Original Article

A Probabilistic Monte Carlo model for pricing discrete barrier and compound real options

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ABSTRACT **We present an original Probabilistic Monte Carlo (PMC) model for pricing European discrete barrier options and compound real options. On the basis of Monte Carlo (MC) simulation, for barrier options the PMC model computes the probability of not crossing the barrier for knock-out options and crossing the barrier for knock-in options. This probability is then multiplied by an average sample discounted payoff of a plain vanilla option that has the same inputs as the barrier option but barrier-free, and to which we have applied a filter. We test the consistency of our model with an analytical solution (Merton, 1973; Reiner and Rubinstein, 1991) adjusted for discretization by Broadie *et al* (1997) and a naïve numerical model using MC simulation presented by Clewlow and Strickland (2000). Our study shows that the PMC model accurately prices barrier options. Moreover, the idea behind the method is simple and can be applied to the pricing of complex derivatives, easing the valuation step significantly: we illustrate the versatility of the PMC model in pricing**

sequential compound real options. Market participants needing to select a reliable, versatile and simple numerical method for pricing options with embedded features will find our article appealing.

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INTRODUCTION

Barrier options are cheaper than plain vanilla options but have a higher risk of loss due to their barrier(s). With a cheap premium, barrier options have been attractive and traded over the counter since 1967 (Haug, 2006). Merton (1973) pioneered their pricing when they are monitored in continuous time. However, the most common traded barrier options are monitored in discrete time and their pricing is more challenging. A few solutions are analytical with a correction for continuity; the most popular solutions are numerical with lattices or Monte Carlo (MC) simulation. Nonetheless, Ahn *et al* (1999) pointed out that lattice-based solutions inaccurately value discretely monitored barrier options. Our article presents an original Probabilistic Monte Carlo (PMC) model for pricing European discrete barrier equity options. On the basis of MC simulation, the PMC model computes the probability of not crossing the barrier for knock-out options and crossing the barrier for knock-in options. This probability is then multiplied by an average sample discounted payoff of a plain vanilla option to which we applied a filter. We test the consistency of our model with the pricing of a European up-and-out discrete barrier equity option obtained with an analytical solution (Merton, 1973; Reiner and Rubinstein, 1991) adjusted for discretization by Broadie *et al* (1997) and a naïve MC simulation presented by Clewlow and Strickland (2000). We apply the PMC model to the computation of four different types of single-barrier options: up-and-out, up-and-in, down-and-in and down-and-out. In addition, to show the versatility and potential of the PMC model, we compute sequential compound real options. The next section will review the literature concerning the pricing of barrier options. The section after that will present the methodology in six steps. The penultimate section will present the results and the final section will wrap up our findings.

LITERATURE REVIEW

Barrier options have been very common on the over-the-counter market since the late 1980s, the main reason being that holders pay lower premiums than for plain vanilla options. We may find barrier options traded on exchanges (Easton *et al*, 2004) but they are not so common. They appeared first in 1991 in the United States on the CBOE (Chicago Board Options Exchange) and on AMEX (American Exchange) but with limited success. They were introduced on ASX (Australian Stock Exchange) in 1998, with better success on the Australian market.

The valuation of simple barrier options relies on closed-form solutions and numerical solutions. Merton (1973), Cox and Rubinstein (1985) and Rubinstein and Reiner (1991) proposed closed-form solutions. These analytical solutions assume that the barrier is monitored continuously but cannot be applied to barrier options where the crossing of the barrier is monitored discretely, for example the options traded on ASX.

Numerical solutions use either lattice or MC simulation. Cox and Rubinstein (1985), Hudson (1992), Boyle and Lau (1994), Derman *et al* (1995), Kat and Verdonk (1995), Ritchken (1995) and Cheuk and Vorst (1996) promoted lattice solutions. Ahn *et al* (1999) pointed out that lattice-based solutions inaccurately value discretely monitored barrier options. 'The cause is often non-linearity or discontinuity in the option payoff that occurs only in a small region'. These authors propose an adaptive mesh model that 'constructs small sections of fine high-resolution mesh in the critical areas and grafts them onto a base lattice with coarser time and price steps elsewhere.'

Broadie *et al* (1997) showed 'that discrete barrier options can be priced with remarkable accuracy using continuous barrier formulas by applying a simple continuity correction to the barrier'. The approximation is remarkably accurate except in extreme cases with H very close to S_o. While very convenient in practice, this analytical approximation is limited to single-barrier options and to the geometric Brownian motion process. Howison and Steinberg (2007) extended the applications of the 'continuity correction' presented by Broadie *et al* (1997) to a wide variety of cases, using a matched asymptotic expansions approach.

