Original Article

Frictional costs of diversification: How many CTAs make a diversified portfolio?

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Bernd Scherer
is CIO at FTC Capital, a Vienna-based CTA. Bernd has been full-time professor of finance at EDHEC Business School and Managing Director at Morgan Stanley. He has published papers in the Journal of Empirical Finance, Journal of Banking and Finance, Journal of Financial Markets, Quantitative Finance, Journal of Economics and Statistics, Financial Analysts Journal, Journal of Portfolio Management, Risk, and many more, and wrote or edited eight books in Quantitative Finance.

Correspondence: Bernd Scherer, FTC Capital Vienna, Prater str.31, 1020 Austria
E-mail: scherer@ftc.at

ABSTRACT **How many commodity trading advisors (CTAs) are needed to arrive at a diversified portfolio? We provide two computational alternatives to find the optimal number of CTAs in a real-world setting where frictional costs of diversification, the amount of assets under management, risk aversion and the state dependence on hedge fund payoffs matter to investors.**

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INTRODUCTION

With the establishment of modern portfolio theory, researchers have started to test how well its normative diversification advice is reflected in observed portfolios. Early studies focused on equity markets and tried to answer the question: 'How many stocks make a diversified portfolio?' Elton and Gruber (1977), Statman (1987), Newbould and Poon (1993), O'Neal (1997) and Statman (2004) all come to different conclusions about the optimal number of stocks in a naively diversified (randomly selected stocks with equal weighting in the absence of conditioning information) portfolio. The recommended holdings range between 10 and 300 stocks. However, even these numbers are high relative to the accounts of individual investors, which often contain only a handful of stocks as well as large holdings in their own company stocks. On the back of these results, Statman (2004) coined the term 'behavioural portfolio theory', that is, the attempt to 'rationalize' the apparent under-diversification of individual investors. In his view, individual investors divide their total wealth into mental buckets according to their investment goals. Equities fall into the top portfolio layer that reflects the investors' demand for lottery tickets. Recent support for this has been provided by Frazzini and Pedersen (2010), who find that leverage aversion will cause investors to arrive at under-diversified portfolios that concentrate on the more volatile stocks.

The popularity of hedge funds, as both an investment vehicle and an object of academic interest, has created interest in the question: 'How many hedge funds (commodity trading advisors (CTAs), managed futures, etc.) make a diversified portfolio?'. Despite well-documented differences in hedge fund return distributions (most notably non-normality and non-linearity with respect to underlying risk factors) and hedge fund investment costs, virtually all studies heavily borrowed the methodologies designed for individual stock portfolios and applied them to hedge funds. In short, this amounts to a two-step procedure:

1. Simulate random portfolios of size $n = 1, \ldots, N$ and record the evolution of volatility, SHARPE-ratio or correlation with an already diversified index to trace out a diversification curve, that is, a functional relationship between portfolio standard deviation and portfolio size.
2. Decide when the marginal improvement in the above-mentioned statistic becomes 'small'. What 'small' means is usually left to eyeballing the diversification curve, that is, it relies on the researcher's subjective statement.

Henker and Martin (1998), Amin and Kat (2002) and Lhabitant and Learned (2002) are examples of this approach. The number of hedge funds they deem optimal ranges between 5 and 25. Despite the arbitrariness of the above approach, Brown *et al* (2011) claim that fund of funds exhibit excess diversification.

We see several shortcomings in the above papers. First, no attempt is made specifying the frictional costs of adding another fund into a portfolio. In the absence of these costs, it is always optimal to naively diversify across all possible investments. Samuelson (1967) made this point early on by stating that investors should diversify as much as possible, aware of the tradeoff between diversification and its costs. Frictional costs arise from fixed monitoring costs per additional funds, as well as the loss of bargaining power for fee rebates when diversifying among too many funds. Second, assets under management do not enter the decision-making problem, even though fixed costs can be spread more easily across a large pool of assets. Clearly it makes a very practical difference whether a decision-maker with 10 million USD or 100 million USD asks for the optimal number of assets to invest in. Third, the reduction in volatility is most valuable for investors with high risk aversion, while investors with low risk aversion will be less willing to incur frictional diversification costs for a reduction in volatility they value only very little. Finally, but most importantly, volatility for an investment will not differ if we reshuffle returns across different states of the world. However, investors have a preference for investments that pay well in bad states (where wealth is down) of the world. Such an investment might be more valuable or offer more protection than an asset that offers a higher SHARPE-ratio or lower volatility. Diversifications studies on hedge funds remain silent on this topic. This is most relevant for CTAs that, due to their trend-following trading style and money management techniques, offer portfolio insurance properties. For a risk-averse investor, it will now matter most how well his portfolio of CTAs performs in those months where he values insurance most highly. Consequently, the normative advice of these papers is limited at best.

