300.67	301.07	301.27
0.8	280.04	283.29	285.75	287.78	289.47	290.86	291.98	292.85	293.49	293.89	294.06
0.9	274.95	278.29	280.82	282.90	284.62	286.03	287.16	288.04	288.67	289.05	289.20
1.0	273.35	276.71	279.24	281.31	283.03	284.43	285.54	286.40	287.00	287.36	287.48

c. Payoff matrix of HPC (H1) subgame (R_{1h1})

Firm 2

Firm 1	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	350.35	351.74	352.73	353.52	354.14	354.61	354.96	355.20	355.32	355.33	355.24
0.1	343.99	345.56	346.69	347.58	348.29	348.84	349.24	349.52	349.68	349.71	349.62
0.2	336.68	338.45	339.72	340.74	341.55	342.18	342.66	342.98	343.17	343.23	343.15
0.3	328.39	330.37	331.81	332.97	333.89	334.62	335.17	335.56	335.79	335.87	335.81
0.4	319.17	321.39	323.01	324.32	325.37	326.20	326.84	327.29	327.57	327.69	327.64
0.5	309.25	311.72	313.54	315.00	316.18	317.13	317.86	318.38	318.72	318.87	318.84
0.6	299.07	301.79	303.81	305.44	306.76	307.82	308.64	309.25	309.64	309.83	309.83
0.7	289.39	292.35	294.55	296.33	297.78	298.96	299.87	300.56	301.01	301.25	301.27
0.8	281.18	284.34	286.70	288.61	290.19	291.46	292.46	293.22	293.73	294.01	294.06
0.9	275.54	278.84	281.31	283.33	284.99	286.34	287.42	288.23	288.79	289.12	289.20
1.0	273.35	276.71	279.24	281.31	283.03	284.43	285.54	286.40	287.00	287.36	287.48

d. Payoff matrix of HPC (H2) subgame (R_{1h2})

Figure 6 Payoff matrices of four subgames.

infinite game to a finite one that would be easier to solve. Upon formulating the game as a linear programming problem afterwards, the optimal strategies and the value of the PC game can be obtained. The method will be discussed in detail in the next numerical section of the paper.

A copycat game in advertising competition

In an industry that is composed of, or dominated by, two competitors of approximately equal market shares and producing products that have a lot of attributes in common such as Coke and Pepsi in the soft drink industry, their advertising budgets and advertising competitive behavior tend to be also similar. We designate such firms by symmetric (or identical) firms. Following Schmalensee,[22] the two firms are assumed to have the same production costs, to be able to acquire real promotion on the same terms, to charge the same fixed price, and to face symmetric demand functions.

In this section, we consider the situation of two identical firms competing according to a variant structure of the PC game depicted in Figure 2. Again by identical competing firms, we also mean that the two firms possess similar advertising budgets, similar advertising response functions, and the magnitudes of their low and high advertising levels are also similar. Although their advertising pulsation policies are considered to be 'copycats' of each other (me-too kind of behavior), they are *only* allowed to be different in terms of the timing of their advertising pulsation sequence. Accordingly, for such a game we may write $f_{l1} = f_{l2} = f_l$ and $f_{h1} = f_{h2} = f_h$. The structure of the copycat PC game dictates that the payoffs are only relevant along the main diagonals of each of the 16 subgames shown in Figure 2.

We introduce below a result related to a PC game for symmetric firms prior of introducing two more results related to the copycat game. This is performed first as the proofs of the copycat game results are highly dependent on the findings of such results.

Result 5 In a two person, zero-sum game for symmetric firms

(i) $R_{1m}(x, y) = -R_{1m}(y, x)$ for $x \neq y$, and $R_{1m}(x, y) = 0$, for $x = y$.
(ii) $R_{1a}(x, y) = -R_{1a}(y, x)$ for $x \neq y$, and $R_{1a}(x, y) = 0$, for $x = y$.
(iii) $R_{1h1}(x, y) = -R_{1h2}(y, x)$, for all $0 \leqslant x, y \leqslant 1$.

Findings 5(i) and 5 (ii) imply that both the Kernels R_{1m} and R_{1a} are skew-symmetric and take on zero values along the main diagonal of the square $0 \leqslant x, y \leqslant 1$. An interesting consequence of the above is that the values of the MPC and APC subgames are zeros, which is a well-known result in game theory. The outcome 5(iii) implies that the Kernel R_{1h2} is derived from the Kernel R_{1h1} through a rather simple mathematical manipulation. We are now in a position to introduce the following findings for a copycat game.