In the 2000s, authors promoted pure jump and jump-diffusion asset pricing models based on Lévy processes. Boyarchenko and Levendorskii (2002) derived explicit formulas for European-type barrier options and touch-and-out options assuming that under a chosen equivalent martingale measure the stock returns follow a Lévy process. Petrella and Kou (2004) provided a comprehensive study of discrete single-barrier options in Merton's and Kou's jump-diffusion models in this framework based on Spitzer's identity. Metwally and Atiya (2002) proposed an algorithm based on the Brownian Bridge to price barrier options in the frameworks of Merton (1976). They observed that between two jumps, the underlying asset follows a classic geometric Brownian motion. This method provides non-biased estimators and is faster than the naïve simulation. Joshi and Leung (2007) improved their method by adding a preferential sampling process that saves computation time for trajectories with a zero payoff when the barrier is knocked out. They noted that the simulation speed is increased for algorithms involving weak jump frequency. Ross and Ghamani (2010) proposed an alternative variance reduction technique to the method of Metwally and Atiya (2002). Feng and Linetsky (2008) presented a Lévy process-based model solution to price discretely monitored single-and double-barrier options. 'The method involves a sequential evaluation of Hilbert transforms of the product of the Fourier transform of the value function at the previous barrier monitoring date and the characteristic function of the (Esscher transformed) Levy process'. In addition to the promoters of the Lévy process, other authors proposed the application of regime-switching models to investigate option valuation problems, especially barrier options. The basic idea of regime-switching models is to allow the model parameters to change over time according to a state process, which is usually modeled as a Markov chain. A key advantage of regime-switching models is the incorporation of the impact of structural changes in economic conditions on the price dynamics. Guo (2001) applied a regime-switching model to the pricing of options. Elliott *et al* (2014) presented a solution for pricing 'both European-style and American-style barrier options in a Markovian, regime-switching, Black–Scholes–Merton economy, where the price process of an underlying risky asset is governed by a Markovian, regime-switching, geometric Brownian motion'.

Finally, the trend of the 2010s has been to apply the SABR (Stochastic, Alpha, Beta, Rho) stochastic volatility model to the pricing of options, particularly barrier options. The SABR model attempts to capture the volatility smile. Tian *et al* (2012) priced barrier and American options by the least squares MC method under the SABR model. Shiraya *et al* (2012) provided a numerical model for pricing double-barrier call options with discrete monitoring under Heston and λ-SABR models.

METHODOLOGY

We price a 6-month European up-and-out discrete barrier equity option with the PMC model.

Our benchmarks are the analytical solution of Merton (1973) and Reiner and Rubinstein (1991) and a naïve numerical model using MC simulation presented by Clewlow and Strickland (2000). Our methodology presents six steps.

Step 1

By simulating the stock price using the following Brownian motion:

$$S_{t+1} = S_t e^{\left((r-\delta-0.5\sigma^2)dt+\sigma\sqrt{dt}\varepsilon\right)}, \qquad (1)$$

we price a plain vanilla European call option with the exact same parameters as the barrier option but with no barrier. The algorithm is borrowed from Clewlow and Strickland (2000). For N trajectories of the stock prices simulated over the life of the option with a maturity of T years (in our example $T=0.5$ year$=125$ steps, assuming a year equal to 250 business days), we count the number of times the stock price has reached maturity of the option without hitting the barrier, that is, the probability P that the option is not knocked out during its life.

Step 2

We filter $c_{Ti}=(S_{Ti}-X)^+$, the option value of a plain vanilla call option at maturity of the ith trajectory, S_{Ti} being the stock price at maturity of the option and X the strike price: given H the up-and-out barrier, if $(S_{Ti}-X)^+ \geqslant (H-X)$ then $c_{Ti}=0$, since c_{Ti} cannot have a value equal to or higher than $(H-X)$ otherwise the barrier is activated ($S_{Ti} \geqslant H$) and the option is knocked out.

Step 3

Simulating N trajectories of the stock price and applying the filter to the option value c_{Ti} of a plain vanilla call option, we obtain a sample of N possible values of the option at its maturity c_{TiAF} ($AF=$ *A*fter *F*iltering). We randomly draw without replacement NP option values of c_{TiAF} from the sample.

Step 4

The PMC model computes c_{uo} the value of a T-year European equity up-and-out barrier option using equation 2:

$$c_{uo} = \left[\frac{P}{N}\sum_{i=1}^{NP} c_{TiAF} + K(1-P)\right]e^{-rT} \qquad (2)$$

with r the continuous risk-free rate over time T, c_{TiAF} the option values after filtration in Step 2 and the draws without replacement in Step 3, P the probability that the option is not knocked out and K the rebate offered by the barrier option when the option is knocked out.

Step 5

We extend our reasoning to down-and-out, up-and-in and down-and-in European call options. We obtain Table 1.

The same reasoning applies to put options. We obtain Table 2.