Another more recent motivation of our work is the flood of papers motivated by Demiguel *et al* (2009). The authors show that equal weighting ($1/n$) is preferable to mean variance optimization if the SHARPE ratio differences between assets are small (adjusted for sample size). This situation is likely to be given for CTAs that are notoriously known for both large return dispersion and little to no persistence as documented in Bhardwaj *et al* (2008). None of the $1/n$ papers discusses frictional diversification costs and consequently the optimal number of assets is imposed rather than derived.

The next section first reviews the existing methodology used in diversification studies. We then extend the traditional mean variance framework to account for frictional costs of diversification, differences in assets under management and risk aversion to arrive at a closed form solution for the optimal number of assets. This method works well under the assumptions of normality and for investors who show no interest in the conditional nature of hedge fund returns. However, for investors who care whether losses are realized in good or bad times, we evaluate portfolios of CTAs for different state price deflators (assuming investors exhibit power utility). This allows us to evaluate whether a portfolio of CTAs is properly diversified, where diversification means the extraction of 'CTA-beta', that is, the ability to protect from a fall in risky assets. The latter sections apply both methods on CTAs. The final section concludes.

OPTIMAL DIVERSIFICATION REVISITED

Diversification curves

Diversification curves trace out the relation between the expected risk (or, more generally, performance measure) of a portfolio of hedge funds (or more general assets) that consists of n randomly selected and equally weighted constituents. The brute force approach is to repeatedly sample n funds (out of a total of N funds). For the jth sampling we get a set $S_j(n)$ that contains n index numbers (out of N) to calculate the portfolio return

$$R(n)_{m,j} = \sum_{i \in S_j(n)} \tfrac{1}{n} R_{i,m} \tag{1}$$

for all $m = 1, \ldots, M$ scenarios, that is, observations on a databank. This exercise is repeated many times, that is, we compute the variance of (1), that is,

$$\sigma(n)_j^2 = \sum_{m=1}^{M} \left(R(n)_{m,j} - \tfrac{1}{M} \sum_{m=1}^{M} R(n)_{m,j} \right)^2 \tag{2}$$

and average it over all J samplings. However, it was shown early on by Elton and Gruber (1977) that a simulation is not needed and we can instead replace time-intensive simulations with the formula for the expected variance of a naively diversified portfolio.

$$\sigma^2(n) = \frac{\overline{\sigma}^2}{n} + \left(1 - \frac{1}{n}\right) \overline{\sigma}^2 \overline{\rho} \tag{3}$$

where $\overline{\sigma}^2$ and $\overline{\rho}$ represent average variance and correlation across all N assets, that is, for the whole universe rather than the subset of n stocks. The rationale for this surprising result is that (3) is equivalent to the expectation of all possible n out of N permutations of index numbers. Mathematically, (3) is a monotonically decreasing function of n that converges to $\overline{\sigma}^2 \overline{\rho}$; hence, optimal diversification is only achieved by holding the full universe. A decision to hold less than n assets is typically imposed by *ad hoc* assumptions on the desired degree of reduction in diversifiable risk and is therefore not satisfactory.

Mean variance-based performance measure

Rather than the previous *ad hoc* approach, we first suggest starting to model optimal diversification as a decision-making problem for a standard mean variance investor and ask: 'How many hedge funds make a diversified portfolio?' In contrast to previous work, our decision maker will trade off the marginal benefits from diversification against their marginal costs. For an investor using naive diversification (that is, in the absence of conditioning information) the only choice parameter is the number of equally weighted funds she intends to include, that is, we assume that the investor tries to optimize

$$\mu(n) - \lambda\sigma^2(n) - n\frac{f}{aum} \tag{4}$$

where $\mu(n)$ and $\sigma^2(n)$ are the expected return and risk for an equally weighted portfolio of size n, λ denotes the investor's risk aversion and f/aum represents the additional fixed costs as a fraction of assets under management per additional fund.

