Original Article

Interpolation of forward price from liquidly traded forwards application to energy models

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PRACTICAL APPLICATION The issue of relating the state variables in forward price models to market-observed quantities is addressed in this article. The article links the state variables directly, and explicitly, to forward prices, and establishes that the models are, in fact, affine with respect to a finite number of forward prices. This finding is of significant value in implementing the models in practice. In fact, in an ongoing project, this affine interpolation of the forward price from liquidly traded forwards allows us to calibrate the forward price model using the bond price as observation and the resulting liquidly forwards as partially observed states.

ABSTRACT This article investigates how the infinite dimensional forward price process can be interpolated from finite dimensional standardized liquidly traded forwards. Under suitable restrictions on the forward price volatility and the forward rate volatility, it is shown that the forward price is an affine function of a finite number of some liquidly traded forwards. As a consequence, we find that the forward price is independent of the fluctuations of the interest rate, but still dependent on the structure of the forward rate curve. The results developed in the paper are illustrated by the study of Oil and Electricity forward price model.

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INTRODUCTION

The main financial object under study in this article is forward contracts. In standard textbook

treatments, a forward contract is an agreement between two parties to buy or sell an asset (which can be of any kind) at a pre-agreed future

point in time. It is used to control and hedge risk, for example currency exposure risk (for example, forward contracts on USD or EUR) or commodity prices (for example, forward contracts on oil). There are currently two approaches to the pricing question in the literature. The first starts from a stochastic representation of the spot price and other key variables, such as the convenience yield on the asset and interest rates, and derives the forward prices of contingent claims consistent with the spot process. However, one of the problems of implementing these models is that often the state variables are unobservable – even the spot price is hard to obtain, with the problems exasperated if the convenience yield has to be jointly estimated.

The second stream of the literature models directly the evolution of the forward curve. From a theoretical point of view, forward prices term structures are a more complex object than interest rates term structures or futures prices term structures. The extra complexity results from the fact that forward prices, with maturity T , are martingales under the T -forward measure, Q^T . Under such T -forward measures, zero-coupon bonds (with maturity T) are numeraires, which implies that zero-coupon bond prices volatilities will enter into the dynamics of forward prices. Consequently, in general, term structures of forward prices cannot be studied in isolation; they must be studied under some interest rate setting, and a required input to forward price term structures is the term structure of interest rates.

The theoretical literature on forward prices term structures has mainly focused on understanding under what conditions, on the dynamics of the state space variables (which are assumed to be finite), the term structure is

of an *a priori* given specific functional form. Included in this traditional approach are the studies on affine and quadratic term structures of forward prices.

In this article, we consider the entire term structure of forward prices (or equivalently the entire forward price curve) as the primitive object, and model all forward prices simultaneously. This leads us, as one can see in the following section, to model the forward prices by an infinite dimensional system of stochastic differential equations (SDEs). Here, we do not assume that the state space model is finite, nor that the term structure of forward prices is of a given specific function form. Instead, our aim is to investigate under what conditions on the forward price volatility the corresponding infinite dimensional forward price admits an affine interpolation in terms of a vector of ‘benchmark’ forwards. The maturities of these particular forwards can be chosen in the belief that they mostly effect the evolution of the forward price. Perhaps they correspond to the most liquid maturities or maturities affected by seasonal phenomena, such as agricultural or energy forward markets.

This article is organized as follows. In Term structure of the forward price section, we formulate the probabilistic setup and define the term structure of the forward price under study. In The affine interpolation nature of the forward price section, we discuss the conditions that will be imposed on the volatility processes in order to rewrite the forward price as an affine function of some liquidly traded forwards. Section Application to energy forward models is dedicated to illustrating the techniques developed in the article by the study of concrete applications, including models for oil and electricity forward markets. For the sake

of readability, the proofs of Lemma 3.2, Corollary 3.3 and Proposition 3.1 are presented in the Appendix.

TERM STRUCTURE OF THE FORWARD PRICE

We consider a financial market living on a given complete probability space $\{\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_{t \geq 0}\}$ carrying an n -dimensional Wiener process W . The main assets to be considered are forward contracts written on a given underlying attainable contingent claim X that settles at time T . Let $F_X(t, T)$ denote the forward price at time t of a forward contract with delivery date T . We assume that the market is arbitrage-free in the sense that the probability measure Q is a risk-neutral martingale measure. Let $P(t, T)$ denote the price at time t of a zero-coupon bond maturing at time T .

In addition to the trivial boundary conditions

$$F_X(T, T) = X_T \text{ and } P(T, T) = 1$$

arbitrage arguments yields

$$P(t, T) = E_t^Q \left[e^{-\int_t^T r(s) ds} \right]$$

$$F_X(t, T) = E_t^T [F_X(T, T)] \quad (1)$$

where r is the short interest rate and $E_t^Q[\cdot]$ and $E_t^T[\cdot]$ denote, respectively, expectations, conditional on $\mathcal{F}_t$, under the martingale measure Q and under the forward martingale measure Q^T .

Interest rate term structure

Consider the Heath–Jarrow–Morton (HJM) model for the forward interest rate, driven by a multi-dimensional Wiener process W . Under the

risk-neutral measure Q the forward rate $r(t, T)$ satisfies the stochastic integral equation

$$r(t, T) = r(0, T) + \int_0^t \sigma(s, T) \bar{\sigma}(s, T)^\tau ds + \int_0^t \sigma(s, T) dW_s \quad (2)$$

where

$$\bar{\sigma}(s, T) = \int_s^T \sigma(s, u) du$$

$r(0, T)$ is the initial forward rate curve, $\sigma(t, T)$ is an adapted $\mathbb{R}^n$ -valued forward rate volatility process and τ denotes transpose.

Given such a family of forward rates $r(t, T)$, the T -maturity zero-coupon bond price at time t is then defined by setting

$$r(t, T) = -\frac{\partial}{\partial T} \ln P(t, T) \quad (3)$$

and the corresponding SDE for the bond price, $P(t, T)$, is

$$dP(t, T) = P(t, T)[r(t)dt - \bar{\sigma}(t, T)dW_t] \quad (4)$$

Forward price term structure

Observe that for every fixed T the forward price process is a Q^T -martingale (recall (1)). Thus, considering only non-negative claims, we may assume the forward price to have dynamics of the following form under Q^T

$$dF_X(t, T) = F_X(t, T)\theta(t, T)dW_t^T \quad (5)$$

where for each T , $\theta(t, T)$ is an adapted $\mathbb{R}_+^n$ -valued forward price volatility process and W^T is a Q^T -Wiener process.