The PMC model is benchmarked to two models: (i) a naïve numerical model using MC simulation to price barrier options presented by Clewlow and Strickland (2000); and (ii) the analytical solution developed by Merton (1973) and emphasized by Reiner and Rubinstein (1991). The numerical model assumes that the crossing of the barrier is monitored daily by dividing the maturity of the option of 6 months in 125 steps (1 year = 250 steps). The analytical solution assumes that the crossing of the barrier is monitored continuously. As mentioned in the literature review, Broadie *et al* (1997) developed an approximation for continuity correction to price discrete barrier options. The correction shifts the barrier away from the underlying asset price that reduces the probability of hitting the barrier and that mimics the effect of discrete monitoring that lowers the probability of hitting the barrier compared with continuous monitoring. The barrier H_{AC} after correction is equal to:

$$H_{AC} = He^{\beta\sigma\sqrt{\Delta t}} \tag{3}$$

when $H > S_o$

$$\text{and is equal to :} \quad H_{AC} = He^{-\beta\sigma\sqrt{\Delta t}} \tag{4}$$

when $H < S_o$

with H the initial barrier, $\beta = -\zeta(1/2)/\sqrt{2\pi} = 0.5826$, where ζ is the Riemann zeta function, $\Delta t = 0.5/125 = 0.004$ (option maturity of 6 months divided in 125 days – steps) and σ = volatility (either 0.25 or 0.30 in our example).

Step 6

We explore the potential and versatility of the PMC model in pricing other types of options than barrier options. This step clarifies the intuition behind the PMC model. Moreover, it shows how useful and flexible the model can be in valuing options with embedded features. For this purpose, we choose sequential compound real options. The example is borrowed from Kodukula and Papudesu (2006). A firm applies the options theory to the valuation of a project for a go/no-go investment decision. The project is divided into three sequential phases: (i) permitting; (ii) design and (iii) construction. Each phase has to be completed before the next phase can begin. The company has a maximum of 1 year from today to decide on Phase 1, 3 years on Phase 2 and 5 years on Phase 3. Permitting is expected to cost US\$30 million, design \$90 million and construction another \$210 million. Discounted cash flow analysis using an appropriate risk-adjusted discount rate values the plant, if it existed today, at S_o = \$250 million. The annual volatility σ of the logarithmic returns for the future cash flows for the plant is estimated to be 30 per cent and the continuous annual risk-free interest rate over the next 5 years is r = 6 per cent. Kodukula and Papudesu apply the binomial tree model to calculate the option values for each of the three sequential options available on this project: construction is dependent on design, which in turn is dependent on permitting. The option value calculations are done in sequence, starting with the longest option: simulate S_o = \$250 million over 5 years, in a binomial

Table 1: Methodology of the PMC model applied to single-barrier call options

	Up-and-out call options	*Down-and-out call options*	*Up-and-in call options*	*Down-and-in call options*
	Spot stock price S_o starts below the barrier level H and has to move up for the option to be knocked out: $S_o<H$ and choose $X<H$	Spot stock price S_o starts above the barrier level H and has to move down for the option to be knocked out: $S_o>H$	Spot stock price S_o starts below the barrier level H and has to move up for the option to be knocked in (activated): $S_o<H$	Spot stock price S_o starts above the barrier level H and has to move down for the option to be knocked in (activated): $S_o>H$
Option value c_{Ti} at maturity of the option	$c_{Ti}=(S_{Ti}-X)^+$ Option value c_{Ti} of a plain vanilla call option at maturity T of the option for a given simulated trajectory i among N trajectories	*idem*	*idem*	*idem*
Option value c_{TiAF} after filtration	If $c_{Ti}\geqslant(H-X)$ then $c_{TiAF}=0$ Else $c_{TiAF}=(S_{Ti}-X)^+$	If H>X then $c_{TiAF}=(S_{Ti}-X)^+>(H-X)$ Else $c_{TiAF}=0$ Else: $c_{TiAF}=(S_{Ti}-X)^+$	$c_{TiAF}=(S_{Ti}-X)^+=c_{Ti}$ (no filtration)	$c_{TiAF}=(S_{Ti}-X)^+=c_{Ti}$ (no filtration)
Compute probability P with MC simulation	Probability P that the option is not knocked out during the life of the option	Probability P that the option is not knocked out during the life of the option	Probability P that the option is not knocked in during the life of the option	Probability P that the option is not knocked in during the life of the option
Draws without replacement	Draw (NP) values without replacement from the sample of N c_{TiAF}	*idem*	*idem*	*idem*
Option price c	$c_{uo}=[(P/N)\sum_{i=1}^{NP}c_{TiAF}+K(1-P)]e^{-rT}$	$c_{do}=[(P/N)\sum_{i=1}^{NP}c_{TiAF}+K(1-P)]e^{-rT}$	$c_{ui}=[[((1-P)/N)\sum_{i=1}^{N-NP}c_{TiAF}]+KP]e^{-rT}$	$c_{di}=[[((1-P)/N)\sum_{i=1}^{N-NP}c_{TiAF}]+KP]e^{-rT}$

