EGRIE Keynote Address

The Effect of Uncertainty on Optimal Control Models in the Neighbourhood of a Steady State

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For both discrete and continuous time this paper derives the Taylor approximation to the effect of uncertainty (in the simple sense of risk, *not* Knightian uncertainty) on expected utility and optimal behaviour in stochastic control models when the uncertainty is small enough that one can focus on only the first term that involves uncertainty. There is a close and illuminating relationship between the discrete-time and continuous-time results. The analysis makes it possible to spell out a tight connection between the behaviour of a dynamic stochastic general equilibrium model and the corresponding perfect foresight model. However, the quantitative analytics of the stochastic model local to a certainty model calls for a more thorough investigation of the nearby certainty model than is typically undertaken.

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Introduction

It can be difficult to get good quantitative intuition for dynamic stochastic general equilibrium models. Transparent equations for the behaviour of key variables—and the back-of-the-envelope calculations that they make possible—are often hard to come by. This is particularly true when the effects of future risk on behaviour are of enough interest that certainty-equivalence approximations seem inadequate.

This paper shows how to use perturbation methods to obtain tractable approximations for the effect of uncertainty on optimal control models in the neighbourhood of a certain steady state. The idea is analogous to using a Taylor expansion: taking derivatives at the steady state in the absence of risk to get an approximation to behaviour in a model with a small amount of uncertainty. Working through the details makes it clear that all the terms in the first-order approximation for the effects of uncertainty on the value function and its first derivative can be expressed in terms

of steady-state values of the decision variables and the first three derivatives of the objective and constraint functions. Given the value function, finding decision rules and calculating approximate distributions is then straightforward.

Perhaps the closest work to this is work by Kenneth Judd and Sy-Ming Guu.¹ Among many other differences, the treatment here aims to be less formal and more accessible. To be more specific, this paper differs from their work by treating in depth the effect of uncertainty on an otherwise general one-state-variable dynamic stochastic optimisation problem in the neighbourhood of a stable steady state in both continuous time and discrete time, in a way that aims to maximise the intuition for the quantitative results.

Although the insights here may be of some value for heavy-duty computation, the focus is on back-of-the-envelope approximations and intuition rather than on computational methods. In line with the goal of accessibility, I do not attempt any formal proofs. The results are of the form “If the value function is sufficiently differentiable, then the following formula gives the appropriate approximation”. In other words, equations that must hold given sufficient differentiability provide strong clues about what propositions are worth trying to prove more generally. Besides all the differentiability needed, and the assumption that there is a nearby stable steady state, I will assume that the optimisation problem is (1) a concave problem and (without much additional loss in generality for applied problems) that (2) in any given vector direction the functions being maximised are either linear or have a negative second derivative in that vector direction. Finally, I will assume (3) that in a neighbourhood around the steady state there is always an interior optimum for the control variables.

The focus on approximations is inspired in part by the great scientific advances that have been made with approximations in physics. There are equations in physics that have a much better claim to being exactly true than any equation in economics that is not an identity, yet physicists are unembarrassed to use approximations that give insight into what is going on. Within economics, the inspiration for this paper comes from two methods that have been very valuable to economists in the past: (a) Taylor expansions to study the economics of risk in one- and two-period models (in instances too numerous to list) and (b) phase diagrams to study dynamic problems that have a steady state.

The connection between phase diagrams and value functions is not always fully appreciated. In continuous-time optimal control models with one state variable, the co-state variable λ is the marginal value of that state variable. For a standardised phase diagram, put the state variable k on the horizontal axis and co-state variable λ

¹ Judd's (1992, 1996) and Judd and Guu's (1997, 2001) central applications are analysing the effect of uncertainty on the basic optimal growth model with inelastic labour supply and studying financial markets. Judd (1991) also discusses a wide variety of other useful approximation methods.

on the vertical axis. In the absence of shocks, the solution to the optimisation problem will be on the saddle path at every moment. Therefore, in the absence of shocks, the saddle path is a graph of the marginal value function $V_k(k)$. The slope of the saddle path is the second derivative of the value function $V_{kk}(k)$. The curvature of the saddle path reflects the third derivative of the value function $V_{kkk}(k)$.

The tight connection between discrete-time and continuous-time is also not always fully appreciated. This paper derives all the key results in both continuous and discrete time and shows how to derive key continuous-time equations—including the continuous-time Bellman equation—as the limit of the discrete-time equations as the length of a period goes to zero.²

The first few equations for the certainty case

It provides a good introduction to the equations below to take a few steps through the corresponding, much simpler analysis in a model with no uncertainty. Here is the basic notation in the absence of risk:

- t : time
- k : state variable (one-dimensional)
- x : control variable vector (many-dimensional)
- ρ : utility discount rate (impatience) in the objective function $E_0 \int_0^\infty e^{-\rho t} U dt$
- U : flow utility (felicity)
- V : value function
- A : deterministic part of the accumulation function

For example, for a continuous-time, perfect-foresight version of the Basic Real Business Cycle Model, k is the capital stock and x is the vector $\begin{bmatrix} c \\ n \end{bmatrix}$, where c is consumption and n is labour hours. For this Basic Real Business Cycle model, the flow utility and accumulation functions are

$$U(k, x) = u(c, n),$$

$$A(k, x) = znf\left(\frac{k}{zn}\right) - \delta k - c - g.$$

Here, the constants z and g are the level of labour-augmenting technology and the level of government purchases.

² Contrary to the practice—and perhaps view—of some mathematical economists, my view is that the only economically relevant continuous-time equations are those that arise as the limit of discrete-time equations in this way.

In general, the continuous-time Bellman equation in the absence of risk is

$$\rho V(k) = \max_x U(k, x) + V_k(k)A(k, x).$$

Using the envelope theorem, the derivative of the Bellman equation with respect to k is

$$\rho V_k(k) = U_k(k, x) + V_k(k)A_k(k, x) + V_{kk}(k, x)A(k, x),$$

where it is understood that this is evaluated at the optimising value of the control variable x . At the steady state, $A(k, x) = 0$, so with a bit of rearrangement, the Bellman equation reduces to

$$V(k^*) = \frac{U(k^*, x^*)}{\rho}$$

and the first derivative of the Bellman equation at the steady state reduces to

$$V_k(k^*) = \frac{U_k(k^*, x^*)}{\rho - A_k(k^*, x^*)}$$

unless this would involve dividing by zero. If $A_k(k^*, x^*) = \rho$, then $U_k(k^*, x^*)$ must be zero, but there is no clear implication for $V_k(k^*)$.

A similar analysis can be done for higher derivatives, but let me save that for below, where it is clearer why the higher derivatives are important for understanding the effects of risk.

The convergence rate κ

In addition to the characteristics of the certainty phase diagram mentioned above, the aspect of the certainty solution that matters most for the effects of risk is the exponential convergence rate κ at which the distance away from the steady-state decays towards zero as $t \rightarrow \infty$ when there are no shocks. Of all the results in this paper, the ones I consider most important are the ones that show how the convergence rate in the certainty model matters for the effects of risk. Thus, the persistence of the effects of shocks matters for the effects of risk. The role of the convergence rate in the formulas below shows how the speed of endogenous mean reversion matters even in cases where the underlying shocks themselves are all permanent.

As an illustration of the benefits of thinking about the convergence rate κ , let me derive a formula for $V_k(k^*)$ that is both quite intuitive, and will not involve dividing by zero. The first step is to take the derivative of the Bellman equation without using the envelope theorem. Writing dx/dk for the derivative of the optimal control variable vector with respect to k ,

$$\rho V_k(k) = U_k(k, x) + U_x(k, x) \frac{dx}{dk} + V_k(k) \left[A_k(k, x) + A_x(k, x) \frac{dx}{dk} \right] + V_{kk}(k, x)A(k, x).$$

By definition, the convergence rate κ is minus the total derivative of A with respect to k at the steady state:

$$\kappa = -\frac{dA}{dk} = -\left[A_k(k, x) + A_x(k, x) \frac{dx}{dk}\right].$$

So one can solve for $V_k(k^*)$ as follows:

$$V_k(k^*) = \frac{U_k(k^*, x^*) + U_x(k, x) \frac{dx}{dk}}{\rho + \kappa} = \frac{\frac{dU}{dk}}{\rho + \kappa}.$$

That is, the marginal value of k at the steady state is equal to the *total* effect of k on flow utility (including the part of that effect that operates through changes in the control variables), discounted not only by the utility discount rate but by the rate κ at which a deviation of k from k^* decays.