For our purposes, it is necessary to rewrite the forward price process in a more convenient form. Thus, it turns out that the analysis is considerably simplified if, instead of working

with the process F_X defined in (5), we work with the process f_X defined by

$$f_X(t, T) = \ln F_X(t, T)$$

Using Itô's formula, we obtain the SDE for f_X as

$$df_X(t, T) = -\frac{1}{2} \|\theta(t, T)\|^2 dt + \theta(t, T) dW_t^T \quad (6)$$

or in integral form as

$$f_X(t, T) = f_X(0, T) - \int_0^t \frac{1}{2} \|\theta(s, T)\|^2 ds + \int_0^t \theta(s, T) dW_s^T \quad (7)$$

Finally, to obtain the dynamics (logarithm) of forward price, under the risk-neutral martingale measure Q , we use the change of numeraire techniques. Denoting by L the Radon-Nikodym derivative

$$L(t) = \frac{dQ^T}{dQ} \quad \text{on } \mathcal{F}_t, \quad 0 \leq t \leq T$$

and recalling the money account $B_t = e^{\int_0^t r_s ds}$ is the numeraire under Q , we have

$$L(t) = \frac{P(t, T)}{B_t P(0, T)}$$

Furthermore, the process W^T is given by

$$W_t^T = W_t + \int_0^t \bar{\sigma}(s, T) ds$$

Hence, we can easily obtain the (logarithm of) forward price Q -dynamics

$$f_X(t, T) = f_X(0, T) + \int_0^t \left[\theta(s, T) \bar{\sigma}(s, T)^\tau - \frac{1}{2} \|\theta(s, T)\|^2 \right] ds + \int_0^t \theta(s, T) dW_s \quad (8)$$

Note that the (logarithm of the) forward price equation (8) is linked to the interest rate equation (2) through the forward rate volatility $\sigma(s, T)$. This means that, in general, to study forward price we should study the entire system.

$$\begin{cases} f_X(t, T) = f_X(0, T) + \int_0^t \left[\theta(s, T) \bar{\sigma}(s, T)^\tau - \frac{1}{2} \|\theta(s, T)\|^2 \right] ds + \int_0^t \theta(s, T) dW_s \\ r(t, T) = r(0, T) + \int_0^t \sigma(s, T) \bar{\sigma}(s, T)^\tau ds + \int_0^t \sigma(s, T) dW_s \end{cases} \quad (9)$$

In the sequel our main object of study will be the following forward price system

$$\begin{aligned} \tilde{f}_X(t, T) &= \tilde{f}_X(0, T) + \int_0^t \left[\mu^f(s, T) \right] ds \\ &+ \int_0^t \left[\frac{\theta(s, T)}{\sigma(s, T)} \right] dW_s \end{aligned} \quad (10)$$

where

$$\tilde{f}_X(t, T) = \begin{bmatrix} f_X(t, T) \\ r(t, T) \end{bmatrix}$$

$$\mu^f(s, T) = \left[\theta(s, T) \bar{\sigma}(s, T)^\tau - \frac{1}{2} \|\theta(s, T)\|^2 \right]$$

$$\mu^r(s, T) = \sigma(s, T) \bar{\sigma}(s, T)^\tau$$

Definition 2.1: We find that the forward price model $\tilde{f}_X(t, T)$ has an interpolation in terms of $n_f^* \in \mathbb{N}$ forward prices $f(t, T_1^f), \dots, f(t, T_{n_f^*}^f)$ and $n_r^* \in \mathbb{N}$ forward rates $r(t, T_1^r), \dots, r(t, T_{n_r^*}^r)$, if there exists a deterministic function $\Phi = (\Phi^f, \Phi^r): \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+^2$ such that

for all $t, T \in \mathbb{R}_+$

$$\begin{aligned} \tilde{f}_X(t, T) &= \Phi(t, T, F(t)) \text{ that is } \begin{bmatrix} f_X(t, T) \\ r(t, T) \end{bmatrix} \\ &= \begin{bmatrix} \Phi^f(t, T, F(t)) \\ \Phi^r(t, T, F(t)) \end{bmatrix} \end{aligned} \quad (11)$$

where $F(t) = [f(t, T_1^f), \dots, f(t, T_{n_f}^f), r(t, T_1^r), \dots, r(t, T_{n_r}^r)]^T$ and $d = n_f^* + n_r^*$.

The interpolation is said to be affine if Φ is an affine function in the vector $F(t)$. That is, the forward process can be written in the following form:

$$\begin{aligned} \tilde{f}_X(t, T) &= \Phi(t, T, F(t)) \\ &= \begin{bmatrix} g^f(t, T) + h^f(t, T)F(t) \\ g^r(t, T) + h^r(t, T)F(t) \end{bmatrix} \end{aligned}$$

where $g^f(t, T)$ and $g^r(t, T)$ (resp $h^f(t, T)$ and $h^r(t, T)$) are deterministic $\mathbb{R}$ (resp $\mathbb{R}^d$) - valued functions.

THE AFFINE INTERPOLATION NATURE OF THE FORWARD PRICE

In the study by Chiarella and Kwon,¹ the affine representation problem for interest rate models is solved by setting some restrictions on the forward rate volatility process. In this article, we follow the same methodology to obtain an affine interpolation for the forward process, except this time restrictions are on both the forward price and forward rate volatility process.

Let $n \in \mathbb{N}_+$ and let $\sigma(t, T) = (\sigma_1(t, T), \dots, \sigma_n(t, T))$ be the forward rate volatility process and $\theta(t, T) = (\theta_1(t, T), \dots, \theta_n(t, T))$ be the forward price volatility process.

Assumption 3.1: For each $i \in \{1, \dots, n\}$ there exist

1. $d_i^1 \in \mathbb{N}_+$ such that $\sigma_i(t, T)$ is d_i^1 times differentiable with respect to T and satisfies

a d_i^1 th order homogeneous linear differential equation of the form

$$\begin{aligned} L_i \sigma_i(t, T) &= 0, \quad \text{where } L_i = \frac{\partial^{d_i^1}}{\partial T^{d_i^1}} \\ &\quad - \sum_{j=0}^{d_i^1-1} \kappa_{ij}^1(T) \frac{\partial^j}{\partial T^j} \end{aligned} \quad (12)$$

and $\kappa_{ij}^1(T)$ are continuous deterministic functions.

2. $d_i^2 \in \mathbb{N}_+$ such that $\theta_i(t, T)$ is d_i^2 times differentiable with respect to T and satisfies a d_i^2 th order homogeneous linear differential equation of the form

$$\begin{aligned} L_i \theta_i(t, T) &= 0, \quad \text{where } L_i = \frac{\partial^{d_i^2}}{\partial T^{d_i^2}} \\ &\quad - \sum_{j=0}^{d_i^2-1} \kappa_{ij}^2(T) \frac{\partial^j}{\partial T^j} \end{aligned} \quad (13)$$

and $\kappa_{ij}^2(T)$ are continuous deterministic functions.

Lemma 3.2: Let $\sigma_i(t, T)$ and $\theta_i(t, T)$, for $0 \leq i \leq n$, satisfy Assumption 3.1. There then exist stochastic processes c_{ij} , $\tilde{c}_{ij}$ and deterministic functions σ_{ij} , θ_{ij} such that

$$\begin{aligned} \sigma_i(t, T) &= \sum_{j=1}^{d_i^1} c_{ij}(t) \sigma_{ij}(T) \text{ and } \theta_i(t, T) \\ &= \sum_{j=1}^{d_i^2} \tilde{c}_{ij}(t) \theta_{ij}(T) \end{aligned}$$

Proof: See Appendix A.

