A note on commodity contingent valuation

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Received (in revised form): 17th September, 2007

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Practical applications

Commodity contingent valuation typically involves complex procedures, like limit steps, partial differential equations or approximations. These *ad hoc* solutions can only be used in the concrete problem for which they are developed. This paper tries to review and clarify stochastic calculus in the commodity contingent valuation context, simplifying formulae and deductions. This approach allows for a relatively simple pricing of all sorts of financial derivatives on commodity prices, avoiding limit steps, partial differential equations and approximations. Moreover, it has been shown how to obtain precise estimators of prices for NYMEX WTI crude oil futures contracts, in a two-factor context.

Abstract

Given the complex dynamics of commodity prices, most of the papers on the valuation of commodity contingent claims present ad hoc solutions, which are very complex and sometimes include approximations. This paper shows how well-known techniques and results in stochastic calculus can be used to simplify formulae and deductions. Specifically, we show how to obtain more precise estimators of the parameters in the two-factor model by Schwartz than the approximations given by

the author, which tend to overestimate the parameters. These divergences are important in the valuation of commodity contingent claims.

Journal of Derivatives & Hedge Funds (2008) **13**, 311–320. doi:10.1057/palgrave.jdhf.1850079

Keywords: stochastic calculus; commodity prices; mean-reverting processes; two-factor models; volatility

INTRODUCTION

Since the now classical Black and Scholes¹ paper, Itô calculus has become a dominant approach in dealing with financial markets. Following this approach, there have been many papers assuming that commodity prices follow a geometric Brownian motion, which allows for a straightforward extension of the Black–Scholes procedure (see, eg Brennan and Schwartz,² Paddock *et al.*,³ among others).

As pointed out in Schwartz,⁴ however, the geometric Brownian motion hypothesis implies a constant rate of growth in the commodity price and a variance of future spot prices increasing monotonically with time, which is not a realistic assumption. Even more, all models with a single source of uncertainty are not very realistic since they imply a perfect correlation between futures prices for different maturities.

This is the reason why Gibson and Schwartz,⁵ Schwartz⁴ and Schwartz and Smith⁶ propose a two-factor model allowing for mean reversion in short-term prices and uncertainty in the equilibrium (long-term) price to which prices revert.

Schwartz⁴ and Cortazar and Schwartz⁷ have proposed different kinds of three-factor models. More recently, Cortazar and Naranjo⁸ extended two- and three-factor models to an arbitrary number of factors (N-factor model) given a fix structure.

Due to the complex dynamic of commodity prices, however, the translation of standard valuation techniques, based on the Black–Scholes framework, is not obvious. This could explain why studies on the stochastic behaviour of commodity prices present *ad hoc* solutions that are very complex and sometimes include approximations. This paper shows how well-

known techniques and results in stochastic calculus can be used to simplify formulae and deductions. Specifically, we show how to obtain the expressions in a simpler and accurate way, together with more precise estimates of the parameters than the approximations given by the author in the two-factor model by Schwartz.⁴ In general, the approximations by Schwartz tend to overestimate the parameters. These divergences are important in the valuation of commodity contingent claims.

This paper is organised as follows: The next section shows how to obtain the expressions in a simpler and accurate way and more precise estimates of the parameters in the two-factor model by Schwartz.⁴ The subsequent section contains the empirical implications of the differences between the precise estimates given in this paper and the approximations by Schwartz.⁴ Finally, the last section concludes with a summary and discussion.

THE PRECISE DISCRETE-TIME ESTIMATION OF THE TWO-FACTOR MODEL BY SCHWARTZ

We will focus on the Schwartz⁴ two-factor model. As proposed by the author, the model can be formulated in the following way. The spot price of a commodity at time t , S_t , and its instantaneous convenience yield, δ_t , are assumed to follow the processes:

$$\begin{cases} dS_t = (\mu - \delta_t)S_t dt + \sigma_1 S_t dz_1 \\ d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma_2 dz_2 \end{cases}$$

As can be appreciated, the process for the commodity price allows for a stochastic convenience yield, which follows an Ornstein–Uhlenbeck mean-reverting process. The convenience yield, δ_t , can be interpreted as the benefits achieved by the owner of the physical

commodity that are not obtained by the owner of a futures contract, which may include the ability to profit from temporary local shortages or the ability to keep a production process running.

