Original Article

Optimal cross-hedging under futures mispricing: A note

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ABSTRACT This note provides a theoretical discussion on the optimal hedge ratio when a cross-hedging is performed under futures mispricing. Assuming that the investor's risk measure is the conditional variance, it is showed that the optimal hedge ratio depends on the correlation structure and the relative volatility of the three assets involved: the futures instrument, the underlying asset of the futures contract and the portfolio to be hedged.

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INTRODUCTION

Futures contracts are essential instruments in financial risk management. When traders implement futures hedging strategies, it is the most often case that the underlying asset of the futures contract does not match the portfolio to be hedged. Transaction costs and transitory illiquidity encourages price deviations in the futures market from the theoretical no-arbitrage values. This note attempts to provide a theoretical discussion on the optimal hedge ratio, taking into account these two aspects: (i) mispricing fluctuates randomly around 0 and (ii) the asset to be hedged is not a replicating portfolio of the underlying asset of the futures contract. Assuming a variance-minimizing

strategy, we extend the model proposed in Lafuente and Novales (2003) by introducing mismatching between the asset to be hedged and the underlying asset of the futures contract to show how the nature of the correlations and the relative volatility affect the optimal hedge ratio. The nature of the futures position and the number of futures contract required for hedging are discussed.

THE CROSS-HEDGING MODEL

Let W_t denote the price of the asset to be hedged and S_t the spot price concerning the underlying asset of the futures contract at period t , respectively, and assume that the instantaneous

return of the both assets evolve according to a geometric Brownian motion:

$$dW_t = \mu_{W,t} W_t dt + \sigma_{W,t} W_t dz_{1,t} \quad (1)$$

$$dS_t = \mu_{S,t} S_t dt + \sigma_{S,t} S_t dz_{2,t} \quad (2)$$

where $dz_{1,t} = \xi_1(dt)^{1/2}$ and $dz_{2,t} = \xi_2(dt)^{1/2}$ are Wiener processes. The non-arbitrage cost-of-carry valuation of the futures contract leads to the following relationship between spot and futures prices:

$$F_{t,T} = S_t e^{(r-d)(T-t)} \quad (3)$$

where $r-d$ is the net cost of carry of the underlying stock, and T is the maturity period. This relationship implies that spot and futures markets share a common source of risk. However, traded futures prices depart from this theoretical valuation (see Kawaller and Koch, 1984; Yadav and Pope, 1990, 1994; Draper and Fung, 2003, among others) and spot and futures prices do not share a common ARCH feature (see, for example, Arshanapalli and Doukas, 1994). Following Lafuente and Novales (2003), we model both aspects including a specific noise in the dynamics of the futures prices, which are derived by applying the Ito's lemma from (2) and (3):

$$dF_t = \mu_{F,t} F_t dt + \sigma_{S,t} F_t dz_{2,t} + \sigma_{M,t} F_t dz_{3,t} \quad (4)$$

where σ_M reflects another source of basis risk and $dz_{3,t} = \xi_3(dt)^{1/2}$ is a Wiener process. Let Σ_t denote the variance covariance matrix of the Gaussian noise vector $\xi \equiv (\xi_1, \xi_2, \xi_3)'$, that is:

$$\Sigma_t = E(\xi_t \xi_t') = \begin{pmatrix} 1 & \rho_{12,t} & \rho_{13,t} \\ & 1 & \rho_{23,t} \\ & & 1 \end{pmatrix}$$

Consider an investor who is long c_t units of the asset W at time t and decides to hedge return variability by using a futures contract which underlying asset does not match the previous

one. Assuming that the investor's risk measure is the conditional variance, the optimization problem can be stated as follows:

$$\begin{aligned} \text{Min} \quad & \text{Var}_t \left(c_t \frac{dW_t}{W_t dt} - h_t \frac{dF_t}{F_t dt} \right) \\ & \{h_t\} \end{aligned}$$

s.t.

$$dW_t = \mu_{W,t} W_t dt + \sigma_{W,t} W_t dz_{1,t}$$

$$dF_t = \mu_{F,t} F_t dt + \sigma_{S,t} F_t dz_{2,t} + \sigma_{M,t} F_t dz_{3,t}$$

which amounts to:

$$\begin{aligned} \text{Min} \quad & c_t^2 \sigma_{W,t}^2 + h_t^2 (\sigma_{S,t}^2 + \sigma_{M,t}^2 + 2\rho_{23,t} \sigma_{S,t} \sigma_{M,t}) \\ & - 2h_t c_t (\rho_{12,t} \sigma_{W,t} \sigma_{S,t} + \rho_{13,t} \sigma_{W,t} \sigma_{M,t}) \\ & \{h_t\} \end{aligned}$$