Note that the above lemma allows separation between time dependence and maturity dependence in the components of the forward rate and forward price volatility processes. This separation is essential in the determination of the affine interpolation obtained below.

Corollary 3.3: Suppose Assumption 3.1 holds and let c_{ij} and $\tilde{c}_{ij}$ be the coefficient processes defined in Lemma 3.2. c_{ij} and $\tilde{c}_{ij}$ are then adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

Proof: See Appendix B.

Now, substituting the expressions for $\sigma_i(t, T)$ and $\theta_i(t, T)$ from Lemma 3.2 into the expression for $f_X(t, T)$ in (8) and the expression for $r(t, T)$ in (2) gives, after simple calculations, the following proposition, which affirms that the uncertainty in the forward price is reduced to a finite number of random quantities, and that the forward price is an affine function of these quantities.

Proposition 3.4: Let $\sigma_i(t, T)$ and $\theta_i(t, T)$, for $1 \leq i \leq n$, satisfy Assumption 3.1. The corresponding forward price can then be written as

$$\begin{aligned} f_X(t, T) = & f_X(0, T) - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{d_i^2} \theta_{ij}(T)^2 \psi_j^i(t) \\ & - \sum_{i=1}^n \sum_{1 \leq l < k \leq d_i^2} \theta_{il}(T) \theta_{ik}(T) \phi_{lk}^i(t) \\ & + \sum_{i=1}^n \sum_{j=1}^{d_i^2} \theta_{ij}(T) \varphi_j^i(t) \\ & + \sum_{i=1}^n \sum_{h=1}^{d_i^1} \sum_{j=1}^{d_i^2} \theta_{ij}(T) \bar{\sigma}_{ih}(T) \eta_{jh}^i(t) \end{aligned} \quad (14)$$

and

$$\begin{aligned} r(t, T) = & r(0, T) + \sum_{i=1}^n \sum_{j=1}^{d_i^1} \sigma_{ij}(T) \beta_j^i(t) \\ & + \sum_{i=1}^n \sum_{j,h=1}^{d_i^1} \sigma_{ij}(T) \bar{\sigma}_{ih}(T) \rho_{jh}^i(t) \end{aligned} \quad (15)$$

where

$$\bar{\sigma}_{ih}(t) = \int_0^t \sigma_{ih}(u) du$$

$$\psi_j^i(t) = \int_0^t \tilde{c}_{ij}(s)^2 ds$$

$$\phi_{lk}^i(t) = \int_0^t \tilde{c}_{il}(s) \tilde{c}_{ik}(s) ds$$

$$\varphi_j^i(t) = \int_0^t \tilde{c}_{ij}(s) dW_i(s) - \sum_{h=1}^{d_i^1} \int_0^t \tilde{c}_{ij}(s) c_{ih}(s) \bar{\sigma}_{ih}(s) ds$$

$$\eta_{jh}^i(t) = \int_0^t \tilde{c}_{ij}(s) c_{ih}(s) ds$$

$$\rho_{jh}^i(t) = \int_0^t c_{ij}(s) c_{ih}(s) ds$$

$$\beta_j^i(t) = \int_0^t c_{ij}(s) dW_i(s) - \sum_{h=1}^{d_i^1} \int_0^t c_{ij}(s) c_{ih}(s) \bar{\sigma}_{ih}(s) ds$$

However, some of these random quantities $\{\psi_j^i(t), \phi_{lk}^i(t), \varphi_j^i(t), \eta_{jh}^i(t), \rho_{jh}^i(t), \beta_j^i(t)\}$ turn out to be deterministic in several applications, and it is useful to select their non-deterministic elements.

Let

$$\begin{aligned} A_f = & \left\{ \psi_j^i(t), \phi_{lk}^i(t), \varphi_j^i(t), \eta_{jh}^i(t) \right. \\ & \left. | 1 \leq i \leq n, 1 \leq l < k \leq d_i^2, 1 \leq h \leq d_i^1, 1 \leq j \leq d_i^2 \right\} \\ A_r = & \left\{ \beta_j^i(t), \rho_{jh}^i(t) | 1 \leq i \leq n, 1 \leq j, h \leq d_i^1 \right\} \end{aligned}$$

and let $A_f^* \subset A_f$ and $A_r^* \subset A_r$ be the subsets of non-deterministic elements. Denote by n_r^* the size of the set A_r^* and by n_f^* the size of the set A_f^* . Following this notation, the expression (14) for

the forward price $f_X(t, T)$ can be written as

$$f_X(t, T) = a_0^f(t, T) + \sum_{\chi \in A_f^*} a_\chi^f(T) \chi(t) \quad (16)$$

and the expression (15) for the forward rate $r(t, T)$ can be written as

$$r(t, T) = a_0^r(t, T) + \sum_{\chi \in A_r^*} a_\chi^r(T) \chi(t) \quad (17)$$

so that

$$\tilde{f}_X(t, T) = \begin{bmatrix} a_0^f(t, T) \\ a_0^r(t, T) \end{bmatrix} + \begin{bmatrix} \sum_{\chi \in A_f^*} a_\chi^f(T) \chi(t) \\ \sum_{\chi \in A_r^*} a_\chi^r(T) \chi(t) \end{bmatrix} \quad (18)$$

where a_0^r , a_0^f , a_χ^r and a_χ^f are deterministic functions, and χ is used to denote the elements of the sets A_r^* and A_f^* .

If we denote

$$z(t) = [\chi_1^r(t), \dots, \chi_{n_r^*}^r(t), \chi_1^f(t), \dots, \chi_{n_f^*}^f(t)]$$

$$h^f(t, T) = \begin{pmatrix} 0, \dots, 0, \underbrace{a_{\chi_1}^f(T), \dots, a_{\chi_{n_f^*}^f}^f(T)}_{n_f^* \text{ times}} \end{pmatrix}$$

and

$$h^r(t, T) = \begin{pmatrix} \underbrace{a_{\chi_1}^r(T), \dots, a_{\chi_{n_r^*}^r}^r(T)}_{n_r^* \text{ times}}, 0, \dots, 0 \end{pmatrix}$$

then equation (18) becomes

$$\tilde{f}_X(t, T) = \begin{bmatrix} f_X(t, T) \\ r(t, T) \end{bmatrix} = \begin{bmatrix} a_0^f(t, T) + h^f(t, T)z(t) \\ a_0^r(t, T) + h^r(t, T)z(t) \end{bmatrix}$$

Assumption 3.5: There exist $m_r, m_f \in \mathbb{N}_+$ such that

1. $\theta(t, T)$ is a function of t, T and m_f forward prices $f_X(t, T_1^*), \dots, f_X(t, T_{m_f}^*)$,

so that

$$\theta(t, T) = \theta(t, T, f_X(t, T_1^*), \dots, f_X(t, T_{m_f}^*)) \quad (19)$$