Table 2: Methodology of the PMC model applied to single-barrier put options

	Up-and-out put options	*Down-and-out put options*	*Up-and-in put options*	*Down-and-in put options*
	Spot stock price S_o starts below the barrier level H and has to move up for the option to be knocked out: $S_o<H$	Spot stock price S_o starts above the barrier level H and has to move down for the option to be knocked out: $S_o>H$ and choose $X>H$	Spot stock price S_o starts below the barrier level H and has to move up for the option to be knocked in (activated): $S_o<H$	Spot stock price S_o starts above the barrier level H and has to move down for the option to be knocked in (activated): $S_o>H$
Option value c_{Ti} at maturity of the option	$c_{Ti}=(X-S_{Ti})^+$ Option value c_{Ti} of a PLAIN VANILLA PUT option at maturity T of the option for a given simulated trajectory i among N trajectories	*idem*	*idem*	*idem*
Option value c_{TiAF} after filtration	If X>H *then* $c_{TiAF}=(X-S_{Ti})^+ >(X-H)$ *else* $c_{TiAF}=0$ Else: $c_{TiAF}=(X-S_{Ti})^+$	If $c_{TiAF}=(X-S_{Ti})^+>(X-H)$ *then* $c_{TiAF}=0$ Else: $c_{TiAF}=(X-S_{Ti})^+$	$c_{TiAF}=(X-S_{Ti})^+=c_{Ti}$ (no filtration)	$c_{TiAF}=(X-S_{Ti})^+=c_{Ti}$ (no filtration)
Compute Probability P with MC simulation	Probability P that the option is not knocked out during the life of the option	Probability P that the option is not knocked out during the life of the option	Probability P that the option is not knocked in during the life of the option	Probability P that the option is not knocked in during the life of the option
Draws without replacement	Draw (NP) values without replacement from the sample of N c_{TiAF}	*idem*	*idem*	*idem*
Option price p	$p_{uo}=[(P/N)\sum_{i=1}^{NP}c_{TiAF}+K(1-P)]e^{-rT}$	$p_{do}=[(P/N)\sum_{i=1}^{NP}c_{TiAF}+K(1-P)]e^{-rT}$	$p_{ui}=[[((1-P)/N)\sum_{i=1}^{N-NP}c_{TiAF}]+KP]e^{-rT}$	$p_{di}=[[((1-P)/N)\sum_{i=1}^{N-NP}c_{TiAF}]+KP]e^{-rT}$

tree with S_o going up one step with a value $S_o u$ and going down one step with a value $S_o d$, repeating five steps, and at $T = 5$ years the highest and lowest values of the tree being, respectively, $S_o u^5 = 1120$ and $S_o d^5 = 56$. Applying the payment function $[S_T - X]^+$ at $T = 5$, with $X = \$210$ million the cost of construction, then going backward in the tree, using the binomial formulae:

$$c = (pc_u + (1-p)c_d)e^{-r\Delta t} \tag{5}$$

$$\text{with}\, p = \frac{e^{r\Delta t} - d}{u - d},\, u = e^{\sigma\sqrt{\Delta t}} \text{ and } d = \frac{1}{u}$$

we obtain at $T = 0$ an option value of \$113 million. To simulate the second option over 3 years (Phase 2, design), the initial value is \$113 million, the option value of the construction phase. We again compute $S_o u$ and $S_o d$ and so on over 3 years, that is, three steps, with at $T = 3$ years the highest and lowest values of the tree being, respectively, $S_o u^3 = 429$ and $S_o d^3 = 0$. Applying the payment function $[S_T - X]^+$ at $T = 3$, with $X = \$90$ million the cost of design, then going backward in the tree, using the binomial formulae, we obtain at $T = 0$ an option value of \$63 million. To simulate the first option over 1 year (Phase 1, permitting), the initial value is \$63 million, the option value of the design phase. We again compute $S_o u$ and $S_o d$ over 1 year, with at $T = 1$ year the high and low values of the tree, respectively, $S_o u = 113$ and $S_o d = 16$. Applying the payment function $[S_T - X]^+$ at $T = 1$, with $X = \$30$ million the cost of permitting, then going backward in the tree, using the binomial formulae, we obtain at $T = 0$ an option value of \$41 million, which is the final value of the option. Since it is positive, the firm may go ahead with the project.