Result 6 For a copycat game in advertising competition

(i) $R_{1h1}(x, y) \leqslant R_{1a}(x, y) = R_{1m}(x, y) = m/2 \leqslant R_{1h2}(x, y),\ 0 \leqslant x = y \leqslant 1$
(ii) $R_{1h1}(x, y) + R_{1h2}(x, y) = m, 0 \leqslant x = y \leqslant 1$

Based on the findings of result 6, firm 1 would prefer to engage in a HPC (version H2) subgame in response to the pulsation policy of its rival. Obviously, from 6(ii), $R_{1h1} \leqslant m/2$ and $R_{1h2} \geqslant m/2$. In addition, the equality of R_{1a} to R_{1m} is a direct consequence of result 2 reported earlier. It should be further noted that the inequalities in result 6(i) hold as strict equalities only for the situation in which both firms employ similar UAP policies ($x = y = 1$).

Based on the findings of result 6 and upon employing a maxmin-minimax solution concept, the solution of the copycat advertising game is introduced below.

Result 7 Irrespective of the shape of the advertising response functions, UAP is the optimal policy for both firms competing in a copycat advertising game.

The findings of result 7 are rather striking. They imply, for the first time in the literature, that it is possible for the policy of constant spending over time (UAP) to be optimal for both firms even if their response functions are convex. Such an interesting outcome is mainly attributed to the special properties of the copycat advertising game that result in a peculiar structure for its related payoff matrix. More importantly, the solution of the copycat advertising game would not have been unique had the hybrid competition subgames not been included as integral parts of the PC game as the proof of result 7 is highly dependent on the findings reported in result 6.

The next section illustrates how to solve the PC game in its general form using linear programming and shows how to use maxmin-minimax principle to solve the copycat advertising game.

A numerical investigation

The goals of this section are two-fold: (1) to illustrate how to solve the PC game in its general form numerically using linear programming; and (2) to demonstrate how to obtain a numerical solution for the advertising copycat game introduced in the third section.

As in Mesak and Calloway,[7] the following advertising response function is employed in conducting the numerical analysis pertaining to this section:

$$f_j(u_j) = a_j + b_j u_j^{\delta_j}, a_j, b_j, \delta_j > 0, \qquad j = 1, 2, \qquad (11)$$

where

a_j = is the value of the response function at zero advertising, conceived to represent a measure of 'consumer franchise' or habitual behavior,

b_j = is the measure of advertising effectiveness, and

δ_j = is a measure of the degree of convexity (concavity) of the response function.

The function (11) is concave for $0 < \delta_j < 1$, linear for $\delta_j = 1$, and convex for $\delta_j > 1$.

As has been mentioned earlier, define the policy parameter for firm 1 by x and that for firm 2 by y, both $0 \leqslant x, y \leqslant 1$, such that $D_{l1} = xD_1$ and $D_{l2} = yD_2$, then the different advertising policies that could be employed by firms 1 and 2 can be characterized as follows: UAP is characterized by x (or y) = 1. APMP is characterized by $0 < x$ (or y) < 1 and APP is characterized by x (or y) = 0. For any given $0 \leqslant x, y \leqslant 1, D_1$ and D_2, the high advertising levels are uniquely determined from expression (5).

Assume a market of annual size $m = 800$ million dollars, $D_1 = D_2 = 5$ million dollars, $\tau = 1$ month (0.083 year), $a_1 = a_2 = 0.01, b_1 = b_2 = 0.005$, $\delta_1 = 2.50$, and $\delta_2 = 1.25$. The rationale governing the assignment of the values of m, D_1, D_2, and τ is found in Mesak and Calloway.[7] The chosen values of the parameters a_1, a_2, b_1, and b_2 are within the ranges of the estimated parameters obtained for the application reported in Table 1 in Mesak and Calloway.[7] For different values of x and y ranging between 0 and 1 in increments of 0.1, expressions (7) through (10) were computed using expressions (5) and (11). Due to space limitations, Figure 6 illustrates the obtained payoff matrix for the basic four subgames of the PC game shown schematically in Figure 2. It is noteworthy to mention at this point that the policy variables x and y have been described in the above manner in order to convert the PC infinite game to a finite one that is generally easier to solve.