An important consequence of the equation $V_k = (dU/dx)/(\rho + \kappa)$ is that in a model with a positive marginal value of capital and a stable steady state, the total derivative of flow utility with respect to k is always positive in the neighbourhood of the steady state. That is, *close enough to the steady state, going down to the right along the saddle path will always lead to a higher level of flow utility*, regardless of the number of variables in the control variable vector and the details of functional forms.

The next section turns to the general analysis for a stochastic model local to a stable steady state in continuous time. To fix ideas, the subsequent sections then apply the equations derived in that section to a stochastic version of a Ramsey model: “Continuous-time application: choosing how much to use a risky technology in general equilibrium” analyses the stochastic Ramsey model with a choice of how much risk to take; “Continuous-time application: when the only technology available is risky” analyses the stochastic Ramsey model with risk that is unavoidable except for the fact that it scales up with the level of output. The penultimate section gives the general analysis for a discrete-time dynamic stochastic model local to a stable steady state. Finally, the Conclusion describes the additional examples analysed in the companion working paper³ and offers some parting thoughts.

General analysis for a one-state-variable optimising model in continuous time

Notation

All of the notation for the certainty case carries over to the case of uncertainty. In addition, the following variables are needed to think about the effects of small

³ Kimball (2014).

amounts of risk:

- ω : scalar parameter controlling amount of risk; equal to 1 for the level of risk of ultimate interest
- v : addition to flow utility U when $\omega = 1$
- α : addition to the accumulation function A when $\omega = 1$
- σ : standard deviation of the Brownian motion term when $\omega = 1$
- dz : standard Wiener innovation

The optimisation problem

Other than the constant utility discount rate ρ , time does not explicitly enter the problem. Thus, the agent faces the stationary problem

$$\begin{aligned} \max_{\{x\}} \mathbf{E}_0 \int_0^\infty e^{-\rho t} [U(k, x) + \omega v(k, x)] dt \\ \text{s.t. } dk = [A(k, x) + \omega \alpha(k, x)] dt + \sqrt{\omega} \sigma(k, x) dz. \end{aligned} \quad (1)$$

The parameter ω controls the presence or absence of uncertainty. The case of primary interest is $\omega = 1$, while $\omega = 0$ is the benchmark certainty case. I include a possible dependence of the objective function and the accumulation function on ω through v and α in order to ensure that the optimal value of control variables (such as the degree of employment of the risky technology in the next section) have a nontrivial limit as $\omega \rightarrow 0$.⁴ Under the aegis of assuming that it is a concave problem, I will assume that U , v , A and α are all jointly concave in k and x , and σ^2 is jointly convex in k and x . I am also assuming, with little loss of generality, that at the certainty steady state, $V_k(k^*, 0) > 0$. (If $V_k(k^*, 0) < 0$, make $-k$ the state variable to rectify the sign of the steady-state marginal value of the state variable. But if $V_k(k^*, 0) = 0$, the arguments below would need to be revisited to see where they simplify and where they get more complex.) Given these assumptions about curvature and orientation, standard results then guarantee that $V_{kk} \leq 0$.

The Bellman equation and its derivatives

The current-value Bellman equation is

$$\begin{aligned} \rho V(k, \omega) = \max_x F(k, x, \omega) = \max_x \left\{ U(k, x) + V_k(k, \omega) A(k, x) \right. \\ \left. + \omega \left[v(k, x) + V_k(k, \omega) \alpha(k, x) + V_{kk}(k, \omega) \frac{\sigma^2(k, x)}{2} \right] \right\}. \end{aligned} \quad (2)$$

⁴ Judd (1991, ch. 11) discusses from a technical angle the use of terms like those involving v and α under the heading “Bifurcate Methods”.

By the envelope theorem, the induced variation in the control variable vector x can be ignored when taking the first derivatives of the Bellman equation. Thus:

$$\begin{aligned}\rho V_k(k, \omega) &= F_k(k, x(k, \omega), \omega) \\ &= U_k + V_k A_k + V_{kk} A + \omega \left[v_k + V_k \alpha_k + V_{kk} \sigma \sigma_k + V_{kk} \alpha + V_{kkk} \frac{\sigma^2}{2} \right]\end{aligned}\quad (3)$$

and

$$\begin{aligned}\rho V_\omega(k, \omega) &= F_\omega(k, x(k, \omega), \omega) \\ &= V_{k\omega} A + v + V_k \alpha + V_{kk} \frac{\sigma^2}{2} + \omega \left[V_{k\omega} \alpha + V_{kk\omega} \frac{\sigma^2}{2} \right].\end{aligned}\quad (4)$$

The arguments of the derivatives of V are k and ω , while the arguments of all the other functions on the right-hand side of (3) and (4) are k and x . Note that at the certain steady state, $\omega = 0$ and $A(k^*, 0) = 0$, making many terms drop out of (3) and (4).

Christening the “prevalue function” $F(k, x, \omega)$

In many different contexts, close analysis of the Bellman equation requires careful attention to the properties of the function underneath the maximisation operator on the right-hand side of the Bellman equation. Therefore, the right-hand side of the Bellman equation before maximisation deserves a name. Let me propose the name “prevalue function”. To make the case that this is a reasonable coinage, let me make the analogy to the terminology of physiology, in which a hormone that is converted into another hormone is given the prefix “pro-”. For example, “prothrombin” is converted into thrombin (one of the key agents in blood clotting). Similarly, if maximising a function over x yields the value function, then the original function that is converted into the value function by that maximisation can be called the “prevalue function”. There are two justifications for using *pre*—instead of *pro*—in the phrase “prevalue function”. First, “value” is of Latin rather than Greek derivation, so it is appropriate to prefix it with the Latin “pre-” instead of the Greek “pro-”. Second—and more importantly—political uses of “values” would give the word “provalue” other possible meanings that could be confusing—or at least distracting.

On usage, let me suggest that the phrase “prevalue function” will be most useful if it can be used in a flexible way for the object to be maximised on the right-hand side of any Bellman equation, whether a continuous-time or discrete-time Bellman equation, and regardless of how the Bellman equation is arranged. In other words, regardless of whether terms unaffected by the maximisation operator are put on the left- or right-hand side of the equation, and regardless of whether the equation is divided through by a term unaffected by the maximisation operator, the function under the maximisation operator can be called “the prevalue function”.

(Thus, turning the “prevalue function” into the value function may require some other simple operations beyond maximising over the control variable vector.)

The effect of uncertainty on the level of the value function at the steady state

Since $\Delta\omega = 1$, V_ω at $(k, \omega) = (k^*, 0)$ is the first term in the Taylor approximation for the effect of uncertainty on the level of the value function at the steady state. Eq. (4) implies:

$$V_\omega(k^*, 0) = \frac{v(k^*, x^*) + V_k(k^*, 0)\alpha(k^*, x^*) + V_{kk}(k^*, 0)\frac{\sigma^2(k^*, x^*)}{2}}{\rho}. \quad (5)$$

There is one important interpretive point to make here. $V_\omega(k^*, 0)$ gives the effect of uncertainty on welfare *at* k^* , the steady-state level of k *for the certainty model*. Thus, it is *not* the effect of uncertainty on steady-state welfare, but the effect of the introduction of uncertainty on an expected present value concept of welfare at time zero, *including the present value of all the expected transitional behaviour of the model*. Thus, in the analysis here, it is actually easier to include transitional effects on welfare than to neglect them.