While adding funds to a portfolio, we can expect portfolio risk to fall, while average portfolio return is expected to remain constant, that is $(d\mu(n)/dn) = 0$.[1] We can therefore focus on the impact of increasing portfolio size on portfolio risk and diversification costs. For the optimal number of funds in a portfolio, marginal benefits and costs need to be balanced. Hence we can write

$$\frac{f}{aum} = -\lambda\frac{d\sigma^2(n)}{dn} \tag{5}$$

As seen in the previous section, the expected variance for an equally weighted portfolio can be equally written as $\sigma^2(n) = \bar{\sigma}^2/n + (1-(1/n))\bar{\sigma}^2\bar{\rho}$, so we can find an explicit solution for the marginal change in risk

$$\frac{d\sigma^2(n)}{dn} = \frac{1}{n^2}\bar{\sigma}^2(\bar{\rho}-1) \tag{6}$$

Substituting (6) into (5) we arrive at

$$\frac{f}{aum} = -\lambda\frac{1}{n^2}\bar{\sigma}^2(\bar{\rho}-1) \tag{7}$$

which can be solved for the optimal n.

$$n^* = \sqrt{\lambda\bar{\sigma}^2(1-\bar{\rho})\left(\frac{f}{aum}\right)^{-1}} \tag{8}$$

We see that the optimal number of funds increases with rising risk aversion (λ), rising average volatility ($\bar{\sigma}^2$), falling average correlation ($\bar{\rho}$), falling frictional costs (f) and rising assets under management (*aum*). The objective of (8) is to provide insight into the determinants of naïve diversification with frictional diversification costs. Linearizing (8) via logs will also yield a testable model for observed fund of fund data. The model suffers from two shortcomings. First, how should an investor solve the above problem when instead he has some information (return forecasts) on individual funds? In this case we need to solve a quadratic program with mixed integer constraints using dedicated software like NUOPT for SPLUS/R. A formulation is given in Appendix C. Second, it is clear that (8) will provide little guidance when hedge fund data are non-normally distributed. This is dealt with in the next section.

State price deflators

The previous section established the optimal number of hedge funds in closed form for a mean variance decision-maker. Even if the returns for our assets would all be individually normal (which we know they are not) the approach in the previous section would fail. The main drawback of the solution in the previous sub-section arises from hedge fund returns that display a non-linear relationship to risky assets, as documented by Fung and Hsieh (2001). This is particularly true for CTAs that create part of their attraction from their ability to perform well if risky assets are down for a prolonged period of time. This feature of CTA returns is important to real-world investors but gets lost in SHARPE-ratio or volatility measures. Imagine we simply reshuffle hedge fund returns, that is, we take the returns of hedge fund strategies, and rearrange them randomly (without replacement) along the timeline. This will affect neither their SHARPE-ratio nor their skewness or volatility, but it will alter their attractiveness to real-world investors as suddenly their returns conditioned on the returns of equity and bond markets will change.

We follow Chen and Knez's (1996) seminal work and use state price deflators for performance measurement to address this problem. A state price deflator is a stochastic discount factor that applies a separate discount rate to each state of the world. States where economy-wide wealth is down and therefore marginal utility is up should carry a larger importance for risk-averse investors. Hence we can generically express the state price deflator Λ_m for state m, a negative function of wealth in state m (W_m), and a positive function of risk aversion (γ)

$$\Lambda_m = F\left(\underset{(+)}{\gamma}, \underset{(-)}{W_m}\right) \tag{9}$$

The exact functional form for (9) and its calibration (choice of γ) to real data is described in Appendices A and B. Once we know the state price deflator, we can price any asset (or derivative claim) by multiplying the state price deflator with the corresponding payoff to the asset under consideration and build the expectation under the real-world probability measure P. The value of an n fund portfolio with return $R(n)_{m,j}$ in state m after frictional costs $\varphi(n)$ in simulation run j is given by)

$$V(n)_j = E^P\left(\Lambda_m \cdot \left(1 + R(n)_{m,j}\right)\right) - \phi(n) \tag{10}$$

Averaging across many (J) we get

$$EV(n) = \overline{V}(n) = \tfrac{1}{J}\sum_{j=1}^{J} V(n)_j \tag{11}$$

Consequently, funds that pay off well when wealth is low will get a higher valuation even when their average returns are identical or even lower than those hedge funds that synchronize their losses with losses in the economy.