2. $\sigma(t, T)$ is a function of t, T and m_r forward rates $r(t, T_1'), \dots, r(t, T_{m_r}')$, so that

$$\sigma(t, T) = \sigma(t, T, r(t, T_1'), \dots, r(t, T_{m_r}')) \quad (20)$$

Provided that some invertibility conditions are satisfied, we can express the elements of $A^* = A_r^* \cup A_f^*$ in terms of a certain set of forward prices and forward rates. Furthermore, we establish that the forward price is, in fact, affine with respect to a finite number of forward prices. If we note $A_r^* = \{\chi_1^r, \dots, \chi_{n_r^*}^r\}$ and $A_f^* = \{\chi_1^f, \dots, \chi_{n_f^*}^f\}$, we have the following proposition:

Proposition 3.1: Suppose Assumption 3.1 and 3.5 are satisfied. If there exist $T_1^r, T_2^r, \dots, T_{n_r^*}^r \in \mathbb{R}_+$ and $T_1^f, T_2^f, \dots, T_{n_f^*}^f \in \mathbb{R}_+$ such that the matrices

$$M_r^*(T_1^r, \dots, T_{n_r^*}^r) = \begin{bmatrix} a_{\chi_1^r}(T_1^r) & a_{\chi_2^r}(T_1^r) & \dots & a_{\chi_{n_r^*}^r}(T_1^r) \\ a_{\chi_1^r}(T_2^r) & a_{\chi_2^r}(T_2^r) & \dots & a_{\chi_{n_r^*}^r}(T_2^r) \\ \vdots & \vdots & \ddots & \vdots \\ a_{\chi_1^r}(T_{n_r^*}^r) & a_{\chi_2^r}(T_{n_r^*}^r) & \dots & a_{\chi_{n_r^*}^r}(T_{n_r^*}^r) \end{bmatrix}$$

$$M_f^*(T_1^f, \dots, T_{n_f^*}^f) = \begin{bmatrix} a_{\chi_1^f}(T_1^f) & a_{\chi_2^f}(T_1^f) & \dots & a_{\chi_{n_f^*}^f}(T_1^f) \\ a_{\chi_1^f}(T_2^f) & a_{\chi_2^f}(T_2^f) & \dots & a_{\chi_{n_f^*}^f}(T_2^f) \\ \vdots & \vdots & \ddots & \vdots \\ a_{\chi_1^f}(T_{n_f^*}^f) & a_{\chi_2^f}(T_{n_f^*}^f) & \dots & a_{\chi_{n_f^*}^f}(T_{n_f^*}^f) \end{bmatrix}$$

are invertible. The corresponding forward price model then admits an affine interpolation in terms of the following Markov process:

$$F(t) = [r(t, T_1^r), \dots, r(t, T_{n_r^*}^r), f_X(t, T_1^f), \dots, f_X(t, T_{n_f^*}^f)]^\tau$$

Proof: See Appendix C.

Consequence 3.6 The final representation (31) of the forward price is important, as it shows that the forward price is independent of the fluctuations of the interest rate. More precisely, it is independent of the source of randomness owing to the level shift of the interest rate, which is described by the processes $c_{ij}(t)$. However, the representation (31) still depends on the structure of the forward rate curve described by the functions $\sigma_{ij}(T)$ (note that some of the components of $b_i^f(T)$ in equation (C.3) depend on the $\sigma_{ij}(T)$).

APPLICATION TO ENERGY FORWARD MODELS

The issue of how energy products are priced in spot and forward markets has become one of the most controversial topics facing utilities, energy producers, regulators, political officials, accounting firms and a broad array of financial market participant. In fact, the price of energy products (oil, natural gas, electricity and so on) is an important macroeconomic variable. It reflects demand pressures in the global economy, and potentially directly impacts the inflation rate, at least in the short run. High prices in the energy markets during the first two quarters of 2008 have produced an immediate need for both analysis of the events and efficient modeling of the prices. As the spot price of energy products has risen in recent years, so has the price for delivery in the future.

Application to oil forward model

Oil is traded 'forward', for future delivery, both on recognized futures exchanges and directly

between market participants (known as over-the-counter or OTC trades). In this section, we apply the method developed in the above section to the single-factor model for forward prices developed in the study by Clewlow and Strickland.² The corresponding volatility is designed for the oil forward market, but applicable to any forward market, such as the natural gas market, in which volatility decreases with maturity and varies by the season. We are interested in showing that the Clewlow and Strickland model admits an affine interpolation in terms of some forwards. Recall that according to Clewlow and Strickland² the oil forward model is defined as

$$dF_{Oil}(t, T) = F_{Oil}(t, T)\theta(t, T)dW_t^T \quad (21)$$

where

$$\theta(t, T) = \theta e^{-\alpha(T-t)}$$

The model in equation (21) has two volatility parameters: θ determines the level of spot and forward price return volatility, while α determines the rate at which the volatility of increasing maturity forward prices decline, and is also the speed of mean reversion of the spot price. These parameters can be estimated directly from the prices of options on the spot price of oil or forward contracts using the results in Section 3 of the study by Clewlow and Strickland.²

If we define

$$f_{Oil}(t, T) = \ln F_{Oil}(t, T)$$

then

$$df_{Oil}(t, T) = -\frac{1}{2}\theta^2 e^{-2\alpha(T-t)}dt + \theta e^{-\alpha(T-t)}dW_t^T$$

Note that the forward price equation (8) and the forward rate equation (2) are driven by the

same Wiener process. Hence, for this example we have to consider a one-factor interest rate model.

We have chosen to work with the Ritchken and Sankarasubramanian model,² which is a one-dimensional HJM model corresponding to the forward rate volatility process

$$\sigma(t, T) = g(t, r(t))e^{-\int_t^T \kappa(u)du}$$

where κ and g are deterministic functions, and $r(t)$ is the short rate.

The dynamics of (the logarithm) of forward prices under the risk-neutral martingale measure, Q , is then given by

$$\begin{aligned} f_{Oil}(t, T) = & f_{Oil}(0, T) + \int_0^t \left[\theta e^{-\alpha(T-s)} \bar{\sigma}(s, T) \right. \\ & \left. - \frac{1}{2} \theta^2 e^{-2\alpha(T-s)} \right] ds + \int_0^t \theta e^{-\alpha(T-s)} dW_s \end{aligned}$$

where

$$\begin{aligned} \bar{\sigma}(s, T) = & \int_s^T g(s, r(s)) e^{-\int_s^u \kappa(v)dv} du \\ = & g(s, r(s)) e^{\int_0^s \kappa(v)dv} \int_s^T e^{-\int_0^u \kappa(v)dv} du \end{aligned}$$

Note that as $\sigma(t, T)$ and $\theta(t, T)$ satisfy the equations $\partial \sigma(t, T) / \partial T + \kappa(T) \sigma(t, T) = 0$ and $\partial \theta(t, T) / \partial T + \alpha \theta(t, T) = 0$, respectively, Assumption 3.1 is satisfied, and as the dependence of $\sigma(t, T)$ on the path enters only through the short rate $r(T)$ and $\theta(t, T)$ is deterministic, Assumption 3.5 is also satisfied.