Eq. (5) is important enough that it is worth repeating it in a spare form that omits arguments:

$$V_\omega = \frac{v + V_k\alpha + V_{kk}\frac{\sigma^2}{2}}{\rho}. \quad (5')$$

The effect of uncertainty on the marginal value at the steady state

Taking the “total” derivative of (4) with respect to k —that is, the derivative with respect to k including the induced variation in x —yields the key equation for evaluating the effects of uncertainty on behaviour:

$$\begin{aligned} \rho V_{k\omega}(k, \omega) &= \frac{d}{dk} F_\omega(k, x(k, \omega), \omega) \\ &= V_{kk\omega}A(k, x(k, \omega)) + V_{k\omega} \frac{d}{dk} A(k, x(k, \omega)) + V_{kk}\alpha + V_{kkk} \frac{\sigma^2}{2} \\ &\quad + v_k + V_k\alpha_k + V_{kk}\sigma\sigma_k + \left[\frac{\partial x(k^*, 0)}{\partial k} \right]^T [v_x + V_k\alpha_x + V_{kk}\sigma\sigma_x] \\ &\quad \times \omega \left[V_{kk\omega}\alpha + V_{kkk\omega} \frac{\sigma^2}{2} + V_{k\omega} \frac{d}{dk} \alpha(k, x(k, \omega)) + V_{kk\omega} \frac{d}{dk} \frac{\sigma^2(k, x(k, \omega))}{2} \right], \quad (6) \end{aligned}$$

where $(d/dk)A(k, x(k, \omega)) = A_k + [\partial x(k^*, 0)/\partial k]^T A_x$ is the *total* derivative of the rate of change in k with respect to k itself under the optimal policy when $\omega = 0$. Note that

here the convergence rate discussed in the introduction is

$$\kappa = -\frac{d}{dk}A(k, x(k, 0)) = -A_k - [A_x]^T \frac{\partial x(k, \omega)}{\partial k}. \quad (7)$$

Solving (6) for $V_{k\omega}$ at the certain steady state, where $\omega=0$, $A=0$ and $dA/dk=-\kappa$, one finds that

$$V_{k\omega}(k^*, 0) = \frac{v_k + V_k \alpha_k + V_{kk} \sigma \sigma_k + V_{kk} \alpha + V_{kkk} \frac{\sigma^2}{2}}{\rho + \kappa} + \frac{\left[\frac{\partial x}{\partial k}\right]^T [v_x + V_k \alpha_x + V_{kk} \sigma \sigma_x]}{\rho + \kappa}. \quad (8)$$

Eq. (5), giving the effect of a small amount of risk on the value at the steady state is one of the two central results in this paper. Eq. (8), giving the effect of a small amount of risk on the marginal value at the steady state is the other key result.

Calculating $\partial x/\partial k$ at the certain steady state

The factor $\partial x/\partial k$ in (8) represents what happens to the control variables as one traces the saddle path going to the right—which in the neighbourhood of the certain steady state is known from one's analysis of the certainty model. To find $\partial x(k, \omega)/\partial k$ using the notation here, begin with the first-order condition

$$F_x(k, x(k, \omega), \omega) \equiv 0. \quad (9)$$

Taking the total derivative of this first-order condition with respect to k

$$\frac{d}{dk} F_x(k, x(k, \omega), \omega) = F_{xk}(k, x(k, \omega), \omega) + F_{xx}(k, x(k, \omega), \omega) \frac{\partial x(k, \omega)}{\partial k} = 0, \quad (10)$$

so

$$\frac{\partial x(k, \omega)}{\partial k} = -[F_{xx}(k, x(k, \omega), \omega)]^{-1} F_{xk}(k, x(k, \omega), \omega). \quad (11)$$

At the certain steady state, (11) reduces to

$$\frac{\partial x(k^*, 0)}{\partial k} = -[U_{xx} + V_k A_{xx}]^{-1} [V_{kk} A_x + U_{kx} + V_k A_{kx}]. \quad (12)$$

But what if $F_{xx} = U_{xx} + V_k A_{xx}$ is not invertible, as assumed in Eqs. (11) and (12)? The case where some dimensions of the control variable vector only matter under uncertainty is handled in the following subsections. If, more radically, a dimension (in the sense of a vector direction) of the control variable vector is linear in all the functions U , A , v , α and σ^2 , with a nonzero marginal contribution to at least one of these functions, it is very unlikely that the problem would have both a well-defined steady state when $\omega=0$, and satisfy the assumption of having an interior solution for the control variable vector everywhere in a neighbourhood around that steady state. Any possible exception would have so much structure (beginning with having

to have V_k , V_{kk} or V_{kk}/V_k be a constant to maintain the flatness of F in that dimension needed to get an interior solution at values of k away from the steady state), that it would be relatively easy to analyse.

The case of additive separability between two sets of control variables, only one of which involves uncertainty

A common special case leads to a great simplification in Eq. (8) by making the coefficient of $\partial x/\partial k$ in (8) equal zero. If there is additive separability between two sets of control variables, with only one set entering into the standard deviation function σ , then the functions U and v , A and α can be defined so that the set x^I of control variables affecting U and A is separate from the set x^{II} of control variables affecting v , α and σ . This allows the first-order condition—

$$F_x = U_x + V_k A_x + \omega[v_x + V_k \alpha_x + V_{kk} \sigma \sigma_x] = 0 \quad (13)$$

—to be analysed into two separate (vector) first-order conditions:

$$U_{x^I} + V_k A_{x^I} = 0 \quad (14)$$

and

$$v_{x^{II}} + V_k \alpha_{x^{II}} + V_{kk} \sigma \sigma_{x^{II}} = 0. \quad (15)$$

(Strictly speaking, (15) only has to hold when $\omega \neq 0$, but it must hold in the limit as $\omega \rightarrow 0$.)

A useful trick of rearrangement in the additively separable case

To see how to get the two separate first-order conditions (14) and (15), begin with the two premises of (i) additive separability between x^I and x^{II} and (ii) σ depending only on x^{II} :

$$\begin{aligned} \rho V(k, \omega) &= \max_{x^I, x^{II}} F(k, x^I, x^{II}, \omega) \\ &= \max_{x^I, x^{II}} \left\{ U^I(k, x^I) + U^{II}(k, x^{II}) + V_k(k, \omega) [A^I(k, x^I) + A^{II}(k, x^{II})] \right. \\ &\quad + \omega \left[v^I(k, x^I) + v^{II}(k, x^{II}) + V_k(k, \omega) [\alpha^I(k, x^I) + \alpha^{II}(k, x^{II})] \right. \\ &\quad \left. \left. + V_{kk}(k, \omega) \frac{\sigma^2(k, x^{II})}{2} \right] \right\}. \end{aligned} \quad (16)$$

Next, recall that the aim is to get a good approximation for the behaviour of the Bellman equation when $\omega = 1$. It is possible to construct an approximation for the

Bellman equation when $\omega = 1$ based on any modified prevalue function $\mathcal{F}(k, x^I, x^{II}, \omega)$ satisfying

$$\mathcal{F}(k, x^I, x^{II}, 1) = F(k, x^I, x^{II}, 1), \quad (17)$$

since that leaves the target Bellman equation at $\omega = 1$ unchanged. In particular, it is convenient to choose

$$\begin{aligned} \mathcal{F}(k, x^I, x^{II}, 1) &= U^I(k, x^I) + v^I(k, x^I) + V_k(k, \omega) [A^I(k, x^I) + \alpha^I(k, x^I)] \\ &\quad + \omega \left[U^{II}(k, x^{II}) + v^{II}(k, x^{II}) + V_k(k, \omega) [A^{II}(k, x^{II}) + \alpha^{II}(k, x^{II})] \right. \\ &\quad \left. + V_{kk}(k, \omega) \frac{\sigma^2(k, x^{II})}{2} \right] \\ &= U(k, x^I) + V_k(k, \omega) A(k, x^I) \\ &\quad + \omega \left[v(k, x^{II}) + V_k(k, \omega) \alpha(k, x^{II}) + V_{kk}(k, \omega) \frac{\sigma^2(k, x^{II})}{2} \right], \end{aligned} \quad (18)$$

where

$$\begin{aligned} U(k, x^I) &= U^I(k, x^I) + v^I(k, x^I), \\ A(k, x^I) &= A^I(k, x^I) + \alpha^I(k, x^I), \\ v(k, x^{II}) &= U^{II}(k, x^{II}) + v^{II}(k, x^{II}), \\ \alpha(k, x^{II}) &= A^{II}(k, x^{II}) + \alpha^{II}(k, x^{II}). \end{aligned}$$