We apply a two-step approach with the PMC model to compute the final option value of the project: (i) using MC simulation of equation 1 above with $S_o = \$250$ million, $r = 0.06$, $\sigma = 0.30$, $\Delta t = 1$, $q = 0$, $T = 5$, we compute the probabilities P_1 that $S_1 \geqslant \$30$ million (S_1 the value of the project at $T = 1$), P_2 that $S_3 \geqslant \$90$ million (S_3 the value of the project at $T = 3$) and P_3 that $S_5 \geqslant \$210$ million (S_5 the value of the project at $T = 5$), with \$30, \$90 and \$210 million being, respectively, the costs of Phases 1–3. (ii) Compute the value of the option applying equation 2 above:

$$c = \left[\frac{P}{N}\sum_{i=1}^{NP} c_{TiAF} + K(1-P)\right]e^{-rT} \tag{6}$$

With $K = 0$, $c_{TiAF} = [S_{Ti} - X]^+$ at $T = 5$ years, $X = \$210$ million (X is the construction cost) and $P = P_1 P_2 P_3$. P is the conditional probability that Phase 3 starts if Phase 2 starts if Phase 1 starts. The c_{TiAF} corresponds to NP random draws without replacement among the N possible values $[S_{Ti} - X]^+$ of a plain vanilla option. By using the PMC model, we simplify the valuation of the project to computing the probability that Phase 3 starts conditional on the start of the two previous phases. The PMC model avoids building three binomial trees and solves the whole valuation problem in one simulation.

RESULTS

Pricing barrier options with the PMC model

Tables 3 and 4 gather the results, respectively, for European single-barrier call and put options. We price a barrier option where $S = 100$, $T = 0.5$, $r = 0.08$, $q = 0.04$, rebate $= 3$, $\sigma = 0.25$ or 0.30. We compute the option premiums for different

Table 3: Premiums of European Barrier Call Options ($S=100$, $T=0.5$, $r=0.08$, $q=0.04$, rebate $=3$) computed with four models (for MC and PMC solutions: 1000 simulations)

			Merton et al solution continuous recording		*Merton et al solution +Broadie correction daily recording*		*Naive MC solution daily recording*		*Average simulation time in seconds*			*PMC solution daily recording*		*Average simulation time in seconds*		
Type	*X*	*H*	*σ = 0.25*	*σ = 0.30*	*σ = 0.25 (a) Merton +Broadie*	*σ = 0.30 (b) Merton +Broadie*	*σ = 0.25 (c) Naive MC*	*σ = 0.30 (d) Naive MC*		*Squared error [(a)−(c)]²*	*Squared error [(b)−(d)]²*	*σ = 0.25 (e) PMC*	*σ = 0.30 (f) PMC*		*Squared error [(a)−(e)]²*	*Squared error [(b)−(f)]²*
cdo	90	95	9.0246	8.8334	9.8132	9.7965	9.5641	9.6787	0.2105	0.0621	0.0139	5.8898	5.6865	0.3670	15.3931	16.8921
cdo	100	95	6.7924	7.0255	7.2081	7.6198	7.1549	6.9984	0.2030	0.0028	0.3861	4.6693	4.3796	0.3120	6.4455	10.4989
cdo	110	95	4.8759	5.4137	5.0209	5.7111	4.8899	5.4362	0.1950	0.0172	0.0756	3.3080	4.0481	0.3200	2.9340	2.7656
cdo	90	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.2960	0.0000	0.0000
cdo	100	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.2960	0.0000	0.0000
cdo	110	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.2960	0.0000	0.0000
cuo	90	105	2.6789	2.6341	2.6872	2.6069	2.5976	2.5664	0.4880	0.0080	0.0016	2.7415	2.7000	0.3195	0.0029	0.0087
cuo	100	105	2.3580	2.4389	2.2512	2.3240	2.1510	2.2487	0.4595	0.0100	0.0057	2.1287	2.3165	0.3040	0.0150	0.0001
cuo	110	105	2.3453	2.4315	2.2464	2.3075	2.1877	2.2367	0.4525	0.0034	0.0050	2.2021	2.2252	0.3120	0.0020	0.0068
cdi	90	95	7.7627	9.0093	6.9676	8.0393	3.6459	3.5806	0.2035	11.0337	19.8800	10.2549	11.5281	0.3120	10.8063	12.1717
cdi	100	95	4.0109	5.1370	3.5889	4.5389	0.7898	0.6774	0.1950	7.8350	14.9112	5.9031	6.7304	0.3115	5.3555	4.8027
cdi	110	95	2.0576	2.8517	1.9061	2.5474	0.7898	0.6457	0.2030	1.2461	3.6165	3.7068	4.8809	0.3275	3.2425	5.4452
cdi	90	100	13.8333	14.8816	12.5268	13.4179	8.2121	8.0057	0.1170	18.6166	29.2919	12.6791	15.8472	0.2965	0.0232	5.9015
cdi	100	100	7.8494	9.2045	6.9542	8.1314	0.1268	0.1614	0.1020	46.6134	63.5209	7.4937	12.5064	0.2960	0.2911	19.1406
cdi	110	100	3.9795	5.3043	3.4707	4.6096	0.1585	0.0894	0.1090	10.9707	20.4322	4.5654	6.8170	0.3045	1.1984	4.8726
cui	90	105	14.1112	15.2098	14.0964	15.2299	12.1605	12.5005	0.2105	3.7477	7.4496	11.0642	11.6041	0.3120	9.1942	13.1464
cui	100	105	8.4482	9.7278	8.5485	9.8357	5.0094	5.3417	0.1950	12.5252	20.1960	6.6981	7.1493	0.3045	3.4240	7.2167
cui	110	105	4.5910	5.8350	4.7035	5.9221	0.7254	0.7005	0.2030	15.8253	27.2651	3.4569	3.9312	0.2970	1.5540	3.9637
									RMSE	2.6720	3.3916			RMSE	1.8239	2.4362
									Average RMSE	3.0318				Average RMSE	2.1301	