By taking the first-order necessary condition and solving, the minimum variance hedge ratio is found to be:

$$\frac{h_t}{c_t} = \frac{\rho_{12,t} \sigma_{W,t} \sigma_{S,t} + \rho_{13,t} \sigma_{W,t} \sigma_{M,t}}{\sigma_{S,t}^2 + \sigma_{M,t}^2 + 2\rho_{23,t} \sigma_{S,t} \sigma_{M,t}} \quad (5)$$

The optimal hedge ratio is undetermined only when the futures market returns are non-stochastic ($\rho_{23,t} = -1, \sigma_{S,t} = \sigma_{M,t}$). In this case, risk cannot be hedged regardless of the choice of the ratio to be applied. When the asset to be hedged and the underlying asset of the futures contract share a common noise ($dz_{1,t} = dz_{2,t}$), which implies $\rho_{12,t} = 1$ and $\sigma_{W,t} = \sigma_{S,t}$, the model becomes equivalent to the model originally proposed in Lafuente and Novales (2003), with perfect matching between the asset to be hedged and the underlying asset of the futures contract.

Proposition 1: The optimal hedge ratio is just a function of (i) the correlation between noises, (ii) the relative standard deviation between the specific and common noise of

the futures market and (iii) the relative standard deviation between the common noise in spot and futures markets and the disturbance of the asset to be hedged.

Proof: Equation (5) can be rearranged as follows: $\square$

$$\begin{aligned} \frac{h_t}{c_t} &= \frac{\rho_{12,t} \frac{\sigma_{W,t}}{\sigma_{S,t}} + \rho_{13,t} \frac{\sigma_{W,t} \sigma_{M,t}}{\sigma_{S,t} \sigma_{S,t}}}{\frac{\sigma_{S,t}^2}{\sigma_{S,t}^2} + \frac{\sigma_{M,t}^2}{\sigma_{S,t}^2} + 2\rho_{23,t} \frac{\sigma_{M,t}}{\sigma_{S,t}}} \\ &= \frac{\theta_t (\rho_{12,t} + \rho_{13,t} \phi_t)}{1 + \phi_t^2 + 2\rho_{23,t} \phi_t} \end{aligned} \quad (6)$$

where: $\theta_t = (\sigma_{W,t})/(\sigma_{S,t})$, $\phi_t = (\sigma_{M,t})/(\sigma_{S,t})$

The conditional variance of the hedged position can be written as follows:

$$\begin{aligned} c_t^2 \sigma_{W,t}^2 &\left[1 + \left(\frac{h_t}{c_t} \right)^2 \left(\frac{1}{\theta_t^2} + \left(\frac{\phi_t}{\theta_t} \right)^2 + 2\rho_{23,t} \frac{\phi_t}{\theta_t^2} \right) \right. \\ &\left. - 2 \left(\frac{h_t}{c_t} \right) \left(\rho_{12,t} \frac{\sigma_{S,t}}{\sigma_{W,t}} + \rho_{13,t} \frac{\sigma_{M,t}}{\sigma_{W,t}} \right) \right]. \end{aligned}$$

Using the above expression, and equation (6), the minimized conditional variance can be expressed as follows:

$$\begin{aligned} c_t^2 \sigma_{W,t}^2 &\left[1 - \frac{(\rho_{12,t} + \rho_{13,t} \phi_t)^2}{1 + \phi_t^2 + 2\rho_{23,t} \phi_t} \right] \\ &= c_t^2 \sigma_{W,t}^2 \left[\frac{\left((1 - \rho_{12,t}^2) + \phi_t^2 (1 - \rho_{13,t}^2) \right) + 2\phi_t (\rho_{23,t} - \rho_{12,t} \rho_{13,t})}{1 + \phi_t^2 + 2\rho_{23,t} \phi_t} \right] \end{aligned} \quad (7)$$

where $(1 - \rho_{12,t}^2) + \phi_t^2 (1 - \rho_{13,t}^2) + 2\phi_t (\rho_{23,t} - \rho_{12,t} \rho_{13,t}) / (1 + \phi_t^2 + 2\rho_{23,t} \phi_t)$ is the ratio between the conditional variances of the hedged and

unhedged position. Alternatively, $(\rho_{12,t} + \rho_{13,t} \phi_t)^2 / (1 + \phi_t^2 + 2\rho_{23,t} \phi_t)$ is the difference between both these conditional variances but expressed in percentage of the conditional variance of the unhedged position. Both ratios can be used as a measure of hedging effectiveness (HE). Let us denote the second ratio for further discussion as $HE = (\rho_{12,t} + \rho_{13,t} \phi_t)^2 / (1 + \phi_t^2 + 2\rho_{23,t} \phi_t)$.