That yields this Bellman equation in k, x^I, x^{II} and ω :

$$\begin{aligned} \rho V(k, \omega) &= \max_{x^I, x^{II}} \mathcal{F}(k, x^I, x^{II}, \omega) \\ &= \max_{x^I} \{ U(k, x^I) + V_k(k, \omega) A(k, x^I) \} \\ &\quad + \omega \max_{x^{II}} \left\{ v(k, x^{II}) + V_k(k, \omega) \alpha(k, x^{II}) + V_{kk}(k, \omega) \frac{\sigma^2(k, x^{II})}{2} \right\}. \end{aligned} \quad (19)$$

This trick of rearrangement leaves the situation unchanged when $\omega = 1$, but can change the certainty model corresponding to $\omega = 0$ that forms the anchor for calculations. After this rearrangement, U and A only depend on x^I , while v , α and σ only depend on x^{II} . That leads to the first-order conditions (14) and (15) above.

Note that in practice, given knowledge of this trick, one is likely to have written down a prevalue function with the clean separation exhibited by $\mathcal{F}(k, x^I, x^{II}, \omega)$ in the first place when that is possible, so the procedure is less complex then it appears above. The idea is to (a) take everything that depends on control variables that affect σ and put those terms into the functions v and α , (b) also include in x^{II} any other

control variables nonseparably connected with those that affect σ , and (c) hope that what is left for U and A is enough to yield a meaningful and analysable certainty model when $\omega=0$. To the extent that terms swept into v and α by this procedure appear large, remember (i) that constants can be reallocated at will between U and v and between A and α to normalise the steady-state values of v and α to zero (a procedure that yields a certainty steady state that is closer to the target risky situation) and (ii) that the adequacy of the approximation is likely to depend primarily on second and higher-order derivatives, not so much on the steady-state values of the first derivatives.

Outcomes in the additively separable case

Since the control variables not explicitly mentioned in (14) and (15) do not even enter into the equations where they are omitted, it is also true that

$$U_x + V_k A_x = 0 \quad (20)$$

and

$$v_x + V_k \alpha_x + V_{kk} \sigma \sigma_x = 0. \quad (21)$$

Eq. (21) allows one to simplify (8) to

$$V_{k\omega}(k^*, 0) = \frac{v_k + V_k \alpha_k + V_{kk} \sigma \sigma_k + V_{kk} \alpha + V_{kkk} \frac{\sigma^2}{2}}{\rho + \kappa}. \quad (22)$$

Additive separability between x^I and x^{II} simplifies $\partial x^I / \partial \omega$ to

$$\frac{\partial x^I(k^*, 0)}{\partial \omega} = -[U_{x^I x^I} + V_k A_{x^I x^I}]^{-1} V_{k\omega} A_{x^I} \quad (23)$$

assuming invertibility, which is much more likely to hold here than in Eq. (12). This is essentially the effect on the control variables in the certainty model of raising the costate variable by $V_{k\omega}$ for a fixed k in the neighbourhood of the steady state. Thus, as long as a meaningful certainty steady state exists after all the functions of the control variables in or nonseparable with the control variables in σ are reallocated to v and α , the effect of uncertainty on the control variables that appear in that certainty model all work through the marginal value of k .

A similar calculation for $\partial x^{II}(k^*, 0) / \partial \omega$ based on differentiating (15) with respect to ω yields

$$\frac{\partial x^{II}(k^*, 0)}{\partial \omega} = -[U_{x^{II} x^{II}} + V_k A_{x^{II} x^{II}}]^{-1} [V_{k\omega} A_{x^{II}} + V_{kk\omega} \sigma \sigma_{x^{II}}]. \quad (24)$$

This depends on a relatively high-order derivative of the value function: $V_{kk\omega}$. Therefore, in the sequence of things to compute, this has to be further down the list (indeed, further down than I will compute in this paper). The key to the approach here is to make sure there is a path to computing things in sequence without ever depending on something not yet known. The fact that all the effects of ω working

through x'' are multiplied by the ω that stands in front of the terms involving v , α and σ makes it possible to compute other key magnitudes prior to knowing the value of $\partial x''(k^*, 0)/\partial \omega$.

Control variables that only matter under uncertainty, when separability cannot be assumed

To find $\partial x(k, \omega)/\partial \omega$ when separability cannot be assumed, begin again with the first-order condition

$$F_x(k, x(k, \omega), \omega) \equiv 0.$$

Taking the total derivative of this first-order condition with respect to ω yields

$$\frac{d}{d\omega} F_x(k, x(k, \omega), \omega) = F_{x\omega}(k, x(k, \omega), \omega) + F_{xx}(k, x(k, \omega), \omega) \frac{\partial x(k, \omega)}{\partial \omega} = 0. \quad (25)$$

Recall that

$$F(k, x, \omega) = U(k, x) + V_k(k, \omega)A(k, x) + \omega \left[v(k, x) + V_k(k, \omega)\alpha(k, x) + V_{kk}(k, \omega) \frac{\sigma^2(k, x)}{2} \right].$$

Therefore, at $\omega = 0$, (25) becomes

$$V_{k\omega}A_x + v_x + V_k\alpha_x + V_{kk}\sigma\sigma_x = -[U_{xx} + V_kA_{xx}] \frac{\partial x}{\partial \omega}. \quad (26)$$

If one or more control variables only matters under uncertainty—that is, appears only in the functions v , α or σ —then $U_{xx} + V_kA_{xx}$ will not be of full rank. However, since it is symmetric and negative semidefinite, there will be a rotation of the control variables that allows x , after relabelling, to be separated into two subvectors x^a and x^b , with only x^a generating curvature at $(k, \omega) = (k^*, 0)$:

$$U_{xx} + V_kA_{xx} = \begin{bmatrix} U_{x^a x^a} + V_kA_{x^a x^a} & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, $U_{x^a x^a} + V_kA_{x^a x^a}$ has full rank, given the assumption that in every direction things are either linear or have a negative second derivative.

Note that since $V_k > 0$, if x_i is in the subvector x^b , both $U_{x_i x_i} = 0$ and $A_{x_i x_i} = 0$. Joint concavity of U in k and x requires that $U_{kk}U_{x_i x_i} - [U_{kx_i}]^2 = -[U_{kx_i}]^2 \geq 0$, so $U_{kx_i} = 0$ for any x_i in the subvector x^b , and $U_{kx^b} = 0$. By similar reasoning, $A_{kx^b} = 0$. Finally, if there are linear terms in U or A involving elements of x^b , one can use the trick of rearrangement to shift them into v and α instead, without changing the $\omega = 1$ target model. The assumption that all control variables have an internal optimum in the neighbourhood of the steady state means that the prevalue function must be flat in these elements of x^b when $\omega = 0$ in any case, so shifting these linear terms into v and α will not stop a steady state from being a steady state, and will not cause any change in the relevant $U_{xx} + V_kA_{xx}$ matrix. In combination, these observations imply that to a

second-order approximation the problem can be rewritten so that both U and A are functions only of k and x^a .

Breaking (26) into its two constituent equations,

$$V_{k\omega}A_{x^a} + v_{x^a} + V_k\alpha_{x^a} + V_{kk}\sigma\sigma_{x^a} = -[U_{x^a x^a} + V_k A_{x^a x^a}] \frac{\partial x^a}{\partial \omega}, \quad (27)$$

$$V_{k\omega}A_{x^b} + v_{x^b} + V_k\alpha_{x^b} + V_{kk}\sigma\sigma_{x^b} = v_{x^b} + V_k\alpha_{x^b} + V_{kk}\sigma\sigma_{x^b} = 0. \quad (28)$$

Since $U_{x^a x^a} + V_k A_{x^a x^a}$ has full rank, (27) implies

$$\frac{\partial x^a}{\partial \omega} = -[U_{x^a x^a} + V_k A_{x^a x^a}]^{-1}[V_{k\omega}A_{x^a} + v_{x^a} + V_k\alpha_{x^a} + V_{kk}\sigma\sigma_{x^a}]. \quad (29)$$

Eq. (28) says that the first-order condition for x^b holds in the terms that have x^b as an argument, even after dividing them through by ω .