Table 1 Relevant payoffs of the copycat game

	Subgame					
Policy variable $x = y$	*Matching* R_{1m}	*Alternating* R_{1a}	*Hybrid H1* R_{1h1}	*Hybrid H2* R_{1h2}	*Min*	*Max*
0.0	0.00	0.00	−1.19	1.19	−1.19	1.19
0.1	0.00	0.00	−1.02	1.02	−1.02	1.02
0.2	0.00	0.00	−0.87	0.87	−0.87	0.87
0.3	0.00	0.00	−0.72	0.72	−0.72	0.72
0.4	0.00	0.00	−0.57	0.57	−0.57	0.57
0.5	0.00	0.00	−0.44	0.44	−0.44	0.44
0.6	0.00	0.00	−0.30	0.30	−0.30	0.30
0.7	0.00	0.00	−0.18	0.18	−0.18	0.18
0.8	0.00	0.00	−0.09	0.09	−0.09	0.09
0.9	0.00	0.00	−0.02	0.02	−0.02	0.02
1.0	0.00	0.00	0.00	0.00	0.00	0.00

LP solution of the pulsing game

To formulate the LP program related to the above PC game, let

$i = 1, 2, \ldots, 11$ designate the strategy of firm 1 when $x = 0, 0.1, \ldots, 1$ respectively if the firm employs its H-H-L-L pulsing sequence.

$i = 12, 13, \ldots, 22$ designate the strategy of firm 1 when $x = 0, 0.1, \ldots, 1$ respectively if the firm employs its L-L-H-H pulsing sequence.

$i = 23, 24, \ldots, 33$ designate the strategy of firm 1 when $x = 0, 0.1, \ldots, 1$ respectively if the firm employs its L-H-H-L pulsing sequence.

$i = 34, 35, \ldots, 44$ designate the strategy of firm 1 when $x = 0, 0.1, \ldots, 1$ respectively if the firm employs its H-L-L-H pulsing sequence.

j = strategy of firm 2, $j = 1, 2, \ldots, 44$ defined in a similar way as i.

$R_{1m}(i, j), R_{1a}(i, j), R_{1h1}(i, j), R_{1h2}(i, j)$ = payoff to firm 1 when it chooses strategy i and firm 2 chooses strategy j and the two firms compete in the subgames MPC, APC, HPC (H1), and HPC(H2) respectively.

P_i = probability of firm 1 choosing strategy i.

W_j = probability of firm 2 choosing strategy j.

Having defined the above terms, the details of the related LP are found in the Appendix. Upon solving the LP problems (A1) and (A2) included in the Appendix, using the commercial software package LP88 and a personal computer, it is found that $P_1^* = P_{12}^* = P_{23}^* = P_{34}^* = 0.25$ and $W_1^* = W_{12}^* = W_{23}^* = W_{34}^* = 0.25$ whereas the remaining 80 quantities take on zero values represent one optimal feasible solution. The value of the PC game (entries of the payoff matrix represent deviations from 400) is $V^* = 348.99$. It is interesting to note that the value of the PC game is one-fourth of the sum of the values of the four subgames depicted in Figure 6. This is in conformity with the discussion related to result 4 in conjunction with game G shown in Figure 5.

The above optimal solution implies that the two firms should be engaged in a matching pulsing competitive subgame for which $x^* = y^* = 0$, 25% of the time, alternating pulsing competitive subgame for which $x^* = y^* = 0$, 25% of the time, hybrid pulsing competitive subgame (H1) for which $x^* = y^* = 0$, 25% of the time, and hybrid pulsing competitive subgame (H2) for which $x^* = y^* = 0$, 25% of the time. For all such solutions, $D_{h1} = D_{h2} = 10$ and $D_{l1} = D_{l2} = 0$. The long-run average sales revenue for the first firm is $400 + 348.99 = 748.99$ (equivalent to a market share of 93.62%) while it is 51.01 for the second firm (equivalent to a market share of 6.38%).

It should be noted that the solution for convex advertising response function PC games does not need to be always at $x^* = y^* = 0$. As a matter of fact, when the as and bs have changed to $a_1 = a_2 = 0.25$ and $b_1 = b_2 = 1.5$, the optimal policy of firm 1 turned out to be UAP (for which $x^* = 1$) and the optimal policy of firm 2 turned out to be APP (for which $y^* = 0$). The value of the PC game (V^*) has changed to 300.21. The differences in the optimal solutions are basically attributed to the fact that for the first example, all the subgames shown in Figure 6 have saddle points at the corners ($x = y = 0$). For the second example, however, two of the subgames do not have saddle points (more detail on this point is found in section 1.1 in Mesak and Calloway).[7]

A copycat game example

As discussed in the previous section, the payoffs of an advertising copycat game for two identical firms are derived from a related PC game by simply considering only the main diagonal elements associated with its four subgames.