Proposition 2: If $\phi_t = (\sigma_{M,t})/(\sigma_{S,t}) \rightarrow 0$ the optimal hedge ratio converges to $(\sigma_{W,t})/(\sigma_{S,t})$ and the variance reduction factor approaches to $(1 - \rho_{12,t}^2)$.

Proof: From equation (6), substituting $\phi_t = 0$ the optimal hedge ratio becomes $\theta_t \rho_{12,t} = (\sigma_{W,t})/(\sigma_{S,t})$. Relative to the variance reduction factor, see equation (7). $\square$

As the common noise becomes more representative of the uncertainty of the futures returns, spot and futures market innovations tend to be increasingly similar. In the limit case, no basis risk arises and, consequently, the optimal hedge ratio and the HE also depend on the nature of the relationship between the returns of the asset to be hedged and the returns of the underlying asset of the futures contract. In this case, and denoting mean returns by μ , the optimal hedge ratio can be computed from the following regression:

$$\ln \left(\frac{W_t}{W_{t-1}} \right) - \mu_W = \alpha + \beta \left[\ln \left(\frac{S_t}{S_{t-1}} \right) - \mu_S \right] + \varepsilon_t$$

and the variance reduction factor is simply the variance of the W asset returns that remains unexplained by the variance of S asset returns.

Proposition 3: When Σ_t equals to the identity matrix, the optimal hedge ratio is $\theta_t / 1 + \phi_t$ and full HE is achieved.

Proof: From equations (6) and (7), substituting $\rho_{i,j,t} = 1 \forall i \neq j, i, j = 1, 2, 3$ leads to $h_t/c_t = \theta_t / (1 + \phi_t)$ and $HE = 1$, respectively. $\square$

When the underlying asset returns are perfect and positively correlated with both the derivative asset returns (as implied in the theoretical valuation according to the cash-and-carry model) and the returns of the asset to be hedged, the hedged position becomes a riskless portfolio, regardless of the relative variance between market noises.

Proposition 4: With positive correlations, the optimal hedge ratio increases not only as the common and the specific noise of futures returns become more correlated with the noise of the asset to be hedged, but also as the asset to be hedged becomes more volatile relative to the underlying asset of the futures contract.

Proof: From equation (6), we have: $\square$

$$\frac{\partial \left(\frac{h_t}{c_t} \right)}{\partial p_{12,t}} = \frac{\theta_t}{1 + \phi_t^2 + 2p_{23,t}\phi_t} > 0 \text{ if } p_{23,t} > 0$$

$$\frac{\partial \left(\frac{h_t}{c_t} \right)}{\partial p_{13,t}} = \theta_t \frac{\partial \left(\frac{h_t}{c_t} \right)}{\partial p_{12,t}} = \frac{\theta_t \phi_t}{1 + \phi_t^2 + 2p_{23,t}\phi_t} > 0 \text{ if } p_{23,t} > 0$$

$$\frac{\partial \left(\frac{h_t}{c_t} \right)}{\partial \theta_t} = \frac{p_{12,t} + p_{13,t}\phi_t}{1 + \phi_t^2 + 2p_{23,t}\phi_t} > 0 \text{ if } p_{12,t}, p_{13,t} > 0$$

The above partial derivatives confirm that as the asset to be hedged and the derivative instrument tend to have common fluctuations, the required number of the futures contract

increases. Moreover, when fluctuations of the underlying asset of the futures returns show less intense amplitude relative to the asset to be hedged, a greater number of futures contract is required to immunize the value of the spot position.

Although the above propositions deal with partial derivatives, the optimal hedge ratio depends on five parameters (three correlations and two variance ratios), leading to a continuous range of possibilities. To provide additional insights, we show how the optimal hedge ratio evolves for alternative bivariate parameter structures once the other three parameters take specific values.