Calculating $V_{k\omega}$ when separability cannot be assumed

When the only part of the control variable vector that matters in the certainty model is x^a , the correct version of (12) is

$$\frac{\partial x^a(k^*, 0)}{\partial k} = -[U_{x^a x^a} + V_k A_{x^a x^a}]^{-1}[V_{kk}A_{x^a} + U_{kx^a} + V_k A_{kx^a}]. \quad (30)$$

Breaking (8) into the pieces that involve $\partial x^a/\partial k$ and $\partial x^b/\partial k$:

$$\begin{aligned} V_{k\omega}(k^*, 0) &= \frac{v_k + V_k\alpha_k + V_{kk}\sigma\sigma_k + V_{kk}\alpha + V_{kkk}\frac{\sigma^2}{2}}{\rho + \kappa} + \frac{\left[\frac{\partial x^a}{\partial k}\right]^T [v_{x^a} + V_k\alpha_{x^a} + V_{kk}\sigma\sigma_{x^a}]}{\rho + \kappa} \\ &\quad + \frac{\left[\frac{\partial x^b}{\partial k}\right]^T [v_{x^b} + V_k\alpha_{x^b} + V_{kk}\sigma\sigma_{x^b}]}{\rho + \kappa} \\ &= \frac{v_k + V_k\alpha_k + V_{kk}\sigma\sigma_k + V_{kk}\alpha + V_{kkk}\frac{\sigma^2}{2}}{\rho + \kappa} \\ &\quad - \frac{[V_{kk}A_{x^a} + U_{kx^a} + V_k A_{kx^a}]^T [U_{x^a x^a} + V_k A_{x^a x^a}]^{-1} [v_{x^a} + V_k\alpha_{x^a} + V_{kk}\sigma\sigma_{x^a}]}{\rho + \kappa}. \end{aligned} \quad (31)$$

Eq. (31) shows clearly the extra complications that arise from nonseparability. However, there is one important element of simplicity: calculating $V_{k\omega}$ does not depend on knowledge of $\partial x^b/\partial k$.

Calculating V_k , V_{kk} and V_{kkk} at the certain steady state

In order to make Eqs. (5) and either (8), (22) or (31) operational, it is necessary to calculate the first three derivatives of the value function at the steady state with $\omega = 0$. Because these magnitudes depend only on the certainty model, the only part of the

control variable vector that is relevant is typically x^a . I will assume in this subsection that x^b is totally absent from the certainty model and for notational simplicity, I will write x in place of x^a in this subsection. Also, as a reminder that ω is not in the picture in the certainty model, I will use the notation

$$\frac{dx}{dk} = \frac{\partial x(k^*, 0)}{\partial k},$$

where x on both sides of this equation really means x^a .

The marginal value $V_k(k^*, 0)$ can be found as the steady-state value of the costate variable (“ λ^* ”) using standard techniques. In brief, (3) implies that $\rho V_k = U_k + V_k A_k$ with all functions evaluated at the certain steady state. This co-state equation, the steady-state first-order condition $F_x = U_x + V_k A_x = 0$, and the condition that defines the certain steady state, $A(k^*, x^*) = 0$, can be jointly solved for k^* , x^* and $V_k(k^*, 0)$.

The most general way to calculate $V_{kk}(k^*, 0)$ is actually to use standard techniques to find the slope of the saddle-path at the steady state in (k, λ) space. In the notation here, *totally* differentiating (3) with respect to k yields

$$\rho V_{kk} = F_{kk} + [F_{kx}]^T \frac{dx}{dk} = F_{kk} - [F_{kx}]^T [F_{xx}]^{-1} F_{kx}. \quad (32)$$

Calculating the derivatives in (32) at the certain steady state,

$$\begin{aligned} \rho V_{kk} = & U_{kk} + V_k A_{kk} + 2V_{kk} A_k - [U_{kx} + V_k A_{kx} + V_{kk} A_x]^T [U_{xx} + V_k A_{xx}]^{-1} \\ & \times [U_{kx} + V_k A_{kx} + V_{kk} A_x]. \end{aligned} \quad (33)$$

Eq. (33) is a quadratic equation in $V_{kk}(k^*, 0)$:

$$\begin{aligned} - \left([A_x]^T [U_{xx} + V_k A_{xx}]^{-1} A_x \right) V_{kk}^2 + \left(-\rho + 2A_k - 2A_x [U_{xx} + V_k A_{xx}]^{-1} [U_{kx} + V_k A_{kx}] \right) V_{kk} \\ + U_{kk} + V_k A_{kk} - [U_{kx} + V_k A_{kx}] [U_{xx} + V_k A_{xx}]^{-1} [U_{kx} + V_k A_{kx}] = 0. \end{aligned} \quad (34)$$

Since U and A are each jointly concave in k and x , the coefficient of V_{kk}^2 will be positive and the constant term will be negative, guaranteeing that there is one positive and one negative root. Thus, (34) can be solved by the quadratic formula, taking $V_{kk}(k^*, 0)$ as the negative root. Note that every other quantity in (34) can be calculated prior to calculating V_{kk} .

To calculate $V_{kk}(k^*, 0)$, it is easiest to start with the Bellman equation $\rho V = F$ and totally differentiate without using the envelope theorem:

$$\rho V_k = F_k + F_x \frac{dx}{dk} \quad (35)$$

$$\rho V_{kk} = F_{kk} + 2F_{kx}^T \frac{dx}{dk} + \left[\frac{dx}{dk} \right]^T F_{xx} \frac{dx}{dk} + F_x^T \frac{d^2x}{dk^2} \quad (36)$$

$$\begin{aligned} \rho V_{kkk} = & F_{kkk} + 3[F_{kkx}]^T \frac{dx}{dk} + 3 \left[\frac{dx}{dk} \right]^T [F_{kxx}] \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk} \right]^T F_{xxx_i} \frac{dx}{dk} \right) \\ & + 3 \left[F_{kx} + F_{xx} \frac{dx}{dk} \right]^T \frac{d^2x}{dk^2} + F_x^T \frac{d^3x}{dk^3}. \end{aligned} \quad (37)$$

Having held the envelope theorem in reserve, now it can be applied with a vengeance. The F_x that multiplies d^3x/dk^3 is zero by the first-order condition. In addition, Eq. (10) gives

$$\left(F_{xk} + F_{xx} \frac{dx}{dk} \right) = 0$$

by the total derivative of the first-order condition with respect to k . The quantity d^2x/dk^2 could be calculated from totally differentiating in turn with respect to k , but there is no need, since the coefficient of d^2x/dk^2 is zero. Deleting the last line of (37) in accordance with this curious higher-order envelope theorem and expanding at the certain steady state, (37) becomes

$$\begin{aligned} \rho V_{kkk} = & U_{kkk} + V_k A_{kkk} + 3V_{kk} A_{kk} + 3V_{kkk} A_k + 3[U_{kxx} + V_k A_{kxx} + 2V_{kk} A_{kx} + V_{kkk} A_x]^T \frac{dx}{dk} \\ & + 3 \left[\frac{dx}{dk} \right]^T [U_{kxx} + V_k A_{kxx} + V_{kk} A_{xx}] \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk} \right]^T [U_{xxx_i} + V_k A_{xxx_i}] \frac{dx}{dk} \right) \end{aligned} \quad (38)$$

with

$$\frac{dx}{dk} = -[U_{xx} + V_k A_{xx}]^{-1} [U_{kx} + V_k A_{kx} + V_{kk} A_x],$$

from (12)—or more precisely, from (30) with the x^a written just x .