In this model, μ represents the long-term total return on the commodity, κ is the mean-reverting coefficient, α is the long-term convenience yield and σ_1 and σ_2 are the volatilities of the spot price and the convenience yield, respectively. The standard Brownian motions, dz_1 and dz_2 , are assumed to be correlated, that is $dz_1 dz_2 = \rho dt$.

For the purposes of derivative securities valuation, we need the risk-neutral or risk-adjusted processes for S_t and δ_t . In this model, the commodity is assumed to be an asset paying a stochastic dividend yield, δ_t . Therefore, the risk-adjusted drift of the commodity price process will be $r - \delta_t$, where r is the risk-free interest rate. Given the non-tradable nature of the convenience yield, however, it cannot be hedged and the risk-adjusted convenience yield process will have a market price of risk (λ) linked to it.

In this model, the author assumes for simplicity that the market price of risk, λ , is

Thus, the stochastic process for the two factors under the equivalent martingale measure can be written as:

$$\begin{cases} dS_t = (r - \delta_t)S_t dt + \sigma_1 S_t dz_1^* \\ d\delta_t = [\kappa(\alpha - \delta_t) - \lambda]dt + \sigma_2 dz_2^* \end{cases}$$

where dz_1^* and dz_2^* are the standard (correlated) Brownian motions under the equivalent martingale measure. The correlation between these two Brownian motions does not change, that is, $dz_1^* dz_2^* = \rho dt$.

Defining $Y_t = \ln(S_t)$ and applying Itô's Lemma, under the risk-neutral measure, the model can be expressed as:

$$\begin{cases} dY_t = (r - \delta_t - \sigma_1^2/2)dt + \sigma_1 dz_1^* \\ d\delta_t = [\kappa(\alpha - \delta_t) - \lambda]dt + \sigma_2 dz_2^* \end{cases}$$

Defining the state vector as $X_t = (Y_t, \delta_t)'$ and applying the results in the appendix, it is easy to see that, under the risk-neutral measure, X_t is normally distributed with mean and variance:

$$E^*[X_t] = \begin{pmatrix} (r - \sigma_1^2/2 - \alpha)t + \alpha(1 - e^{-kt})\lambda/k \\ \alpha(1 - e^{-kt}) \end{pmatrix} + \begin{pmatrix} 1 & -(1 - e^{-kt})/k \\ 0 & e^{-kt} \end{pmatrix} X_0$$

$$\text{Var}^*[X_t] =$$

$$\begin{pmatrix} \sigma_1 t + 2\sigma_1 \sigma_2 \rho(1 - e^{-kt} - kt)/k^2 & \sigma_1 \sigma_2 \rho(1 - e^{-kt})/k \\ -\sigma_2^2(3 - 4e^{-kt} + e^{-2kt} - 2kt)/2k^3 & +\sigma_2^2(1 - 2e^{-kt} + e^{-2kt})/2k^2 \\ \sigma_1 \sigma_2 \rho(1 - e^{-kt})/k & \sigma_2^2(1 - e^{-kt})/k \\ +\sigma_2^2(1 - 2e^{-kt} + e^{-2kt})/2k^2 & \end{pmatrix}$$

constant. However, we would expect the market price of risk to be associated to certain variables such as for example the business cycle or the level of inventories. Following the original model, in this paper the market price of risk will be assumed to be constant.

Therefore, under the risk-neutral measure, $Y_t = \ln(S_t)$ is also Gaussian with mean:

$$Y_0 - \delta_0(1 - e^{-kt})/k + (r - \sigma_Y^2/2 - \alpha^*)t + \alpha^*k(1 - e^{-kt})/k^2$$

where $\alpha^* = \alpha - \lambda/\kappa$, and variance:

$$(\sigma_1^2 + \sigma_2^2/k^2 - 2\sigma_1\sigma_2\rho/k)t + (1 - e^{-2\kappa t})\sigma_2^2/2\kappa^3 + 2(\rho\sigma_1\sigma_2 - \sigma_2^2/k)(1 - e^{-\kappa t})/\kappa^2$$

Finally, the spot price, S_t , is lognormally distributed and therefore the futures price can be written as:

$$\begin{aligned} F_{0,T} &= E^*[S_T] = \exp\left(E^*[Y_T] + \frac{1}{2}\text{Var}^*[Y_T]\right) \\ &= \exp\{Y_0 - \delta_0(1 - e^{-\kappa t})/k \\ &\quad + (r - \alpha^* + \sigma_2^2/2k^2 - \sigma_1\sigma_2\rho/k)T \\ &\quad + (1 - e^{-2\kappa t})\sigma_2^2/4\kappa^3 \\ &\quad + (\alpha^*k + \rho\sigma_1\sigma_2 - \sigma_2^2/k)(1 - e^{-\kappa t})/\kappa^2\} \end{aligned}$$

which is the result already obtained in Schwartz,⁴ equation 20, but avoiding unnecessary partial differential equations.