Table 4: Premiums of European Barrier Put Options ($S=100$, $T=0.5$, $r=0.08$, $q=0.04$, rebate $=3$) computed with four models (for MC and PMC solutions: 1000 simulations)

			Merton et al solution continuous recording		*Merton et al solution + Broadie correction daily recording*		*Naive MC solution daily recording*		*Average simulation time in seconds*			*PMC solution daily recording*		*Average simulation time in seconds*		
Type	*X*	*H*	$\sigma=0.25$	$\sigma=0.30$	$\sigma=0.25$ *(a) Merton +Broadie*	$\sigma=0.30$ *(b) Merton +Broadie*	$\sigma=0.25$ *(c) Naive MC*	$\sigma=0.30$ *(d) Naive MC*		*Squared error* $[(a)-(c)]^2$	*Squared error* $[(b)-(d)]^2$	$\sigma=0.25$ *(e) PMC*	$\sigma=0.30$ *(f) PMC*		*Squared error* $[(a)-(e)]^2$	*Squared error* $[(b)-(f)]^2$
pdo	90	95	2.2798	2.4170	2.1575	2.2956	2.0350	2.2425	0.4840	0.0150	0.0028	2.0407	2.2713	0.3115	0.0136	0.0006
pdo	100	95	2.2947	2.4258	2.1857	2.3144	2.1574	2.2646	0.4760	0.0008	0.0025	2.2260	2.3146	0.3120	0.0016	0.0000
pdo	110	95	2.6252	2.6246	2.6319	2.6013	2.5498	2.4458	0.4835	0.0067	0.0242	2.8313	2.7852	0.3200	0.0398	0.0338
pdo	90	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.3120	0.0000	0.0000
pdo	100	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.3120	0.0000	0.0000
pdo	110	100	3.0000	3.0000	3.0000	3.0000	3.0000	3.0000	0.0930	0.0000	0.0000	3.0000	3.0000	0.3120	0.0000	0.0000
puo	90	105	3.7760	4.2293	3.8193	4.3792	3.5835	4.4614	0.4600	0.0556	0.0068	2.7533	2.8946	0.3200	1.1364	2.2040
puo	100	105	5.4932	5.8032	5.7966	6.2554	5.4966	5.8779	0.4520	0.0900	0.1425	3.6586	3.8920	0.3200	4.5710	5.5857
puo	110	105	7.5187	7.5649	8.1849	8.3981	8.4449	8.2022	0.4605	0.0676	0.0384	5.1286	5.4700	0.3120	9.3410	8.5738
pdi	90	95	2.9586	3.8769	3.0745	3.9915	0.7725	0.6607	1.1540	5.2992	11.0942	2.1701	3.3584	0.3195	0.8179	0.4008
pdi	100	95	6.5677	7.7989	6.6703	7.9034	4.8786	5.1644	0.4680	3.2102	7.5021	4.4645	5.9924	0.3200	4.8656	3.6519
pdi	110	95	11.9752	13.3078	11.9621	13.3242	11.7010	12.6445	0.4835	0.0682	0.4620	9.2133	9.5404	0.3200	7.5559	14.3171
pdi	90	100	2.2845	3.3328	2.4078	3.4509	0.1499	0.1066	0.2190	5.0981	11.1843	2.5901	3.9574	0.3120	0.0332	0.2565
pdi	100	100	5.9085	7.2636	6.0319	7.3816	1.1663	1.3578	0.2105	23.6741	36.2862	5.3388	6.7993	0.3040	0.4804	0.3391
pdi	110	100	11.6465	12.9713	11.7450	13.0711	10.2182	10.5818	0.2180	2.3311	6.1966	9.2165	11.1055	0.2960	6.3933	3.8636
pui	90	105	1.4653	2.0658	1.4154	1.9089	0.7350	0.6244	0.4680	0.4629	1.6499	2.5959	2.8700	0.3435	1.3936	0.9237
pui	100	105	3.3721	4.4226	3.0622	3.9634	0.7062	0.7177	0.4680	5.5507	10.5346	5.9681	6.1137	0.3275	8.4443	4.6238
pui	110	105	7.0846	8.3686	6.4119	7.5284	3.6323	3.5316	0.4760	7.7262	15.9744	9.4208	9.8662	0.3275	9.0535	5.4653
									RMSE	1.7265	2.3700			RMSE	1.7343	1.6707
									Average RMSE	2.0483				Average RMSE	1.7025	