Grouping terms in (38) according to the derivative of V in each,

$$\begin{aligned} \rho V_{kkk} = & U_{kkk} + 3U_{kxx}^T \frac{dx}{dk} + 3 \left[\frac{dx}{dk} \right]^T U_{kxx} \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk} \right]^T U_{xxx_i} \frac{dx}{dk} \right) \\ & + V_k \left[A_{kkk} + 3A_{kkx}^T \frac{dx}{dk} + 3 \left[\frac{dx}{dk} \right]^T A_{kxx} \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk} \right]^T A_{xxx_i} \frac{dx}{dk} \right) \right] \\ & + 3V_{kk} \left[A_{kk} + 2A_{kx}^T \frac{dx}{dk} + \left[\frac{dx}{dk} \right]^T A_{xx} \frac{dx}{dk} \right] + 3V_{kkk} \left[A_k + A_x^T \frac{dx}{dk} \right]. \end{aligned} \quad (39)$$

Since $\kappa = -A_k - [A_x]^T dx/dk$, the coefficient of V_{kkk} is $-3\kappa V_{kkk}$. Thus, solving (39) for V_{kkk} yields

$$\begin{aligned} V_{kkk}(k^*, 0) = & (\rho + 3\kappa)^{-1} \left\{ U_{kkk} + 3U_{kx}^T \frac{dx}{dk} + 3 \left[\frac{dx}{dk} \right]^T U_{kxx} \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk} \right]^T U_{xxx_i} \frac{dx}{dk} \right) \right. \\ & + V_k \left[A_{kkk} + 3A_{kx}^T \frac{dx}{dk} + 3 \left[\frac{dx}{dk} \right]^T A_{kxx} \frac{dx}{dk} + \sum_i \frac{dx_i}{dk} \left(\left[\frac{dx}{dk} \right]^T A_{xxx_i} \frac{dx}{dk} \right) \right] \\ & \left. + 3V_{kk} \left[A_{kk} + 2A_{kx}^T \frac{dx}{dk} + \left[\frac{dx}{dk} \right]^T A_{xx} \frac{dx}{dk} \right] \right\}. \end{aligned} \quad (40)$$

The nature of the denominator $\rho + 3\kappa$ on the right-hand side of (40) would be more evident if both sides of the equation were multiplied by $(k - k^*)^3$; it is exactly what one would expect for the present value of terms involving $(k - k^*)^3$, with k following a continuous-time AR(1) process that reverts to k^* at the rate κ . In the numerator within the curly brackets, the first line is the third derivative of the flow utility function U with respect to the state variable k , including indirect effects through changes in the vector of control variables *evaluated as if the derivative of the control variables with respect to the state variable k were a constant*. The second line is the third derivative of the accumulation function A with respect to k (including indirect effects through changes in the control variables evaluated as if the derivative of the control variables with respect to k were a constant) *multiplied by the steady-state marginal value, which converts from units of production to units of utility*. The third line is three times the second derivative of the accumulation function with respect to k (including indirect effects through changes in the control variables evaluated as if the derivative of the control variables with respect to k were a constant) multiplied by the rate at which marginal value declines with k .

Intuitively, in determining the degree of convexity of the marginal value function, the first line on the right-hand side of (40) (within the curly brackets) is the contribution of the convexity of the underlying marginal utility function. The second line is the contribution of the convexity of the marginal accumulation function. The third line is the contribution of the interaction between declining marginal value and declining marginal product. With both the value function and accumulation function A concave, this third “covariance” term will be positive, since marginal value and marginal product will tend to be high together (when k is low) and low together (when k is high).

The central role of dx/dk

In relation to the certainty model, $\partial x(k^*, 0)/\partial k = dx/dk$ describes how the control variables change as the point (k, λ) moves down a saddle path on the phase diagram.

(Remember that $\lambda = V_k(k, 0)$ is the saddle-path in the certainty model, while $V_{kk} \leq 0$ is the slope of the saddle path.) The central role of the way the control variable vector changes as one moves down the saddle path is made clear by the fact that all the key calculations above remain true if restricted to the subset of control variables that change when moving down the saddle path. To see this, note first that at $(k, \omega) = (k^*, 0)$, Eq. (5) indicates V_ω does not depend on dx/dk at all. Second, mechanically, by (7), the convergence rate κ only depends on those control variables that have nonzero elements in dx/dk . Third, if the vector of control variables that matter under certainty x^a is divided into two subvectors x_1^a and x_2^a , with $(dx_2^a)/(dk) = 0$, then with the superscript a suppressed, Eq. (10) can be broken into the two equations

$$F_{kx_1} + F_{x_1x_1} \frac{dx_1}{dk} + F_{x_1x_2} \frac{dx_2}{dk} = F_{kx_1} + F_{x_1x_1} \frac{dx_1}{dk} = 0 \quad (41)$$

$$F_{kx_2} + F_{x_2x_1} \frac{dx_1}{dk} + F_{x_2x_2} \frac{dx_2}{dk} = F_{kx_2} + F_{x_2x_1} \frac{dx_1}{dk} = 0. \quad (42)$$

Thus, for this special case when $dx_2/dk = 0$, Eq. (41) gives the same equation for dx_1/dk as if x_2 had never entered into the Bellman equation. Fourth, the middle expression in Eq. (32) makes it clear that the equation for V_{kk} must hold whether or not the elements of x that have zero entries in dx/dk are included. Fifth, given all these magnitudes, Eq. (40) implies that $V_{kkk}(k^*, 0)$ depends only on the nonzero entries in dx/dk . Sixth, it is clear that Eqs. (8), (22) and (31) for $V_{k\omega}(k^*, 0)$ depend only on nonzero parts of dx/dk . Finally, recall from Eqs. (5) and (31) that calculating V_ω and $V_{k\omega}$ do not depend on knowledge of $\partial x^b / \partial k$ for the control variables in x^b that don't appear in the certainty model.

Continuous-time application: Choosing how much to use a risky technology in general equilibrium

Consider the Ramsey model, augmented to allow for a choice between a risky technology and a safe technology. In addition to being an interesting model in its own right, the calculations for the Ramsey model will also give the appropriate results for the Basic Real Business Cycle Model augmented to allow a risky technology choice *when the parameters of the Basic Real Business Cycle model happen to make labour hours constant along the saddle path* (to a first-order approximation in the neighbourhood of the certain steady state).

The risky technology produces $(1+\mu)$ times as much on average, but also gives a standard deviation to the accumulation equation proportional to output.

The stochastic optimisation problem is then⁵

$$\begin{aligned} \max_{\{c, \pi\}} \mathbf{E}_0 \int_0^{\infty} e^{-\rho t} [u(c)] dt \\ \text{s.t.} \quad dk = [(1 - \pi)f(k) + \pi(1 + \omega\mu)f(k) - \delta k - c - g]dt + \sqrt{\omega}\pi Sf(k)dz, \end{aligned} \quad (43)$$

where c is consumption, π is the fractional employment of the risky technology, k is the capital stock and g is a constant level of government spending.⁶ Defining

$$\vartheta = f(k)\pi, \quad (44)$$

this is the same problem as

$$\begin{aligned} \max_{\{c, \vartheta\}} \mathbf{E}_0 \int_0^{\infty} e^{-\rho t} [u(c)] dt \\ \text{s.t.} \quad dk = [f(k) - \delta k - c - g + \omega\vartheta\mu]dt + \sqrt{\omega}\vartheta Sdz. \end{aligned} \quad (45)$$

The correspondence between (45) and the generic problem above in “General analysis for a one-state-variable optimising model in continuous time” is as follows:

$$\begin{aligned} k: & \quad k \\ x^I: & \quad c \\ x^{II}: & \quad \vartheta \\ U: & \quad u(c) \\ A: & \quad f(k) - \delta k - c - g = A(k, c) \\ v: & \quad 0 \\ \alpha: & \quad \vartheta\mu \\ \sigma: & \quad \vartheta S \end{aligned}$$

⁵ Note that to keep the problem simple, a bad realisation of dz is allowed to cause a reduction in the capital stock greater than depreciation. So the risky technology *must be putting the capital stock at risk*. Furthermore, the risk to the capital stock when the risky technology is used must be systematic, or it would wash out in the aggregate. I think there are possible microfoundations for this. For example, random fashions in preferences for goods within the consumption bundle, and capital that has to be designed for producing particular varieties of consumption goods might do the trick. But I will not try to pursue microfoundations for the social planner’s optimisation problem here.