We can now rewrite the forward price volatility and the forward rate volatility in the form

$$\begin{aligned} \sigma(t, T) = & g(t, r(t)) e^{\int_0^t \kappa(u)du} \cdot e^{-\int_0^T \kappa(u)du} \\ = & c_{11}(t) \cdot \sigma_{11}(T) \end{aligned}$$

$$\theta(t, T) = \theta e^{\alpha t} \cdot e^{\alpha T} = \tilde{c}_{11}(t) \cdot \theta_{11}(T)$$

It follows from Proposition 3.4 that the state variables are

$$\bar{\sigma}_{11}(t) = \int_0^t \sigma_{11}(u) du = \int_0^t e^{-\int_0^u \kappa(s)ds} du$$

$$\psi_1^1(t) = \int_0^t \tilde{c}_{11}(s)^2 ds = \int_0^t \theta^2 e^{2\alpha s} ds = \theta^2 \left(\frac{e^{2\alpha t} - 1}{2\alpha} \right)$$

$$\varphi_1^1(t) = \int_0^t \tilde{c}_{11}(s) dW_s - \int_0^t \tilde{c}_{11}(s) c_{11}(s) \bar{\sigma}_{11}(s) ds$$

$$\begin{aligned} = & \theta \int_0^t e^{\alpha s} dW_s - \int_0^t \theta e^{\alpha s} g(s, r(s)) e^{\int_0^s \kappa(u)du} \\ & \times \left[\int_0^s e^{-\int_0^u \kappa(v)dv} du \right] ds \end{aligned}$$

$$\begin{aligned} \eta_{11}^1(t) = & \int_0^t \tilde{c}_{11}(s) c_{11}(s) ds \\ = & \int_0^t \theta e^{\alpha s} g(s, r(s)) e^{\int_0^s \kappa(u)du} ds \end{aligned}$$

$$\rho_{11}^1(t) = \int_0^t c_{11}(s) c_{11}(s) ds = \int_0^t g(s, r(s))^2 e^{2 \int_0^s \kappa(u)du} ds$$

$$\begin{aligned} \beta_1^1(t) = & \int_0^t g(s, r(s)) e^{\int_0^s \kappa(u)du} dW_s \\ & - \int_0^t g(s, r(s))^2 e^{2 \int_0^s \kappa(u)du} \\ & \times \left[\int_0^s e^{-\int_0^u \kappa(u)du} du \right] ds \end{aligned}$$

For this example, there are some state variables that are deterministic so that

$$A_f^* = \{\varphi_1^1(t), \eta_{11}^1(t)\}$$

$$A_r^* = \{\rho_{11}^1(t), \beta_1^1(t)\}$$

Consequently, the forward price model can be written as

$$f_{Oil}(t, T) = a_0^f(t, T) + \sum_{i=1}^2 a_i^f(T) \chi_i^f(t)$$

$$r(t, T) = a_0^r(t, T) + \sum_{i=1}^2 a_i^r(T) \chi_i^r(t)$$

where

$$a_0^f(t, T) = f_{Oil}(0, T) - \frac{1}{2} \theta_{11}(T)^2 \psi_1^1(t)$$

$$a_0^r(t, T) = r(0, T)$$

and

$$\chi_1^f(t) = \varphi_1^1(t) \quad ; \quad a_{\chi_1^f}(T) = \theta_{11}(T)$$

$$\chi_2^f(t) = \eta_{11}^1(t) \quad ; \quad a_{\chi_2^f}(T) = \sigma_{11}(T) \bar{\sigma}_{11}(T)$$

$$\chi_1^r(t) = \beta_1^1(t) \quad ; \quad a_{\chi_1^r}(T) = \sigma_{11}(T)$$

$$\chi_2^r(t) = \rho_{11}^1(t) \quad ; \quad a_{\chi_2^r}(T) = \sigma_{11}(T) \bar{\sigma}_{11}(T)$$

There now exist T_1^f, T_2^f and T_1^r, T_2^r such that the matrices

$$M_f^*(T_1^f, T_2^f) = \begin{bmatrix} a_{\chi_1^f}(T_1^f) & a_{\chi_2^f}(T_1^f) \\ a_{\chi_1^f}(T_2^f) & a_{\chi_2^f}(T_2^f) \end{bmatrix}$$

$$M_r^*(T_1^r, T_2^r) = \begin{bmatrix} a_{\chi_1^r}(T_1^r) & a_{\chi_2^r}(T_1^r) \\ a_{\chi_1^r}(T_2^r) & a_{\chi_2^r}(T_2^r) \end{bmatrix}$$

are invertible and the state variables can be written as

$$[\chi_1^f(t), \chi_2^f(t)]^\tau = M_f^*(T_1^f, T_2^f)^{-1} [\hat{f}_{Oil}(t, T_1^f), \hat{f}_{Oil}(t, T_2^f)]^\tau$$

$$[\chi_1^r(t), \chi_2^r(t)]^\tau = M_r^*(T_1^r, T_2^r)^{-1} [\hat{r}(t, T_1^r), \hat{r}(t, T_2^r)]^\tau$$

where $\hat{f}_{Oil}(t, T) = f_{Oil}(t, T) - a_0^f(t, T)$, and $\hat{r}(t, T) = r(t, T) - a_0^r(t, T)$. The oil forward price and the forward rate can be rewritten in terms of the forward prices $f_{Oil}(t, T_i^f)$, $i = 1, 2$ and the forward rates $r(t, T_i^r)$, $i = 1, 2$

$$\begin{aligned} f_{Oil}(t, T) &= a_0^f(t, T) + [a_{\chi_1^f}(T), a_{\chi_2^f}(T)] \\ &\quad M_f^*(T_1^f, T_2^f)^{-1} [\hat{f}_{Oil}(t, T_1^f), \hat{f}_{Oil}(t, T_2^f)]^\tau \end{aligned} \quad (22)$$

$$\begin{aligned} r(t, T) &= a_0^r(t, T) + [a_{\chi_1^r}(T), a_{\chi_2^r}(T)] \\ &\quad M_r^*(T_1^r, T_2^r)^{-1} [\hat{r}(t, T_1^r), \hat{r}(t, T_2^r)]^\tau \end{aligned} \quad (23)$$