Specifically, Schwartz⁴ obtains this result by solving the partial differential equation that must be satisfied by the futures price:

$$\begin{aligned} \frac{1}{2}\sigma_1^2 S^2 F_{SS} + \sigma_1\sigma_2\rho SF_{S\delta} + \frac{1}{2}\sigma_2^2 F_{\delta\delta} + (r - \delta)SF_S \\ + (\kappa(\alpha - \delta) - \lambda)F_\delta - F_T = 0 \end{aligned}$$

With the terminal boundary condition: $F(S_0, \delta_0, 0) = S_0$.

This partial differential equation can be obtained in a similar way as in the Black–Scholes

It is clear that the procedure presented here is easier and more straightforward than the one used by Schwartz.⁴

Moreover, using the result in the appendix, the squared volatility of futures returns is:

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 0 & e^{-\kappa T} \end{pmatrix} \begin{pmatrix} 1 & (e^{-\kappa T} - 1)/\kappa \\ \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (e^{-\kappa T} - 1)/\kappa & e^{-\kappa T} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \sigma_1^2 + (1 - e^{-\kappa T})^2 \sigma_2^2/\kappa^2 \\ &\quad - 2(1 - e^{-\kappa T})\rho\sigma_1\sigma_2/\kappa \end{aligned}$$

which is exactly the formula in Schwartz,⁴ equation 40.

Following the appendix, the discrete-time version of this model can be expressed as:

$$\text{where } X_t = c_t + M_t X_{t-1} + \psi_t$$

$$c_t = \left(\frac{(\mu - \sigma_1^2/2 - \alpha)\Delta t + \alpha(1 - e^{-\kappa\Delta t})\lambda/k}{\alpha(1 - e^{-\kappa\Delta t})} \right) \quad (1)$$

$$M_t = \begin{pmatrix} 1 & (e^{-\kappa\Delta t} - 1)/\kappa \\ 0 & e^{-\kappa\Delta t} \end{pmatrix} \quad (2)$$

and ψ_t is an n -vector of serially uncorrelated Gaussian disturbances with zero mean and variance:

$$\begin{aligned} \text{Var}[\psi_t] = & \begin{pmatrix} \sigma_1\Delta t + \frac{2\sigma_1\sigma_2\rho(1 - e^{-\kappa\Delta t} - \kappa\Delta t)}{k^2} - \frac{\sigma_2^2(3 - 4e^{-\kappa\Delta t} + e^{-2\kappa\Delta t} - 2\kappa\Delta t)}{2k^3} & \frac{\sigma_1\sigma_2\rho(1 - e^{-\kappa\Delta t})}{k} + \frac{\sigma_2^2(1 - 2e^{-\kappa\Delta t} + e^{-2\kappa\Delta t})}{2k^2} \\ \frac{\sigma_1\sigma_2\rho(1 - e^{-\kappa\Delta t})}{k} + \frac{\sigma_2^2(1 - 2e^{-\kappa\Delta t} + e^{-2\kappa\Delta t})}{2k^2} & \frac{\sigma_2^2(1 - e^{-\kappa\Delta t})}{k} \end{pmatrix} \quad (3) \end{aligned}$$

context, that is, we must form an instantaneously riskless portfolio, whose return, in the absence of arbitrage opportunities, must be the risk-free interest rate.