⁶ There are some subtleties with units. μ is a pure number. As for S , because it multiplies a flow, but must have an effect on a stock, the units for S are the square root of time rather than the more familiar reciprocal of the square root of time for standard deviations in a continuous-time stochastic optimisation problem. But this gives the Sharpe ratio μ/S the usual units: the reciprocal of the square-root of time. The production functions implicitly have labour in them, but the quantity of labour is always equal to 1.

Calculations

The fact that $A(k, c)$ is linear in c , while U is concave in c but does not depend on k greatly simplifies many equations. For example, $A_c = -1$, $A_{cc} = 0$, $A_{kc} = 0$, $A_k = f'(k) - \delta$, $A_{kk} = f''(k)$ and $U_k = 0$. The first-order condition for optimal consumption is $V_k(k) = u'(c(k))$. Therefore,

$$\frac{dc}{dk} = \frac{V_{kk}(k)}{u''(c)},$$

or in the neighbourhood of the certainty steady state,

$$\frac{dc}{dk} = \frac{V_{kk}(k^*)}{u''(c^*)}. \quad (46)$$

Substituting these functional forms into Eq. (34) and solving indicates that at $(k, \omega) = (k^*, 0)$, where $c = c^* = f(k^*) - \delta k^* - g$ and $f'(k^*) - \delta = \rho$,

$$V_{kk} = \frac{u''}{2} \left[\rho + \sqrt{\rho^2 + 4 \frac{u' f''}{u''}} \right], \quad (47)$$

so

$$\frac{dc}{dk} = \frac{\rho + \sqrt{\rho^2 + 4 \frac{u' f''}{u''}}}{2} = \rho + \kappa, \quad (48)$$

since by (7) the convergence rate is

$$\kappa = - \left[f' - \delta - \frac{dc}{dk} \right] = \rho - \frac{V_{kk}(k^*)}{u''(c^*)} = \frac{-\rho + \sqrt{\rho^2 + 4 \frac{u' f''}{u''}}}{2}. \quad (49)$$

Substituting into Eq. (40) is a pleasure because of the number of terms that are zero. The result is

$$V_{kkk} = \frac{u''' \left[\frac{dc}{dk} \right]^3 + V_k f''' + 3V_{kk} f''}{\rho + 3\kappa}. \quad (50)$$

The first-order condition for ϑ is

$$V_k(k, \omega)\mu + V_{kk}(k, \omega)\vartheta S^2 = 0. \quad (51)$$

Solving (51) for ϑ ,

$$\vartheta = \frac{-V_k(k, \omega) \mu}{V_{kk}(k, \omega) S^2}. \quad (52)$$

Note that in (52), ϑ will have the well-defined limit $(-V_k(k, 0)/V_{kk}(k, 0))(\mu/S^2)$ as $\omega \rightarrow 0$. In the limit as $\omega \rightarrow 0$, the “risky” technology is in use, but becomes identical to the safe technology so that the certain steady state is indistinguishable from the certain steady state of a model with only the safe technology.

Following (5) and using (52), the effect of the option of using the risky technology on *welfare* with ω going from zero to one is approximately given by

$$\begin{aligned}
 V_{\omega}(k^*, 0) &= \rho^{-1} \left[V_k(k^*, 0) \vartheta \mu + V_{kk}(k^*, 0) \frac{\vartheta^2 S^2}{2} \right] \\
 &= \rho^{-1} \left[V_k \left(\frac{-V_k}{V_{kk}} \frac{\mu}{S^2} \right) \mu + V_{kk} \frac{\left(\frac{-V_k}{V_{kk}} \frac{\mu}{S^2} \right)^2 S^2}{2} \right] \\
 &= \frac{-V_k^2}{V_{kk}} \frac{\mu^2}{2\rho S^2}.
 \end{aligned} \tag{53}$$

Both (52) and (53) indicate the importance of the absolute risk tolerance of the value function. Its value at $(k, \omega) = (k^*, 0)$ can be calculated as follows:

$$-\frac{V_k}{V_{kk}} = \left(\frac{-u'}{u''} \right) \frac{2}{\rho + \sqrt{\rho^2 + 4 \frac{u'f''}{u''}}} = \left(\frac{-u'}{c^* u''} \right) \frac{2c^*}{\rho + \sqrt{\rho^2 + 4 \frac{u'f''}{u''}}}. \tag{54}$$

In words, the absolute risk tolerance of the social planner's value function is equal to the *relative* risk tolerance of the underlying utility function over consumption $\gamma^* = -c^* u''/u'$, times steady-state consumption, divided by the general equilibrium marginal propensity to consume $(\rho + \sqrt{\rho^2 + 4((u'f'')/u'')})/2$. Note that the units are right: the social planner's risk tolerance is a stock, since consumption is a flow and the marginal propensity to consume is a rate. The optimal use of the risky technology is

$$\vartheta = \left(\frac{\mu}{\gamma^* S^2} \right) \frac{2c^*}{\rho + \sqrt{\rho^2 + 4 \frac{u'f''}{u''}}}. \tag{55}$$

or

$$\pi = \frac{\vartheta}{f} = \frac{\vartheta}{y^*} = \left(\frac{\mu}{\gamma^* S^2} \right) \frac{2 \frac{c^*}{y^*}}{\rho + \sqrt{\rho^2 + 4 \frac{u'f''}{u''}}}, \tag{56}$$

where y^* for steady-state output is a natural way to interpret $f(k^*)$.⁷

⁷ The units are tricky here, but with μ a pure number and S in units of the square root of time, everything works out.

In order to interpret the overall value of the option to use the risky technology, the natural comparison is to a 1 per cent increase in consumption forever, which is $\approx 0.01(c^*u'(c^*))(\rho)$. Using (53),

$$V_\omega = \frac{-V_k^2 \mu^2}{V_{kk} 2\rho S^2} = \frac{\left(\frac{\mu}{S}\right)^2}{\gamma \left(\rho + \sqrt{\rho^2 + 4 \frac{u'f''}{u''}}\right)} \left[\frac{c^* u'}{\rho}\right]. \quad (57)$$

In words, the underlying risk tolerance of the utility function, times the squared Sharpe ratio, divided by twice the general equilibrium marginal propensity to consume, gives the benefit of the option of using the risky technology in units of a permanent proportional increase in consumption. For comparison, in the corresponding decision-theory setting, with a constant distribution of returns, the utility discount rate ρ would appear in place of the general-equilibrium marginal propensity to consume. Therefore, the general equilibrium curvature arising from f'' , which makes the general equilibrium marginal propensity to consume greater than ρ , reduces the benefit.

Following (22), and using (52), the effect of the option of using the risky technology on the *marginal value of k* with ω going from zero to one is approximately

$$V_{k\omega}(k^*, 0) = (\rho + \kappa)^{-1} \left[V_{kk} \vartheta \mu + V_{kkk} \frac{\vartheta^2 S^2}{2} \right] = \frac{V_k}{\rho + \kappa} \frac{\mu^2}{2S^2} \left[\frac{V_{kkk} V_k}{V_{kk}^2} - 2 \right], \quad (58)$$

since neither $v(=0)$ nor $\alpha(=\vartheta\mu)$ nor $\sigma(=\vartheta S)$ is a function of the state variable k . The effect on the marginal value of k is easiest to interpret when expressed in proportion to the initial marginal value of k . Substituting in from the equations above (including the implication from (47) and (48) that $V_{kk} = (\rho + \kappa)u''$) yields

$$\frac{V_{k\omega}}{V_k} = \frac{\left(\frac{\mu}{S}\right)^2}{2(\rho + \kappa)} \left[\frac{\left(\frac{-c^* u'''}{u''}\right)(\rho + \kappa)^3 + \frac{(c^*)^2 f'''}{\gamma} - 3c^* (\rho + \kappa) f''}{\gamma^* (\rho + 3\kappa)(\rho + \kappa)^2} - 2 \right], \quad (59)$$

where $\gamma^* = -c^* u'''/u''$ as above.