Application to electricity forward model

The deregulation of electricity markets around the world has raised many new challenges for all stakeholders. The modeling of electricity markets is a basic yet critical problem for all stakeholders, and provides the foundation for many decision-making problems within electricity markets. The spot market is usually organized as an auction, which manages the distribution of power in the near future, that is, one day ahead. Empirical studies show that spot prices exhibit seasonalities on different time scales and a strong mean-reversion, and are very volatile and spiky in nature. Because of the inherent properties of electricity as an almost non-storable commodity, such a price behavior has to be expected. Forward prices are heavily influenced by the length of their delivery period

and their time to maturity. Here, we are interested in showing that the two-factor-model for electricity forward contracts developed in the study by Kiesel *et al.*,³ which captures the main features of the market and fits the term structure of volatility, admits an affine interpolation in terms of some forwards. The model is defined as

$$dF_{Elc}(t, T) = F_{Elc}(t, T) [\theta_1(t, T) dW_1^T(t) + \theta_2(t, T) dW_2^T(t)] \quad (24)$$

where

$$\theta_1(t, T) = \theta_1 e^{-\alpha(T-t)} \text{ and } \theta_2(t, T) = \theta_2$$

If we define

$$f_{Elc}(t, T) = \ln F_{Elc}(t, T)$$

then

$$df_{Elc}(t, T) = -\frac{1}{2} [\theta_1^2 e^{-2\alpha(T-t)} + \theta_2^2] dt + \theta_1 e^{-\alpha(T-t)} dW_1^T(t) + \theta_2 dW_2^T(t)$$

For this example we have to consider a two-factor interest rate model. We propose to work with a Gaussian two-factor HJM model corresponding to the forward rate volatility processes

$$\sigma_1(t, T) = \sigma_1 \text{ and } \sigma_2(t, T) = \sigma_2 e^{-\kappa(T-t)}$$

This specification of volatility essentially leads to a combination of the Ho-Lee and Vasicek models.

The dynamics (logarithm) of forward prices under the risk-neutral martingale measure, Q , is

then given by

$$f_{Oil}(t, T) = f_{Oil}(0, T) + \int_0^t \left[\theta_1 e^{-\alpha(T-s)} \bar{\sigma}_1(s, T) + \theta_2 \bar{\sigma}_2(s, T) - \frac{1}{2} (\theta_1^2 e^{-2\alpha(T-s)} + \theta_2^2) \right] ds + \int_0^t [\theta_1 e^{-\alpha(T-s)} dW_1(s) + \theta_2 dW_2(s)] ds \quad (25)$$

where

$$\begin{aligned} \bar{\sigma}_1(s, T) &= \int_s^T \sigma_1 du = \sigma_1(T-s) \\ \bar{\sigma}_2(s, T) &= \int_s^T \sigma_2 e^{-\kappa(T-u)} du \\ &= \sigma_2 \left(\frac{1 - e^{-\kappa(T-s)}}{\kappa} \right) \end{aligned}$$

Note that as $\sigma_2(t, T)$ and $\theta_2(t, T)$ satisfy the equations $\partial \sigma_2(t, T) / \partial T + \kappa \sigma_2(t, T) = 0$ and $\partial \theta_2(t, T) / \partial T + \alpha \theta_2(t, T) = 0$, respectively, and $\sigma_1(t, T)$ and $\theta_1(t, T)$ satisfy $\partial \sigma_1(t, T) / \partial T = 0$ and $\partial \theta_1(t, T) / \partial T = 0$, respectively, Assumption 3.1 is satisfied, and as $\theta_i(t, T)$, $\sigma_i(t, T)$ $i = 1, 2$ are deterministic, Assumption 3.5 is also satisfied.

We can now rewrite the forward price volatility and the forward rate volatility in the form

$$\sigma_1(t, T) = \sigma_1 \cdot 1 = c_{11}(t) \cdot \sigma_{11}(T)$$

$$\sigma_2(t, T) = \sigma_2 e^{\kappa t} \cdot e^{-\kappa T} = c_{21}(t) \cdot \sigma_{21}(T)$$

$$\theta_1(t, T) = \theta_1 \cdot 1 = \tilde{c}_{11}(t) \cdot \theta_{11}(T)$$

$$\theta_2(t, T) = \theta_2 e^{\alpha t} \cdot e^{-\alpha T} = \tilde{c}_{21}(t) \cdot \theta_{21}(T)$$

It follows from Proposition 3.4 that

$$\bar{\sigma}_{11}(t) = \int_0^t \sigma_{11}(u) du = t$$

$$\bar{\sigma}_{21}(t) = \int_0^t \sigma_{21}(u) du = \int_0^t e^{-\kappa u} du = \frac{1 - e^{-\kappa t}}{\kappa}$$

$$\psi_1^1(t) = \int_0^t \tilde{c}_{11}(s)^2 ds = \theta_1^2 t$$

$$\psi_1^2(t) = \int_0^t \tilde{c}_{21}(s)^2 ds = \int_0^t \theta_2^2 e^{2\alpha s} ds = \theta_2^2 \left(\frac{e^{2\alpha t} - 1}{2\alpha} \right)$$

$$\begin{aligned} \varphi_1^1(t) &= \int_0^t \tilde{c}_{11}(s) dW_1(s) - \int_0^t \tilde{c}_{11}(s) c_{11}(s) \bar{\sigma}_{11}(s) ds \\ &= \theta_1 W_1(t) - \theta_1 \sigma_1 t \end{aligned}$$

$$\begin{aligned} \varphi_1^2(t) &= \int_0^t \tilde{c}_{21}(s) dW_2(s) \\ &\quad - \int_0^t \tilde{c}_{21}(s) c_{21}(s) \bar{\sigma}_{21}(s) ds \\ &= \theta_2 \int_0^t e^{\alpha s} dW_2(s) \\ &\quad - \theta_2 \sigma_2 \int_0^t e^{(\alpha+\kappa)s} \left(\frac{1 - e^{-\kappa s}}{\kappa} \right) ds \end{aligned}$$

$$\eta_{11}^1(t) = \int_0^t \tilde{c}_{11}(s) c_{11}(s) ds = \sigma_1 \sigma_2 t$$

$$\begin{aligned} \eta_{11}^2(t) &= \int_0^t \tilde{c}_{21}(s) c_{21}(s) ds = \sigma_2 \theta_2 \int_0^t e^{(\alpha+\kappa)u} du \\ &= \sigma_2 \theta_2 \frac{e^{(\alpha+\kappa)t} - 1}{\alpha + \kappa} \end{aligned}$$

$$\rho_{11}^1(t) = \int_0^t c_{11}(s) c_{11}(s) ds = \sigma_1^2 t$$

$$\begin{aligned} \rho_{11}^2(t) &= \int_0^t c_{21}(s) c_{21}(s) ds = \sigma_2^2 \int_0^t e^{2\kappa u} du \\ &= \sigma_2^2 \frac{e^{2\kappa t} - 1}{2\kappa} \end{aligned}$$

$$\begin{aligned} \beta_1^1(t) &= \int_0^t c_{11}(s) dW_1(s) - \int_0^t c_{11}(s) c_{11}(s) \\ &\quad \times \bar{\sigma}_{11}(s) ds = \sigma_1 W_1(t) - \sigma_1^2 t \end{aligned}$$

$$\begin{aligned} \beta_1^2(t) &= \int_0^t c_{21}(s) dW_2(s) - \int_0^t c_{21}(s)^2 \bar{\sigma}_{21}(s) ds \\ &= \int_0^t e^{-\kappa s} dW_2(s) - \sigma_2^2 \int_0^t \frac{e^{2\kappa s} - e^{\kappa s}}{\kappa} ds \end{aligned}$$