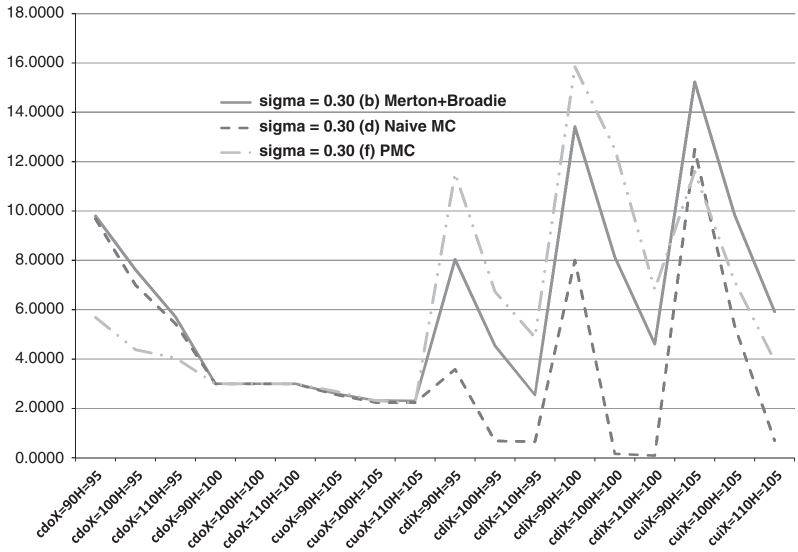

Figure 1: Premiums of European discrete barrier call options ($S=100$, $T=0.5$, $r=0.08$, $q=0.04$, $\sigma=0.30$) computed with PMC and MC models versus the benchmark Merton *et al* (1973) +Broadie *et al* (1997); for MC and PMC solutions: 1000 simulations.

levels of strike price X and barrier H, as suggested in Haug (2006). For MC and PMC solutions, we simulate 1000 trajectories for each option valuation.

On the basis of the criteria of RMSE (Root Mean Square Error), using as a benchmark the analytical solution developed by Merton (1973) and emphasized by Reiner and Rubinstein (1991), adjusted for discretization by Broadie *et al* (1997), the PMC model beats the naïve numerical model using MC simulation, whatever the call or put options. However, the naïve MC model applies to the pricing of barrier put options when the volatility σ is equal to 0.25 has a slightly lower RMSE (1.7265 versus 1.7343) than the PMC model.

Figures 1 and 2 illustrate call and put options prices when $\sigma=0.30$. Overall, we observe the superior accuracy of the PMC model compared with the naïve MC model when pricing single-barrier options.

However, looking in detail at Figures 1 and 2 and Tables 3 and 4, we observe that the naïve MC model outperforms the PMC model for down-and-out and up-and-out calls and puts (lower sum of squared errors). It is the opposite for down-and-in and up-and-in calls and puts, where the PMC model performs best.

Investigating the power of convergence of the naïve MC model versus the PMC model, we choose an up-and-out call with $H=105$, $X=100$ and $\sigma=0.30$. On the basis of Figure 3, we observe that the PMC model converges faster than the naïve MC model.

Pricing sequential compound real options with the PMC model

For an MC simulation of 10 000 trajectories with 5 steps (1 step = 1 year), we obtain a value of 40.31 with the probabilities $P_1 = 10\,000/10\,000$

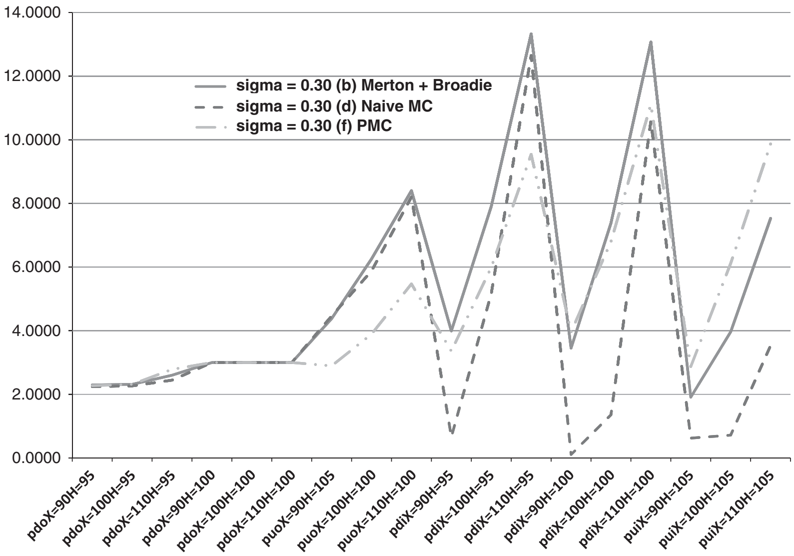

Figure 2: Premiums of European discrete barrier put options ($S=100$, $T=0.5$, $r=0.08$, $q=0.04$, $\sigma=0.30$) computed with PMC and MC models versus the benchmark Merton *et al* (1973) +Broadie *et al* (1997); for MC and PMC solutions: 1000 simulations.