In order to model the frictional costs of diversification, we introduce two cost functions. The first simply models the increase in fixed costs that arise when an additional fund is added to a portfolio. These costs can be thought of as monitoring costs (hire qualified analysts, consultants, buy software, data, or incur other due diligence costs like costs of flights, hotel). All this is consumed in annual fixed costs f per additional fund; hence, costs per months as a fraction of assets under management (aum) are given by

$$\phi(n) = -n\frac{f \cdot aum^{-1}}{12} \tag{12}$$

In addition, we introduce rebates (θ) to reflect buying power if an investor focuses on investing larger sums in a smaller number of funds.

$$\phi(n) = -n\frac{f \cdot aum^{-1}}{12} + b \cdot \log\left(\frac{aum}{n}\right) \cdot \frac{\theta}{12} \quad (13)$$

The size of the rebate depends on $\log(aum/n)$, that is, it rises with the average fund size. We apply a scaling factor b to control the range of rebates. Typically one would set b equal to the inverses of the log of the maximum mandate size in a given study, such that the maximum mandate gets a full rebate $\theta/12$ while all other mandates get a lower rebate. We use this valuation measure to calculate where $(\mathrm{d}EV(n)/\mathrm{d}n) = 0$ to arrive at the optimal number of CTAs in a portfolio.

CASE STUDY #1: OPTIMAL NUMBER OF CTAs IN A MEAN VARIANCE FRAMEWORK

Our first case study looks at potential differences in optimal diversification when diversifying across large or small CTAs. The data for this case study are summarized in Table 1. We use monthly data from the Barclays Managed Futures database stretching from January 2007 to January 2012, where size is defined as average assets under management (AUM) between January 2011 and January 2012. Small funds are defined as funds with AUM smaller than 100 million. No attempt has been made to filter out micro CTAs (CTAs with less than 5 million under management) so the number of funds is large and investing in this quintile might not be entirely realistic. Medium-size CTAs are defined by AUM ranging between 100 and 750 million, while large CTAs are defined by AUM above 750 million.[2]

Comparing CTAs across the size dimension reveals some interesting differences. First, large CTAs display the lowest average volatility (13 per cent) and the highest average correlation (0.8). The high correlation might be due to the

Table 1: Managed futures – size versus key characteristics

	Small	*Medium*	*Large*
Definition	AUM smaller than 100 million	AUM between 100 and 750 million	AUM larger than 750 million
# of funds	332	137	60
Average volatility	19.98	14.55	13.0
Average internal correlation	0.33	0.46	0.80
Average Sharpe	0.50	0.62	0.56
Median Sharpe	0.55	0.56	0.53
5% Percentile Sharpe	−0.21	0.08	0.07
95% Percentile Sharpe	1.22	1.40	1.25
$1/n$ Sharpe	0.96	0.91	0.78

The table presents average volatility (aggressiveness), average correlation (diversification benefit), average SHARPE (skill) as well as the SHARPE of an equally weighted portfolio for managed futures bucketed into three size categories. Our data stretch from January 2007 to January 2012. Size is defined as average AUM between January 2011 and January 2012.
Source: Barclays Managed Futures Database.

fact that large CTAs originate from the same set of ideas, as some of them are just early offspring of other large CTAs. Smaller CTAs try instead to bring in new ideas as they can't afford to be perceived as 'me too' products. This is reflected in their low internal correlation (0.33). Small CTAs want to grow with performance and their large volatility (19.98 per cent) reflects the call-options-like feature of hedge fund business models. Large funds, on the other hand, display lower volatility. This reflects the fact that they have more to lose from drawdowns (large redemptions), as well as the fact that their size makes it difficult to operate with the same level of aggressiveness as their smaller peers.

All this should clearly affect the optimal number of CTAs in an investor's portfolio. The basic information in Table 1 contains all inputs needed to apply (8). We start with an example. Imagine a small investor with 10 million assets under management willing to invest in large CTAs. Her risk aversion is 2 and she faces fixed costs of 15 000 USD per additional fund.[3] How many CTAs will this investor find optimal? The optimal solution becomes

$$\sqrt{2 \cdot 0.13^2 \cdot (1-0.8)\left(\frac{0.015}{10}\right)^{-1}} = 2.12 \qquad (14)$$

We can broaden our calculations to varying levels of assets under management and risk aversion, as displayed in Figure 1 where various 'iso-diversification'-curves are shown. These curves describe the location of aum and risk aversion combinations that would yield the same optimal number of funds. Each curve describes the tradeoff between risk aversion and assets under management. A small number of assets in a portfolio are constant with either a very large investor and very low risk aversion or a very small investor and very high risk aversion.