For this example, there are some state variables that are deterministic so that

$$A_f^* = \{\varphi_1^1(t), \varphi_1^2(t)\}$$

$$A_r^* = \{\beta_1^1(t), \beta_1^2(t)\}$$

Consequently, the forward price model can be written as

$$f_{Elc}(t, T) = a_0^f(t, T) + \sum_{i=1}^2 \int_{\mathcal{X}_i^f} a_i^f(T) \chi_i^f(t)$$

$$r(t, T) = a_0^r(t, T) + \sum_{i=1}^2 \int_{\mathcal{X}_i^r} a_i^r(T) \chi_i^r(t)$$

where

$$\begin{aligned} a_0^f(t, T) &= f_{Elc}(0, T) - \frac{1}{2} \theta_{11}(T)^2 \psi_1^1(t) \\ &\quad - \frac{1}{2} \theta_{21}(T)^2 \psi_1^2(t) + \theta_{11}(T) \bar{\sigma}_{11}(T) \eta_{11}^1(t) \\ &\quad + \theta_{21}(T) \bar{\sigma}_{21}(T) \eta_{11}^2(t) \\ a_0^r(t, T) &= r(0, T) + \sigma_{11}(T) \bar{\sigma}_{11}(T) \rho_{11}^1(t) \\ &\quad + \sigma_{21}(T) \bar{\sigma}_{21}(T) \rho_{11}^2(t) \end{aligned}$$

and

$$\begin{aligned} \chi_1^f(t) &= \varphi_1^1(t) \quad ; \quad a_{\chi_1^f}(T) = \theta_{11}(T) \\ \chi_2^f(t) &= \varphi_1^2(t) \quad ; \quad a_{\chi_2^f}(T) = \theta_{21}(T) \\ \chi_1^r(t) &= \beta_1^1(t) \quad ; \quad a_{\chi_1^r}(T) = \sigma_{11}(T) \\ \chi_2^r(t) &= \beta_1^2(t) \quad ; \quad a_{\chi_2^r}(T) = \sigma_{21}(T) \end{aligned}$$

There now exist T_1^f, T_2^f and T_1^r, T_2^r such that the matrices

$$\begin{aligned} M_f^*(T_1^f, T_2^f, T_3^f) &= \begin{bmatrix} a_{\chi_1^f}(T_1^f) & a_{\chi_2^f}(T_1^f) \\ a_{\chi_1^f}(T_2^f) & a_{\chi_2^f}(T_2^f) \end{bmatrix} \\ M_r^*(T_1^r, T_2^r) &= \begin{bmatrix} a_{\chi_1^r}(T_1^r) & a_{\chi_2^r}(T_1^r) \\ a_{\chi_1^r}(T_2^r) & a_{\chi_2^r}(T_2^r) \end{bmatrix} \end{aligned}$$

are invertible; the state variables can be written as

$$\begin{aligned} [\chi_1^f(t), \chi_2^f(t)]^\tau &= M_f^*(T_1^f, T_2^f)^{-1} [\hat{f}_{Elc}(t, T_1^f), \hat{f}_{Elc}(t, T_2^f)]^\tau \\ [\chi_1^r(t), \chi_2^r(t)]^\tau &= M_r^*(T_1^r, T_2^r)^{-1} [\hat{r}(t, T_1^r), \hat{r}(t, T_2^r)]^\tau \end{aligned}$$

where $\hat{f}_{Elc}(t, T) = f_{Elc}(t, T) - a_0^f(t, T)$, and $\hat{r}(t, T) = r(t, T) - a_0^r(t, T)$. The electricity forward price and the forward rate can be rewritten in terms of the forward prices $f_{Elc}(t, T_i^f)$, $i = 1, 2$, and the forward rates

$$r(t, T_i^r), \quad i = 1, 2$$

$$\begin{aligned} f_{Elc}(t, T) &= a_0^f(t, T) + \begin{bmatrix} a_{\chi_1^f}(T), a_{\chi_2^f}(T) \end{bmatrix} \\ &\quad M_f^*(T_1^f, T_2^f)^{-1} [\hat{f}_{Elc}(t, T_1^f), \hat{f}_{Elc}(t, T_2^f)]^\tau \end{aligned} \quad (26)$$

$$\begin{aligned} r(t, T) &= a_0^r(t, T) + \begin{bmatrix} a_{\chi_1^r}(T), a_{\chi_2^r}(T) \end{bmatrix} \\ &\quad M_r^*(T_1^r, T_2^r)^{-1} [\hat{r}(t, T_1^r), \hat{r}(t, T_2^r)]^\tau \end{aligned} \quad (27)$$

ECONOMIC INTERPRETATION AND CONCLUSION

We should stress that the choice of the forward rate is of utmost importance. For instance, even the proposed model for oil forward price is one dimensional; it can be interpolated from two forwards. On the other hand, the electricity model also admits an interpolation in terms of two forwards, even though it is a two-dimensional model. However, note that the choice of the forward rate plays an important role in the determination of dimension of the state space. If we compare the forward rates in the two examples, we note that the volatility of the forward rate chosen for oil price is random, and depends on a source of uncertainty, the instantaneous interest rate, whereas for the electricity model, the volatility of the chosen forward rate model is deterministic. Thus, depending on the empirical observations and studies made for each market, we can choose the adequate forward rate model.

Equations (21) and (26) affirm that the oil and electricity forward prices can be written as an affine function of a vector of two forward prices, so that the path history of the forward price is captured by the source of randomness described by some particular forward prices. In practice, we can use this result so that the components of

this vector of benchmark forward prices can be chosen to correspond to the mean prices of each season of the year, or can be chosen to correspond to the most liquidly maturities. We conclude that the representation of the forward prices in terms of some specific market-observed forwards obtained in this article is of significant value in implementing models in practice.

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APPENDIX A

PROOF OF LEMMA 3. 2

It is well known that the theory of equation (12) can be reduced to the theory of a system of d_i^1 first-order differential equations (the same

argument for the equation (13)). Hence, the vector equation is associated with (12).

$$\frac{\partial}{\partial T} = \hat{\sigma}_i(t, T) = A(T)\hat{\sigma}_i(t, T) \quad (\text{A.1})$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \kappa_{i0} & \kappa_{i1} & \kappa_{i2} & \kappa_{i3} & \cdots & \kappa_{id_i^1-1} \end{bmatrix}$$

Furthermore, from standard theory of ordinary differential equation we have the following result: the set of all solutions of (28) on the interval of all maturities I , form a d_i dimensional vector space, in particular a set of d_i^1 linearly independent solutions $\hat{\sigma}_{i1}, \dots, \hat{\sigma}_{id_i^1}$ must be exhibited such that every other solution of (28) is a linear combination (with real coefficients, which are stochastic processes in our case, as the equation (A.1) is parametrized by t and the source of randomness $\omega \in \Omega$) of these $\hat{\sigma}_{ij}$.