Arbitrage trading tends to produce positive correlations between the derivative and its underlying asset. Moreover, in a context of increasing globalization of the world economy, shocks in financial markets disseminate in a similar way, and assets tend to share common trends. This way, let us assume not only that the conditional variances of the three assets observed arises from joint dependence on a common factor, leading to common volatility movements across returns, but also that the relative importance of the mispricing fluctuations is negligible in terms of the volatility of the common noise between the derivative and its underlying asset. The next two figures display the optimal ratio as a function of $\rho_{12,t}$ and $\rho_{23,t}$ under alternative scenarios for the relative volatility between the asset to be hedged and the underlying asset of the futures contract.

As expected, when returns of the asset to be hedged are relatively more volatile than returns of the underlying asset of the futures contract, the number of derivative contracts required is clearly higher (compare Figures 1 and 2).

Figure 1: Optimal hedge ratio with $\theta_t=1$, $\phi_t=0.1$, $\rho_{13,t}=0.9$.

Figure 2: Optimal hedge ratio with $\theta_t=1.5$, $\phi_t=0.1$, $\rho_{13,t}=0.9$.

Figure 3: Optimal hedge ratio with $\theta_t = 1, 0$, $\phi_t = 0.1$, $\rho_{13,t} = 0.9$.

When the price discovery process of the futures market is not in the way expected under rational expectations, spot and futures returns could be negatively correlated. In this case, there are combinations for the three correlations involved that lead to a long rather than short position in the futures market to perform the hedging strategy. Observe from equation (6) that, under positive correlation between the common and the specific noise of the futures returns, the optimal hedge ratio is negative when $\rho_{12,t} < -\rho_{13,t}\phi_t$. Figure 3 provides insights into this aspect by allowing negative values of parameter $\rho_{12,t}$ in parameterization benchmark used in Figure 1.

Finally, we show the joint effect of the relative noise variance ratios on the optimal

hedge ratio under positive and non-perfect correlation for all the assets involved in the problem (Figure 4).

Given the correlations, the optimal ratio depends on the relative variance noise ratios in a nonlinear way. Consistently with Figure 2, as the relative volatility of the asset to be hedged in terms of the volatility of the common noise increases a higher number of derivative contracts is needed. However, higher volatility of the specific noise relative to the volatility of the common noise rebalances the intensity of fluctuations between the asset to be hedged and the derivative asset, reducing the number of contracts involved in the hedging operation and its cost.

Figure 4: Optimal hedge ratio with positive correlations $\rho_{12}, t = \rho_{13}, t = \rho_{23}, t = 0.5$.

CONCLUSIONS

This note provides a theoretical discussion for the optimal cross-hedge ratio under futures mispricing. Assuming that assets evolve according to a geometric Brownian motion, it is showed that the optimal ratio that minimizes the conditional variance of the hedged position depends on the correlation matrix and the relative volatility of the asset to be hedged and the hedging instrument in terms of the underlying asset of the futures contract. We analyze bivariate joint effects of correlations and noise variance ratios for the assets involved in the hedging operation. It is showed that, under positive correlations, the cost of hedging a portfolio that mismatches the underlying asset of the futures contract increases as the relative

volatility of the underlying asset of the futures market decreases. However, the relative volatility of the specific noise in the futures market could be exploited to reduce the number of futures contracts required.

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References

- Arshanapalli, B. and Doukas, J. (1994) Common volatility in S&P 500 stock index and S&P 500 index futures prices during October 1987. *Journal of Futures Markets* 14(8): 915–925.
- Draper, P. and Fung, J.K.W. (2003) Discretionary government intervention and the mispricing of index futures. *Journal of Futures Markets* 23(12): 1159–1189.
- Kawaller, I. and Koch, T.W. (1984) Cash and carry trading and the pricing of treasury bill futures. *Journal of Futures Markets* 4(2): 115–123.
- Lafuente, J.A. and Novales, A. (2003) Optimal hedging under departures from the cost of carry valuation: Evidence from the Spanish stock index futures market. *Journal of Banking and Finance* 27(6): 1053–1078.
- Yadav, P.K. and Pope, P.F. (1990) Stock index futures arbitrage: International evidence. *Journal of Futures Markets* 10(6): 573–603.
- Yadav, P.K. and Pope, P.F. (1994) Stock index futures mispricing: Profit opportunities or risk premia. *Journal of Banking and Finance* 18(5): 921–953.