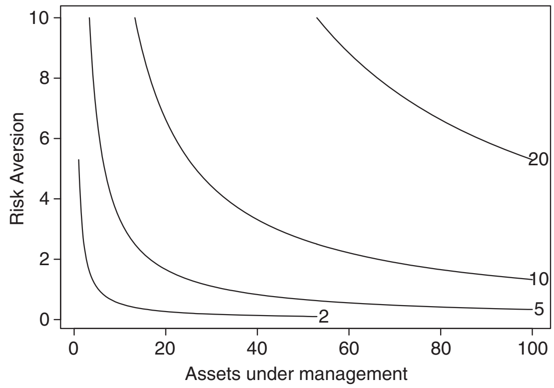

Figure 1: Optimal number of CTAs for a mean variance investor.
The contour plot displays the optimal number of CTAs using (8) as a function of both risk aversion and assets under management for the universe of large CTAs, that is, $\overline{\rho}=0.80$, $\overline{\sigma}^2=0.13$ and $f=0.015$.

Investing in large funds – that are often lookalikes – has limited payoffs and so the number of funds should be kept small unless risk aversion becomes large.

CASE STUDY #2: OPTIMAL NUMBER OF CTAs IN CONTINGENT CLAIMS FRAMEWORK

In this section we apply the more general state price deflator approach to a set of CTA returns in order to find the optimal number of CTAs. Our starting point is using the implied risk aversion of a power utility investor who invests 100 per cent of his wealth in the US equity market to define a representative investor. We use 10 years of monthly data from January 2002 to December 2011 where the one-month

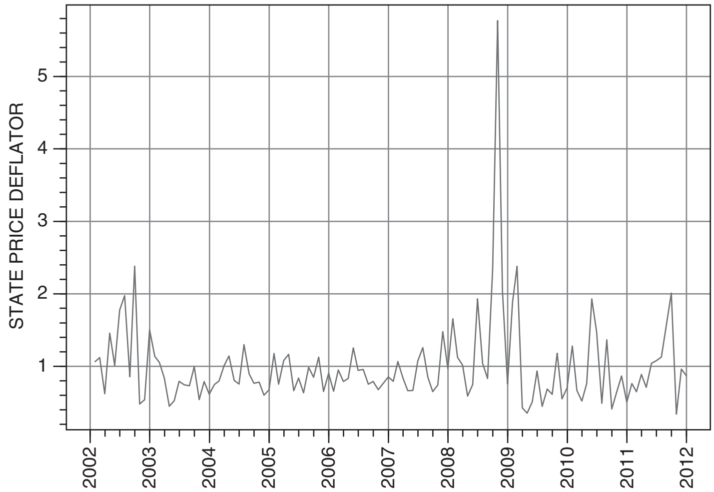

Figure 2: State price deflator.
We plot the implied state price deflator for an economy where the representative display power utility investor finds it optimal to hold 100 per cent in US equities. Our data span 10 years of monthly data from January 2002 to December 2011. The one-month US risk-free rate and US equity market excess returns come from the FAMA/FRENCH research database.

US risk-free rate and US equity market excess returns come from the Fama and French research database. We calibrate the state price deflator as described in Appendix B and plot it across time. The result of this is shown in Figure 2.

The importance of performing well in down markets in order to score a high valuation measure is evident from the figure. Hedge funds that perform poorly in times of down markets (in general short volatility strategies) will be penalized for this behaviour with respect to investor valuation. This effect will become even more pronounced for higher risk aversions, where our valuation measure becomes more and more centric in a few down markets.

Suppose we now want to use the above state price deflators to find the optimal number of CTAs for the period January 2002 to December 2011. First, we select all funds from the Barclays Managed Futures databank that show a complete 10-year series of monthly returns and are described in the databank as technical/systematic/diversified funds. Second, we also require those funds to be denominated in USD. After applying these criteria we are left with 74 funds. Imagine we use cost function (12), that is, a cost function that only incorporates the stepwise increase in fixed costs once a new fund is added to the portfolio. We set $\gamma = 2.3$, $f = 0.01$, $aum = 50$ and $J = 50\,000$, that is, we perform 50 000 simulations for a power utility investor with risk aversion of 2.3, 10 000 USD stepwise fixed costs and 50 million assets under management. Figure 3 plots the distribution (as a boxplot) of our valuation metrics given as a function of n, adding two new funds at a time.