Which implies that every solution of (12) is a linear combination of d_i^1 independent solutions of (12). (For more details, refer to Coddington and Levinson⁴ Chapter 3, Section 6, Theorem 6.1.)

APPENDIX B

PROOF OF COROLLARY 3. 3

As $\sigma_i(t, T)$ and $\alpha_i(t, T)$ check the same equation, it is sufficient to prove the result for $\sigma_i(t, T)$. Let $i \in \{1, \dots, n\}$, as $(\sigma_{i1}, \dots, \sigma_{idi})$ is a linearly independent set of functions, there then exist

$T_1, \dots, T_{d_i^1}$ such that the matrix

$$M_i(T_1, \Lambda, T_{d_i^1}) = \begin{bmatrix} \sigma_{i1}(T_1) & \sigma_{i2}(T_1) & \cdots & \sigma_{id_i^1}(T_1) \\ \sigma_{i1}(T_2) & \sigma_{i2}(T_2) & \cdots & \sigma_{id_i^1}(T_2) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{i1}(T_{d_i^1}) & \sigma_{i2}(T_{d_i^1}) & \cdots & \sigma_{id_i^1}(T_{d_i^1}) \end{bmatrix}$$

is invertible. Substituting $T_1, \dots, T_{d_i^1}$ successively into the expression of $\sigma_i(t, T)$ in Lemma 3.2 and inverting the resulting equation gives

$$\begin{aligned} [c_{i1}(t), \dots, c_{id_i^1}(t)]^\tau &= M_i(T_1, \dots, T_{d_i^1})^{-1} \\ &\quad [\sigma_i(t, T_1), \dots, \sigma_i(t, T_{d_i^1})]^\tau \end{aligned}$$

As $\sigma(t, T)$ is adapted to $\{F_t\}_{0 \leq t \leq T}$ for all $T \in \mathbb{R}_+$, the right-hand side is adapted to $\{F_t\}_{0 \leq t \leq T}$ and, consequently, the processes c_{ij} are adapted to $\{F_t\}_{0 \leq t \leq T}$ for all $1 \leq j \leq d_i^1$.

APPENDIX C

PROOF OF PROPOSITION 3.1

We apply the same argument used in Corollary 3.3 to (16) and (17) to show that for all $T \in \mathbb{R}_+$

$$\begin{aligned} [\chi_1^f(t), \dots, \chi_{n_f^*}^f(t)]^\tau &= M_f^*(T_1^f, \dots, T_{n_f^*}^f)^{-1} \\ &\quad [\bar{f}_X(t, T_1^f), \dots, \bar{f}_X(t, T_{n_f^*}^f)]^\tau \end{aligned} \quad (C.1)$$

and

$$\begin{aligned} [\chi_1^r(t), \dots, \chi_{n_r^*}^r(t)]^\tau &= M_r^*(T_1^r, \dots, T_{n_r^*}^r)^{-1} \\ &\quad [\bar{r}(t, T_1^r), \dots, \bar{r}(t, T_{n_r^*}^r)]^\tau \end{aligned} \quad (C.2)$$

where $\bar{f}_X(t, T) = f_X(t, T) - a_0^f(t, T)$ and $\bar{r}(t, T) = r(t, T) - a_0^r(t, T)$. In particular, each $\chi \in A_f^*$ is an affine function in the forward prices $f_X(t, T_i^f)$ and each $\chi \in A_r^*$ is an affine function in the forward rates $r(t, T_i^r)$. Thus, we can rewrite (16) and (17) to show that the underlying forward price (respectively the forward rate)

equation admits an n_f^* (resp. n_r^*)-dimensional affine interpolation in terms of the forward prices $f_X(t, T_i^f)$ (resp. the forward rates $r(t, T_i^r)$) as follows:

$$f_X(t, T) = b_0^f(t, T) + \sum_{i=1}^{n_f^*} b_i^f(T) f_X(t, T_i^f) \quad (C.3)$$

$$r(t, T) = b_0^r(t, T) + \sum_{i=1}^{n_r^*} b_i^r(T) r(t, T_i^r) \quad (C.4)$$

where b_i^r and b_i^f are the deterministic functions obtained by substituting $\chi_i(T)$ from (29)–(30) into (16)–(17) and collecting the terms in $f_X(t, T_i^f)$ and $r(t, T_i^r)$.

Now, it can be shown that the set of forward prices and forward rates

$$(r(t, T_1^r), \dots, r(t, T_{n_r^*}^r), f_X(t, T_1^f), \dots, f_X(t, T_{n_f^*}^f))$$

forms an F_t -Markov process.

By using (31), the forward prices $f_X(t, T_i^*)$ in (19) can be replaced by the forward prices $f_X(t, T_i^f)$, so that for some $\theta \cdot$ we can write

$$\theta(t, T) = \theta'(t, T, f_X(t, T_1^f), \dots, f_X(t, T_{n_f^*}^f)) \quad (C.5)$$

and by using (32), the forward rates $r(t, T_i^r)$ in (20) can be replaced by the forward rates $r(t, T_i^r)$, so that for some σ' we can write

$$\sigma(t, T) = \sigma'(t, T, r(t, T_1^r), \dots, r(t, T_{n_r^*}^r)) \quad (C.6)$$

Next, from (8), $\{f_X(t, T_k^f), k = 1, \dots, n_f^*\}$ satisfies the following SDE

$$\begin{aligned} f_X(t, T_k^f) &= f_X(0, T_k^f) + \int_0^t \left[\theta(s, T_k^f) \bar{\sigma}(s, T_k^f)^\tau \right. \\ &\quad \left. - \frac{1}{2} \|\theta(s, T_k^f)\|^2 \right] ds + \int_0^t \theta(s, T_k^r) dW_s \end{aligned}$$

and from (2), $\{r(t, T_k^r), k = 1, \dots, n_r^*\}$ satisfies the following SDE

$$r(t, T_k^r) = r(0, T_k^r) + \int_0^t \sigma(s, T_k^r) \tilde{\sigma}(s, T_k^r) ds + \int_0^t \sigma(s, T_k^r) dW_s$$

In view of (33) and (34) these equations are of the following form :

$$df_X(t, T_k^f) = \mu_f(t, T_k^f, f_X(t, T_1^f), \dots, f_X(t, T_{n_f}^f)) dt + \sigma_f(t, T_k^f, f_X(t, T_k^f), \dots, f_X(t, T_{n_f}^f)) dW_t$$

$$dr(t, T_k^r) = \mu_r(t, T_k^r, r(t, T_1^r), \dots, r(t, T_{n_r}^r)) dt + \sigma_r(t, T_k^r, r(t, T_k^r), \dots, r(t, T_{n_r}^r)) dW_t$$

and it follows that

$$F(t) = [r(t, T_1^r), \dots, r(t, T_{n_r}^r), f_X(t, T_1^f), \dots, f_X(t, T_{n_f}^f)]^\tau$$

is an F_t -Markov process.