For $m = 400$, $D_1 = D_2 = 4$, $\tau = 2/12$, $a_1 = a_2 = 0.01$, $b_1 = b_2 = 0.001$, and $\delta_1 = \delta_2 = 2.75$ the relevant payoffs related to firm 1 are shown in Table 1. We observe from Table 1 that $R_{1h1} \leqslant R_{1m} = R_{1a} \leqslant R_{1h2}$, in conformity with result 5. Upon identifying the maximum and minimum of entries lying in the last two columns of Table 1, it becomes evident upon applying the maxmin-minimax solution concept that the optimal policy is a UAP for both firms ($x^* = y^* = 1$). This is the case despite the convexity of the advertising response functions, in conformity with result 7. We note that the PC game has four subgames, each repeated four times (see Figure 2). Therefore, there should be four equal values for each entry appearing in each row of Table 1. Obviously, including one value of each should be sufficient for our analysis.

Realising that the payoffs reported in Table 1 are deviations from m/2, the long-run average sales revenue for each firm at equilibrium is 200 for the analysed copycat game. We reiterate that the solution of the copycat game would have not been unique had the hybrid subgames not been included in the PC game (main diagonal elements of the MPC and APC subgames are all zeros).

Summary and discussion

Advertising plays a key role in the struggle for success in competitive markets. The issues of allocating the advertising budget over time and its relative timing with respect to its main rival(s) are of managerial relevance to any firm.

This paper extends the earlier works of Mesak and Calloway[6,7] by introducing and analyzing, for the first time in the literature, the hybrid pulsing competition (HPC) subgames in addition to the MPC and APC subgames considered previously. Based on the finding summarised in result 1, the article reports several interesting relationships among the payoffs related to the four subgames mentioned above. These relationships are summarised in results 2, 3, and 5.

The solution of a special case of practical relevance related to the PC game is documented in result 4.

This article, also for the first time, introduces and solves a copycat game (or me-too kind of game) in advertising competition. Based on analytical findings summarised in results 6 and 7, the uniform advertising policy UAP is found to be the optimal policy for the game, irrespective of the shape of the advertising response functions of the rivals. The managerial implications of such findings are quite clear. Each firm should spend its advertising budget evenly over time without any need to identify the shape of its advertising response function or that of the competition.

The copycat game would be appropriate in modeling competitive situations for which the two rivals share a lot of relevant things in common. Plausible examples in this regard include the rivalry between Coke and Pepsi in the soft-drink industry and the competition between two major candidates running for an office, where both belong to the same political party and have successfully raised similar funds for their election campaigns. As pointed out by Coyte and Landon,[23] the relationship between political parties and firms is not as obtuse as it may at first seem. The goal of parties, like firms, is to sell a product using a constrained supply of resources.

Using linear programming (LP), this study further illustrated how to numerically solve the game of pulsing competition (PC) for which the hybrid subgames are integral subgames. A numerical example illustrating the procedure of solving the advertising copycat game has been also demonstrated.

Following what have been previously reported in Mesak and Calloway,[7] future research directions may consider: (1) relaxing the assumption of a market with fixed size; (2) incorporating other marketing mix variables in the modeling effort; (3) proposing a stochastic relationship between sales and advertising; (4) dealing with asymmetric structure response of sales to an increase versus decrease in advertising level; (5) treating the advertising budget for each firm as a decision variable; (6) applying a measure of performance for which a discount factor is used; and (7) incorporating more than two firms in the game of pulsing competition.

One immediate research direction that is closely related to the work reported herein would be to study in more depth the properties of a generalized copycat game for which only the policy variables x and y are required to be similar, whereas advertising budgets or the advertising response functions are not required to be similar for both firms. Introducing and proving theorems regarding the existence, uniqueness and characterization of solutions in pure or mixed strategies for such games would offer a significant contribution to the literature. Another important direction for future research is to incorporate production cost in the modeling effort as a function of sales or the advertising levels being alternated. It would be interesting to find out how sensitive are the conclusions arrived at in this paper to variations in the manufacturing cost. The review article of Eliashberg and Steinberg[24] discussing the interface between the marketing and production functions should be helpful in this respect.