The highest portfolio valuation is reached for four funds, that is, our investor finds it optimal to allocate 12.5 million per fund. We can interpret the valuation measure as monthly alpha of

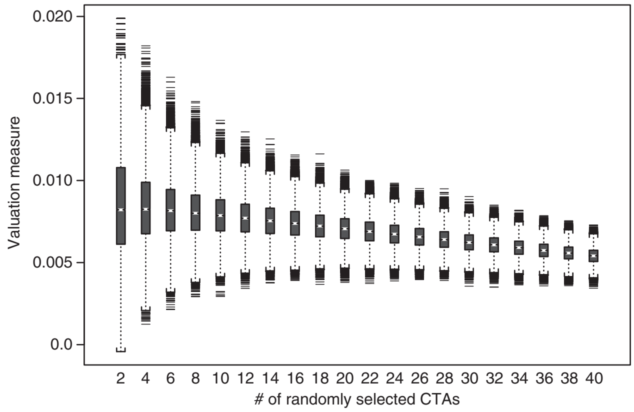

Figure 3: State price deflator.
Optimal diversification curve according to our valuation metric in (10) for $\gamma = 2.3$, $f = 0.01$, $aum = 50$ and $J = 50000$.

around 70 bps versus a 100 per cent investment in US equities with alpha of 0 per cent. In order to generalize this result, we repeat the above analysis for investors with 10, 50, 100 and 500 million assets under management, and risk aversions of 2.3, 10 and 50 for both cost functions.[4] The results are shown in Table 2.

Panel A provides several interesting features. As in the mean variance case, we see that higher AUM allow a larger number of CTAs. Equally, as the fixed costs of investing into an additional CTA simply become lower (as a percentage of AUM), the number of funds increases with assets under management. Higher risk aversion also means that an investor is willing to give up more return (pay for additional CTAs) in order to reduce portfolio volatility. The optimal number of CTAs rises with risk. For large risk aversions this seems excessive. An investor with 10 million AUM and risk aversion of 50 would still be willing to invest in 14 funds, that is, to incur costs of 1.4 per cent of portfolio size without expectations of increased returns. Where does this come from? Note that higher risk aversion

Table 2: How many CTAs make a diversified portfolio?

	Risk aversion		
	2.3	*10*	*50*
Panel A			
AUM			
10	2	4	14
50	4	10	36
100	8	12	>40
500	22	>40	>40
Panel B			
AUM			
10	2	4	18
50	2	8	24
100	2	10	22
500	4	12	>40

We display the optimal number of CTAs for investors with varying risk aversion and assets under management. Frictional costs of diversification are assumed to follow either (12) with 10 000 USD per additional fund (Panel A) or (13) with a rebate θ of 50bps and maximum account size for rebates of 250 million (Panel B).

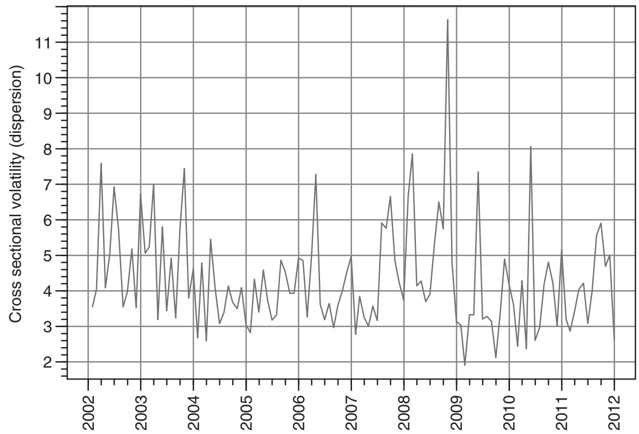

Figure 4: Cross-sectional volatility (dispersion) of CTA returns.
Cross-sectional volatility (dispersion) of CTA returns from January 2012 to December 2011. All funds are selected from the Barclays Managed Futures databank and show a complete 10-year series of monthly returns.

also means that the state price deflator becomes very large in a few extreme down markets. Why is diversification so important in those markets, that is, why do tail risk-sensitive investors need many CTAs? The answer to this can be found in Figure 4, which plots the cross-sectional volatility of CTA returns over time.