In Schwartz,⁴ a discrete-time version of the model is used, which is not the precise one presented above. Specifically, Schwartz⁴ uses a linear approximation to the precise expressions

presented in this paper. It is easy to show with a Taylor expansion that when $\Delta t \rightarrow 0$ equations (1–3) converge to the linear approximations by Schwartz⁴ (equations 35 in Schwartz⁴):

$$c_t = \begin{pmatrix} (\mu - \sigma_1^2/2)\Delta t \\ \alpha k \Delta t \end{pmatrix},$$

$$M_t = \begin{pmatrix} 1 & -\Delta t \\ 0 & 1 - \kappa \Delta t \end{pmatrix} \text{ and}$$

$$\text{Var}[\psi_t] = \begin{pmatrix} \sigma_1^2 \Delta t & \sigma_1 \sigma_2 \rho \Delta t \\ \sigma_1 \sigma_2 \rho \Delta t & \sigma_2^2 \Delta t \end{pmatrix}$$

Therefore, Schwartz⁴ uses a discrete-time version of the model, which is an approximation to the precise one presented above. As we will see below, this accurate discrete-time version of the model, given by expressions (1–3), will have consequences in terms of commodity contingent valuation.

EMPIRICAL IMPLICATIONS

In order to appreciate the differences between the precise version of the estimates given in this paper and the approximate version in Schwartz,⁴ we have estimated both versions using weekly observations of prices for NYMEX WTI crude oil futures contracts with 1, 3, 5, 7 and 9 months to maturity. Futures from 1/1/1985 to 13/02/1995 have been used with a total of 529 observations. This is one of the data sets used in Schwartz.⁴ In that paper, however, the data set includes 510 observations, instead of 529. This is the reason why the results presented here for the Schwartz approximation are not exactly the same as the ones presented in Schwartz.⁴

Table 1 contains some descriptive statistics of the data.

WTI futures prices with one month to maturity are depicted in Figure 1.

Table 1: Data (NYMEX WTI crude oil futures contracts, From 1/1/1985 to 13/02/1995, Weekly Observations)

<i>Futures contract</i>	<i>Mean (\$/bbl)</i>	<i>Standard deviation (\$/bbl)</i>
F1	19.96	4.51
F3	19.63	4.07
F5	19.43	3.74
F7	19.29	3.51
F9	19.20	3.35

Table 2 presents both estimates of the parameters of the two-factor model by Schwartz.⁴ As in Schwartz,⁴ the parameters have been estimated through the Kalman filter methodology. Looking at the table, we can appreciate that the greatest differences are found in the values of the mean-reverting parameter, κ , in the convenience yield volatility, σ_2 , and in the market price of risk associated with the convenience yield, λ . In fact, we find a value for κ considerably lower than the value found with the Schwartz approximation (1.5433 and 1.8855, respectively). Moreover, the value of λ found with the precise version is also lower than the value found with the Schwartz approximation (0.2181 and 0.2558, respectively). Finally, the value of σ_2 with the precise and approximate versions is 0.3967 and 0.4622, respectively. In general, it is found that the approximate version by Schwartz⁴ tends to overestimate the parameters, since all the values found with the approximate version are higher than the values found with the precise version.

To the best of our knowledge, there is no reliable index that reflects the WTI crude oil spot price. The best approximation for it is the price of NYMEX WTI crude oil futures

Figure 1: One-month WTI futures price

Table 2: Precise and approximate estimates in the two-factor model by Schwartz⁴

	Precise version	Schwartz approximation
μ	0.1629 (0.0725)	0.1678 (0.0732)
k	1.5433 (0.0318)	1.8855 (0.0356)
α	0.1458 (0.0558)	0.1496 (0.0545)
σ_1	0.3278 (0.0073)	0.3293 (0.0072)
σ_2	0.3967 (0.0113)	0.4622 (0.0119)
ρ	0.8073 (0.0104)	0.8084 (0.0107)
λ	0.2181 (0.0864)	0.2558 (0.1029)

Standard errors in parentheses.

contracts with one month to maturity. Table 3 presents the differences between one-month WTI futures prices and the spot price calculated with both the precise and the approximated estimates. The predictive ability of these estimates is compared through two metrics: the mean error (defined as the average of the series of one-month futures prices minus estimated

spot prices) and the root mean squared error (RMSE).

It is found that in the full sample period, the precise estimates outperform the approximation by Schwartz,⁴ using the two metrics. They are also the best-performing estimates in all the annual periods considered in the table. Interestingly, the best performance of the precise estimates is found in 1985 and 1990, years that are characterised by higher volatility (see Figure 1). This is not surprising since one of the main advantages of the precise methodology is that it provides a more accurate estimation of the variance of the residual, $\text{Var}[\psi_t]$, given by expression (3). Moreover, both estimates tend to overestimate spot prices since the mean error is negative in the whole sample period. It is also the case in all the annual periods, except for 1993 and 1994.