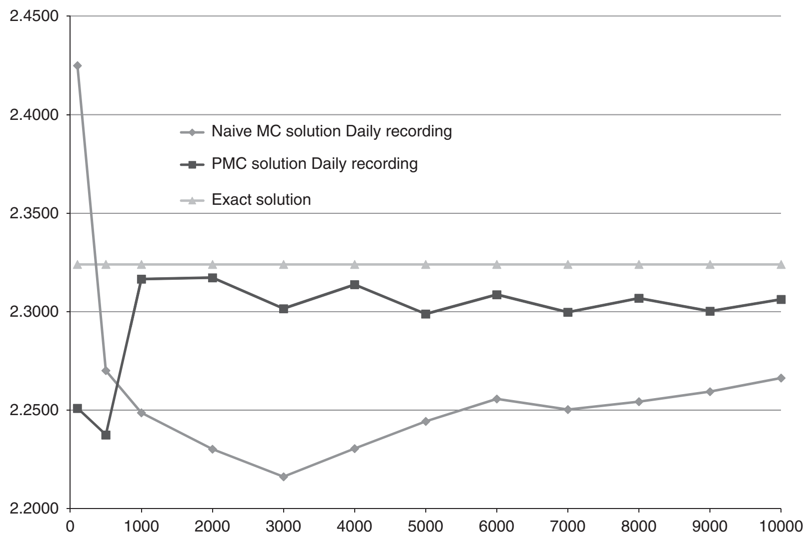

Figure 3: Illustration of the convergence for an up-and-out barrier call option ($S=100$, $T=0.5$, $r=0.08$, $q=0.04$, $H=105$, $X=100$, $\sigma=0.30$) computed with PMC and MC models versus the benchmark Merton *et al* (1973)+Broadie *et al* (1997) that we call exact solution. We increase the number of simulations from 100 to 10 000.

that $S_1 \geqslant \$30$ million (S_1 the value of the project at $T=1$), $P_2=9\,787/10\,000$ that $S_3 \geqslant \$90$ million (S_3 the value of the project at $T=3$), and $P_3=6342/10\,000$ that $S_5 \geqslant \$210$ million (S_5 the value of the project at $T = 5$), with $P = P_1 P_2 P_3 = 0.62$.

$$c = \left[\frac{P}{N}\sum_{i=1}^{NP} c_{TiAF} + K(1-P)\right]e^{-rT}$$

$$= \left[\frac{0.62}{10000}(877,626) + 0.(1-0.62)\right]e^{-0.06(5)}$$

$$= 40.31$$

CONCLUSION

We present an original PMC model based on MC simulation that prices discrete barrier options: we call it probabilistic since it computes the probability of not crossing the barrier for knock-out options and crossing the barrier for knock-in options. This probability is then multiplied by an average sample discounted payoff of a plain vanilla option that has the same inputs as the barrier option but without a barrier and to which we have applied a filter. We test the consistency of our model with an analytical solution (Merton, 1973; Reiner and Rubinstein, 1991) adjusted for discretization by Broadie *et al* (1997) and a naïve numerical model using MC simulation. Overall, based on call and put options and all types of single-barrier European options (up-and-out, down-and-out, up-and-in and down-and-in) and based on the criteria of RMSE, the PMC model is superior to the naïve MC simulation. We show through an example that it also converges faster towards the exact solution. However, the naïve MC simulation offers better results for down-and-out and up-and-out calls and puts (lower sum of squared errors). It is the opposite for down-and-in and up-and-in calls and puts, where the PMC model performs best.

We explore the potential and versatility of the PMC model in pricing other types of options than barrier options. We choose to price sequential compound real options since these options are more complex options than barrier options and are involved in project valuation. We show that the PMC model offers a simpler methodology than the binomial model; the latter involves building three different trees whereas the PMC model computes the sequential compound real options with one simulation only. Whenever there is a way to measure the probability that an event attached to an option will occur, for example the crossing of a barrier or the value of a project lower than its cost – or whatever the feature embedded in the option – there is a way to implement the PMC model. We have shown that by computing two distant types of options, the PMC model is quiet versatile in valuing options with distinctive embedded features.

Further works will extend the range of applications of the PMC model and will investigate variance reduction techniques applied to this model.

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