We see that the dispersion of returns is highest (that is, the correlation between CTAs is lowest) in extreme market states. This is worrying for CTA investors as they can't be sure that diversifying into only a few CTAs will provide protection for asset owners. In extreme months, some funds will have the optimal trades on, while some are still on the other side as their long-term models are caught out by trend reversals. Markets with high volatility, that is, large unanticipated swings in asset prices, will amplify even small differences in positions and the CTA universe will display inhomogeneous performance in times of crisis.

Investors with high risk aversion should allocate into CTAs as a group (as CTAs provide tail insurance) and invest in many CTAs of this group to ensure that a CTA portfolio actually provides tail insurance in down markets.

CONCLUSION

This article shows that in order to find the optimal number of CTAs in a portfolio (or any group of assets) it does not suffice to simply calculate/plot portfolio volatility as a function of portfolio size and choose the optimal number where additional diversification becomes 'small'. Surely we must be able to do better than that. Instead, the frictional costs of diversification, the amount of assets under management, the degree of risk aversion and the state dependence on hedge fund payoffs matter to investors. This article presented two practical and easy to implement methods that allow us to calculate the optimal number of equally weighted CTAs for investors who can't distinguish between CTAs. We found that portfolios containing more than 40 CTAs do not necessarily display over-diversification. They can simply arise out of a combination of large assets under management, high risk aversion and low frictional diversification costs. We also find that valuation measures based on realistic preferences lead investors to portfolios containing more rather fewer funds. This contrasts with earlier claims that adding more funds increases tail risk. We find that that CTA dispersion increases in extreme down markets as small differences in positions become amplified. Highly risk-averse investors should therefore diversify more.

NOTES

1 Assume a universe of hedge funds with average return μ. Sampling all funds without replacement generates in every sampling the average universe return such that the average trivially equals μ again. On the other hand, sampling one fund many times creates a return that is different for every sampling, but the average converges against μ, where convergence depends on the number of samplings.

2 Clearly our choice of data induces survivorship bias, but the case studies' aim is to add an example to our concept rather than deal with the many implementation details real-world investors have to face.

3 What does a risk aversion of 2 practically mean? We know from standard portfolio theory that a mean variance investor optimally allocates a fraction $w = \mu\lambda^{-1}\sigma^{-2}$ of his wealth into the risky asset, where μ and σ^2 denote expected return and risk, while λ describes risk aversion. For $\mu = 0.08$, $\sigma = 0.2$, a risk aversion $\lambda = 2$ translates into an optimal allocation of 100 per cent into the risky asset. Low risk aversions model investors that are willing to take substantial risks, while high risk aversions lead to less risky portfolios.

4 Note that for higher risk aversions our valuation can no longer be interpreted as alpha. Instead we can interpret our deflators as a subjective valuation rather than a no arbitrage valuation measure.

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APPENDIX A

State price deflators for performance measurement

Let Λ_m denote the state price deflator for $m=1, \ldots, M$ equally likely states of the world, defined by the marginal utilities for a representative investor

$$\Lambda_m = \frac{u'(W_m)}{\frac{1}{M}\sum_{m=1}^{M} u'(W_m)}(1+c)^{-1} \tag{A.1}$$

where $u()$ denotes the preferences of a representative investor and W_m denotes the wealth of the representative investor in state m. The riskless rate is given by c. We use equally likely states of the world as this facilitates our empirical work using equally time spaced return from a data bank. The valuation of a *single* CTA with return R_m in state m in (A.1) is found by applying the state price deflator across all states of the world

$$V = \frac{1}{m}\sum_{m=1}^{M} \Lambda_m(1+R_m) - 1 \tag{A.2}$$

This valuation measure will put more weight on returns that occur in states where Λ_m is high, that is, in those states of the world where marginal utility is high or equivalently where wealth in the economy is low. Consequently, funds that pay off well when wealth is low will get a higher valuation – even when their average returns are identical – than those hedge funds that synchronize their losses with losses in the economy. We need to find $EV(n)$, that is, our expected valuation measure for a n fund portfolio. Suppose in the jth draw we sample n funds, that is, we sample a set $S_j(n)$ that contains n index numbers, where each index number identifies a given CTA. For a single draw j we can now value a CTA portfolio with n randomly drawn funds according to

$$V(n) = \frac{1}{m}\sum_{m=1}^{M} \Lambda_m \left(1 + \sum_{i \in S_j(n)} \frac{1}{n} R_{i,m} - \phi(n)\right) - 1 \tag{A.3}$$