Table 4 shows the differences between one-month WTI futures prices and spot prices calculated with both the precise and the

Table 3: Differences between real and estimated spot price by year

	<i>Mean error</i>		<i>Root mean squared error</i>	
	<i>Precise</i>	<i>Schwartz approximation</i>	<i>Precise</i>	<i>Schwartz approximation</i>
1985–1995	−0.1559	−0.1718	0.4424	0.5031
1985	−0.3456	−0.4158	0.4926	0.5694
1986	0.0148	0.0221	0.2092	0.2320
1987	−0.0728	−0.0860	0.1098	0.1218
1988	−0.0010	0.0059	0.1295	0.1372
1989	−0.2936	−0.3533	0.3249	0.3897
1990	−0.7200	−0.7866	1.1552	1.3055
1991	−0.2336	−0.2610	0.3188	0.3652
1992	−0.1290	−0.1195	0.1691	0.1719
1993	0.1376	0.1836	0.1889	0.2322
1994	0.0816	0.0906	0.1969	0.2226
1995	−0.0280	−0.0356	0.0898	0.1009

The table reports the differences between the one-month futures price and the spot price calculated with precise and approximated estimates, by year.

approximated estimates, by month. As before, the precise estimates outperform the approximation by Schwartz⁴ using the two metrics in all months, except for March with the mean error measure.

Figure 2 compares the improvement in the RMSE, defined as the RMSE computed with the Schwartz approximation minus the RMSE computed with the precise version of the estimates (expressed in percentage), and the standard deviation of one-month futures prices, by month. It is interesting to observe that the months characterised by the highest degree of variance (October and November) show the highest improvement in the RMSE. Again, this is related to the fact that one of the main differences between both methodologies lies on a more accurate estimation of the variance of the residual, $\text{Var}[\psi_t]$, given by expression (3). It

should be noted, however, that there are also months with no such high variance showing a high improvement in the RMSE (January and December).

CONCLUSIONS

In recent years, there have been many papers analysing the stochastic behaviour of commodity prices. Given the complex dynamics of commodity prices, however, the application of the standard Black–Scholes analysis is not straightforward. In fact, these studies present *ad hoc* solutions, which are very complex and sometimes include approximations.

This paper shows how well-known techniques and results in stochastic calculus can be used to simplify formulae and deductions. Specifically, we have shown how to obtain the

Table 4: Differences between real and estimated spot prices by month

	<i>Mean error</i>		<i>Root mean squared error</i>	
	<i>Precise</i>	<i>Schwartz approximation</i>	<i>Precise</i>	<i>Schwartz approximation</i>
All months	−0.155884852	−0.171811758	0.442441699	0.503075855
January	−0.148361764	−0.176975534	0.345867414	0.412837467
February	−0.057035493	−0.063956222	0.226770214	0.255777129
March	−0.021821794	−0.018843631	0.220816037	0.249461653
April	−0.048881713	−0.057364135	0.211696474	0.249994592
May	−0.081754769	−0.085655195	0.161309614	0.193322674
June	−0.047580096	−0.050466782	0.198748078	0.240786749
July	−0.092252959	−0.095384407	0.261123270	0.301347232
August	−0.147982930	−0.153883187	0.297399519	0.329944708
September	−0.134288566	−0.142203942	0.293639070	0.327363269
October	−0.436105927	−0.482717149	0.908492618	1.018960284
November	−0.317861875	−0.347944277	0.690954641	0.778313141
December	−0.195527578	−0.222481222	0.461966337	0.535579146

The table reports the differences between the one-month futures price and the spot price calculated with both precise and approximated estimates, by month.

Figure 2: Comparison of the improvement in the RMSE and one-month futures price standard deviation by month

The figure shows the improvement (expressed in percentage) in the RMSE, defined as the RMSE computed with the Schwartz approximation minus the RMSE computed with the precise version of the estimates, and one-month futures price standard deviation, by month.

expressions in a simpler and accurate way, together with more precise estimates of the parameters than the approximations given by the author in the two-factor model by Schwartz.⁴ It has been shown that the approximations by Schwartz tend to overestimate the parameters and fit the spot prices worse than the precise version. The highest improvements in the RMSE are observed in periods-time characterised by higher volatility. This is related to the fact that one of the main advantages of the precise methodology is that it provides a more accurate estimation of the residual variance in the state-space. These divergences are important in the valuation of commodity contingent claims.