Appendix

Formulating the pulsation game as a linear program

Designating the value of the PC game by V and denoting $V' = 1/V, P_i' = P_i/V$ and $W_j' = W_j/V$, the LP of firm 1 is formulated as follows:

$$\min V' = \sum_{i=1}^{44} P_i'$$

subject to

$$\sum_{i=1}^{11} R_{1m}(i,j)P_i' + \sum_{i=12}^{22} R_{1a}(i,j)P_i' + \sum_{i=23}^{33} R_{1h2}(i,j)P_i' + \sum_{i=34}^{44} R_{1h1}(i,j)P_i' \geqslant 1; \quad j = 1, 2, \ldots, 11;$$

$$\sum_{i=1}^{11} R_{1a}(i,j)P_i' + \sum_{i=12}^{22} R_{1m}(i,j)P_i' + \sum_{i=23}^{33} R_{1h1}(i,j)P_i' + \sum_{i=34}^{44} R_{1h2}(i,j)P_i' \geqslant 1; \quad j = 12, 13, \ldots, 22;$$

$$\sum_{i=1}^{11} R_{1h1}(i,j)P_i' + \sum_{i=12}^{22} R_{1h2}(i,j)P_i' + \sum_{i=23}^{33} R_{1m}(i,j)P_i' + \sum_{i=34}^{44} R_{1a}(i,j)P_i' \geqslant 1; \quad j = 23, 24, \ldots, 33;$$

$$\sum_{i=1}^{11} R_{1h2}(i,j)P_i' + \sum_{i=12}^{22} R_{1h1}(i,j)P_i' + \sum_{i=23}^{33} R_{1a}(i,j)P_i' + \sum_{i=34}^{44} R_{1m}(i,j)P_i' \geqslant 1;$$

$j = 34, 35, \ldots, 44.$

and $P_i' \geqslant 0, \quad i = 1, 2, \ldots, 44.$ (A1)

The LP problem of firm 2 is formulated as follows:

$$\max V' = \sum_{j=1}^{44} W_j'$$

subject to

$$\sum_{j=1}^{11} R_{1m}(i,j)W_j' + \sum_{j=12}^{22} R_{1a}(i,j)W_j' + \sum_{j=23}^{33} R_{1h1}(i,j)W_j' + \sum_{j=34}^{44} R_{1h2}(i,j)W_j' \leqslant 1; \quad i = 1, 2, \ldots, 11;$$

$$\sum_{j=1}^{11} R_{1a}(i,j)W_j' + \sum_{j=12}^{22} R_{1m}(i,j)W_j' + \sum_{j=23}^{33} R_{1h2}(i,j)W_j' + \sum_{j=34}^{44} R_{1h1}(i,j)W_j' \leqslant 1; \quad i = 12, 13, \ldots, 22;$$

$$\sum_{j=1}^{11} R_{1h2}(i,j)W_j' + \sum_{j=12}^{22} R_{1h1}(i,j)W_j' + \sum_{j=23}^{33} R_{1m}(i,j)W_j' + \sum_{j=34}^{44} R_{1a}(i,j)W_j' \leqslant 1; \quad i = 23, 24, \ldots, 33;$$

$$\sum_{j=1}^{11} R_{1h1}(i,j)W_j' + \sum_{j=12}^{22} R_{1h2}(i,j)W_j' + \sum_{j=23}^{33} R_{1a}(i,j)W_j' + \sum_{j=34}^{44} R_{1m}(i,j)W_j' \leqslant 1;$$

$i = 34, 35, \ldots, 44.$

and $W_j' \geqslant 0, \quad j = 1, 2, \ldots, 44.$ (A2)

Upon solving (A1) and (A2), the optimal strategies of firms 1 and 2 are given by $P_i^* = P_i'^*/V'^*$ and $W_j^* = W_j'^*/V'^*$ and the value of the PC game is given by $V^* = 1/V'^*$.

After identifying the related x^* and y^* values, the corresponding low advertising levels $D_{lj}^*, j = 1, 2,$ are given by $D_{l1}^* = x^*D_1$ and $D_{l2}^* = y^*D_2$. The high advertising levels $D_{hj}^*, j = 1, 2,$ are obtained from (5) afterwards as $D_{h1}^* = (2 - x^*)D_1$ and $D_{h2}^* = (2 - y^*)D_2$.

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