where $\phi(n)$ is defined as in (12) or (13). All draws are sampled independently as CTA returns do neither exhibit statistically significant serial correlation (in which case we would have employed sampling from blocks of random lengths) nor temporal dependency in their second moment. This process is repeated across $j=1, \ldots, J$ samplings and the expected value of an n fund portfolio is given by

$$EV(n) = \frac{1}{J}\sum_{j=1}^{J} \left(\frac{1}{M}\sum_{m=1}^{M} \Lambda_m \times \left(1 + \sum_{i \in S_j(n)} \frac{1}{n} R_{i,m} - \phi(n)\right) - 1\right) \tag{A.4}$$

We use this valuation measure to calculate where $(\mathrm{d}EV(n)/\mathrm{d}n) = 0$ to arrive at the optimal number of CTAs in a portfolio. Note that the only randomness in our simulations arises from $S_j(n)$.

APPENDIX B

Calibration of a state price deflator

To apply state price deflators to real-world data we need to make assumptions on the investor's utility function and on the asset our representative investor holds. Without apology, we assume power utility, that is, $u(1+R_{US,m})=(1/1-\gamma)(1+R_{US,m})^{1-\gamma}$ where marginal utility is given by

$$u'=\left(1+R_{US,m}\right)^{-\gamma} \quad \text{(B.1)}$$

Our representative investor is further assumed to hold 100 per cent of his wealth in US equities. Calibration involves choosing γ to satisfy the first-order condition of a utility-maximizing investor as given in (B.2)

$$\frac{1}{m}\sum_{m=1}^{M}\left(1+R_{US,m}\right)^{-\gamma}\left(R_{US,m}-c\right)=0 \quad \text{(B.2)}$$

which is a simple numerical exercise. We arrive at $\gamma=2.3$ for the above-described data set. Our implied risk aversion is low, as the past 10 years of US equity returns provide a SHARPE-ratio of just 0.22. Any investor willing to accept that position must display low aversion to risk, that is, be willing to accept a small return per unit of risk. Substituting the expression for marginal utility given by (B.1) into (A.1) and noting that $W_m=1+R_{US,m}$, we get (B.3)

$$\Lambda_m=\frac{\left(1+R_{US,m}\right)^{-2.3}}{\frac{1}{M}\sum_{m=1}^{M}\left(1+R_{US,m}\right)^{-2.3}}(1+c)^{-1} \quad \text{(B.3)}$$

This exercise can be applied to other one parameter utility functions. Researchers can also use bootstrapping methods to test for the statistical significance of estimates.

APPENDIX C

Frictional diversification costs and optimal number of assets with return information and mean variance preferences

Let μ_i and w_i denote the expected return and weight for asset i and σ_{ij} the covariance between asset i and j. We now need to maximize

$$\max_{w,\delta}\sum_{i=1}^{n}w_i\mu_i-\left(\sum_{i=1}^{n}\delta_i\right)\frac{f}{aum} \quad \text{(C.1)}$$

under the following constraints

$$\sum_{i=1}^{n}\sum_{i=1}^{n}w_iw_j\sigma_{ij}\leqslant\overline{\sigma} \quad \text{(C.2)}$$

$$\sum_{i=1}^{n}w_i=1 \quad \text{(C.3)}$$

$$w_i\leqslant M\delta_i \quad \text{(C.4)}$$

$$0\leqslant w_i\leqslant 1 \quad \text{(C.5)}$$

$$\delta_i\in\{0,1\} \quad \text{(C.6)}$$

Equation (C.1) models the expected portfolio return given return forecasts as well as the number of funds ($\Sigma_{i=1}^{n}\delta_i$) multiplied by the monitoring/selection costs, f/aum, per fund (as a percentage of AUM). Here δ_i is defined as an integer variable (taking on either a value of 0 or 1) as in (C.6). As an asset is either in or out we pin down the value of this integer value by (C.4), where M is a 'large' number (for example, 10). If an asset is included (even at tiny size), the inequality is only satisfied for $\delta_i=1$. As soon as it leaves the optimal solution δ_i must assume a value of zero. Equation (C.2) and (C.5) are the usual budget and individual position constraints. The system (C.1)–(C.6) can be solved with a quadratic solver allowing integer constraints as in NUOPT™ for SPLUS™.