Disclaimer: This paper is the sole responsibility of its authors and the views represented here do not necessarily reflect those of the Banco de España.

Acknowledgments

We wish to thank Ángel León Valle, Alfonso Novales, Andrés Ubierna Gorricho, Juan Manuel Martín Prieto, Gabriel Paulik and Ana Puente Pérez for their help. We especially thank Repsol YPF for its support. Most of this research was conducted while Javier Población was working in Repsol YPF. We also thank participants in the XXXI Simposio de Análisis Económico in Oviedo. Gregorio Serna acknowledges the financial support provided by the Ministerio de Educación y Ciencia grant SEJ2005-08931-C02-01 and Junta de Castilla-La Mancha grant PAI05-074. Any errors that remain are, however, entirely our own.

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APPENDIX

Futures Contract Valuation

Most of the models proposed in the literature assume that the dynamics of a commodity price (or its log) is given by a linear stochastic differential system:

$$\begin{cases} dX_t = (b + AX_t)dt + R dW_t \\ Y_t = cX_t \end{cases}$$

where Y_t is the commodity price (or its log), b , A , R and c are deterministic parameters independent of t ($b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $c \in \mathbb{R}^n$) and W_t is an n -dimensional canonical Brownian motion (ie all components uncorrelated and its variance equal to unity).

The solution of that problem is:

$$X_t = e^{At} \left[X_0 + \int_0^t e^{-As} b ds + \int_0^t e^{-As} R dW_s \right] \quad (A1)$$

It is easy to prove that the solution is unique.⁹ Moreover, even in the case that b , A and R were functions of t , if A_t and $\int_0^t$ As commute, the solution of that problem is (A1).

Accordingly, X_t is normally distributed with mean and variance:

$$E[X_t] = e^{At} \left[X_0 + \int_0^t e^{-As} b ds \right]$$

$$\text{Var}[X_t] = e^{At} \left[\int_0^t e^{-As} R R' e^{-As'} ds \right] e^{At'}$$

With this result, it is easy to prove that the price of a futures contract traded at time t with maturity at time $t + T$, $F_{t,T}$ can be computed as:

$$F_{t,T} = \exp[ce^{AT} X_t + g(T)] \quad (\text{A2})$$

where g is a, probably complicated, deterministic function.

Volatility of Futures Returns

The square volatility of a futures contract traded at time t with maturity at time $t + T$ can be defined as:

$$\lim_{h \rightarrow 0} \frac{\text{Var}[\log F_{t+h,T} - \log F_{t,T}]}{h}$$

Equivalently, it is also possible to define it as

$$\lim_{h \rightarrow 0} \frac{\text{Var}[\log F_{t+h,T-h} - \log F_{t,T}]}{h}.$$

It can be proved that it is the expected value of the square of the coefficient of the Brownian

motion (σ_t) in the expansion $d \log(F_{t,T}) = \mu_t ds + \sigma_t dW_t^F$, where W_t^F is a scalar canonical Brownian motion.

Hence, taking logarithms and differentials on both sides of Equation (A2), it follows that:

$$d \log(F_{t,T}) = ce^{AT} dX_t = ce^{AT} [b + AX_t] dt + ce^{AT} R dW_t.$$

Therefore, the squared volatility is:

$$ce^{AT} R R' e^{AT'} c' \quad (\text{A3})$$

Note that R need not to be computed as RR' is the noise covariance matrix.

Empirical Models

In the general model presented above, it is easy to prove that knowing X_{t-1} , X_t can be written as:

$$X_t = c_t + M_t X_{t-1} + \psi_t \quad (\text{A4})$$

where $c_t = e^A [\int_{t-1}^t e^{-As} b^\# ds]$, $M_t = e^A$ and ψ_t is an n -vector of serially uncorrelated Gaussian disturbances with zero mean and covariance matrix: $e^A [\int_{t-1}^t e^{-As} R R' e^{-As'} ds] e^{A'}$.

Uses and Remarks

This general methodology can be used in all kind of problems, is much simpler than the *ad hoc* solutions presented in the literature, which can only be used in the concrete problem for which they were developed, needing complex procedures like limit steps⁸ or partial differential equations,⁴ and avoids approximations like in Schwartz.⁴