Further interpretation in terms of economic parameters

It is possible to home in on the economics more by studying u''' , f'' and f''' more closely. The ratio $-c^* u'''/u''$ is the relative prudence of the underlying utility function.⁸ It can also be expressed in terms of the derivative of γ with respect

⁸ See Kimball (1990).

to c at c^* :

$$\frac{d \ln(\gamma(c))}{d \ln c} \Big|_{c=c^*} = \frac{c^* \gamma'(c^*)}{\gamma^*} = \frac{d}{d \ln c} [\ln c + \ln(-u''(c)) - \ln(u'(c))] = \left[1 + \frac{c^* u'''}{u''} - \frac{c^* u''}{u'} \right].$$

Therefore,

$$\frac{-c^* u'''}{u''} = \gamma^* + 1 - \frac{c^* \gamma'(c^*)}{\gamma^*}. \quad (60)$$

To gain a deeper understanding of derivatives of the production function f , start by denoting capital's share as θ . Then

$$\theta(k) = \frac{kf'(k)}{f(k)}. \quad (61)$$

Using R for the rental rate, W for the real wage and $n = 1$ for the quantity of labour,

$$\frac{\theta}{1-\theta} = \frac{Rk}{Wn}.$$

The elasticity of capital-labour substitution ς is measured as minus the elasticity of k/n with respect to R/W , so $-1/\varsigma$ is the elasticity of R/W with respect to k/n . That makes the elasticity of Rk/Wn with respect to k/n equal to $1-(1/\varsigma)$. Thus, since $n = 1$ makes $k/n = k$,

$$1 - \frac{1}{\varsigma} = \frac{d}{d \ln k} \ln \left(\frac{\theta(k)}{1-\theta(k)} \right) = \frac{k\theta'(k)}{\theta(k)} - \left(\frac{-k\theta'(k)}{1-\theta(k)} \right) = \frac{k\theta'}{\theta(1-\theta)}.$$

Thus,

$$\begin{aligned} (1-\theta) \left(1 - \frac{1}{\varsigma} \right) &= \frac{k\theta'}{\theta} = \frac{d}{d \ln k} \ln(\theta) = \frac{d}{d \ln k} [\ln(k) + \ln f'(k) - \ln f(k)] \\ &= 1 + \frac{kf''}{f'} - \frac{kf'}{f}. \end{aligned} \quad (62)$$

Solving for $-kf''/f'$ by using the fact that $\theta = kf'/f$,

$$\frac{-kf''}{f'} = \frac{1-\theta}{\varsigma}, \quad (63)$$

—or solving for $-f''$:

$$-f'' = \frac{(1-\theta)f'}{k\varsigma}. \quad (64)$$

Thus,

$$\begin{aligned}\frac{-kf'''}{f''} &= -\frac{d \ln(-f''(k))}{d \ln k} = \frac{d}{d \ln k} [\ln(k) + \ln(\zeta) - \ln(1-\theta) - \ln(f')] \\ &= 1 + \frac{k\zeta'}{\zeta} + \frac{k\theta'}{1-\theta} + \left(\frac{-kf''}{f'}\right) \\ &= 1 + \frac{k\zeta'}{\zeta} + \theta\left(1 - \frac{1}{\zeta}\right) + \frac{1-\theta}{\zeta} = 1 + \theta + \frac{1-2\theta}{\zeta} + \frac{k\zeta'}{\zeta},\end{aligned}\quad (65)$$

and by combining (64) and (65),

$$f''' = \left(\frac{(1-\theta)f'}{k^2\zeta}\right) \left[1 + \theta + \frac{1-2\theta}{\zeta} + \frac{k\zeta'}{\zeta}\right]. \quad (66)$$

Substituting into Eqs. (48) and (49), for example:

$$\frac{dc}{dk} = \rho + \kappa = \frac{\rho + \sqrt{\rho^2 + 4 \frac{c^*(\rho+\delta)(1-\theta^*)}{\gamma^* k^* \zeta^*}}}{2}. \quad (67)$$

$$\kappa = \frac{-\rho + \sqrt{\rho^2 + 4 \frac{c^*(\rho+\delta)(1-\theta^*)}{\gamma^* k^* \zeta^*}}}{2}. \quad (68)$$

And substituting into Eq. (59),

$$\frac{V_{k\omega}}{V_k} = \frac{\left(\frac{\mu}{S}\right)^2}{2(\rho + \kappa)} \cdot \left[\frac{\begin{aligned} &\left(\gamma^* + 1 + \frac{c^* \gamma'(c^*)}{\gamma^*}\right)(\rho + \kappa)^3 \\ &+ \left(\frac{(c^*)^2(1-\theta^*)(\rho + \delta)}{\gamma^* k^* \zeta^*}\right) \left[1 + \theta + \frac{1-2\theta}{\zeta^*} + \frac{k^* \zeta'(k^*)}{\zeta^*}\right] \\ &+ \frac{3c^*(\rho + \kappa)(1-\theta)(\rho + \delta)}{k^* \zeta^*} \end{aligned}}{\gamma^*(\rho + 3\kappa)(\rho + \kappa)^2} - 2 \right]. \quad (69)$$

Continuous-time application: When the only technology available is risky

There are economies of scope in examining several models that all reduce to the same model under certainty. One model that has an especially low marginal cost given the analysis above is one in which, instead of having a choice between a risky and a safe

technology, the economy is stuck with only a risky technology. To model this, modify (43) by setting $\pi = 1$ and $\mu = 0$:

$$\begin{aligned} \max_{\{c\}} \mathbf{E}_0 \int_0^{\infty} e^{-\rho t} [u(c)] dt \\ \text{s.t.} \quad dk = [f(k) - \delta k - c - g]dt + \sqrt{\omega} Sf(k)dz. \end{aligned} \quad (70)$$

The only correspondences that differ from the previous section are as follows:

$$\begin{aligned} x: & c \\ v: & 0 \\ \alpha: & 0 \\ \sigma: & Sf(k) = Sy \end{aligned}$$

Substituting these into (5) and (8) and using the expressions for the certainty derivatives of V from the previous section yields

$$V_{\omega} = \frac{V_{kk} S^2 (y^*)^2}{2c^* V_k} \left[\frac{c^* u'}{\rho} \right] = - \frac{\gamma^* (\rho + \kappa) S^2}{2 \left(\frac{c^*}{y^*} \right)^2} \left[\frac{c^* u'}{\rho} \right] \quad (71)$$

and

$$\begin{aligned} \frac{V_{k\omega}}{V_k} &= \frac{\frac{V_{kk}}{V_k} S^2 f f' + \frac{V_{kkk}}{V_k} \frac{S^2 f^2}{2}}{\rho + \kappa} \\ &= -\gamma^* \left(\frac{y^*}{c^*} \right) (\rho + \delta) S^2 + \left(\frac{S^2 (y^*)^2}{2(\rho + \kappa)(\rho + 3\kappa)} \right) \cdot \left\{ \gamma^* \left(\gamma^* + 1 - \frac{c^* \gamma'(c^*)}{\gamma^*} \right) (\rho + \kappa)^3 \right. \\ &\quad \left. + \frac{(1 - \theta)(\rho + \delta)}{(k^*)^2 \zeta^*} \left[1 + \theta^* + \frac{1 - 2\theta^*}{\zeta^*} + \frac{k^* \zeta'(k^*)}{\zeta} \right] + \frac{3\gamma^* (\rho + \kappa)(\rho + \delta)(1 - \theta^*)}{c^* k^* \zeta^*} \right\}. \end{aligned} \quad (72)$$

Discrete time

Many economists are more comfortable with discrete-time models than with continuous-time models. Let me discuss discrete-time models in two ways. First, I will show how a discrete-time model limits into the continuous-time model as the length of a period goes to zero. Second, I will show how to find equations analogous to the key equations above for a discrete-time model with a fixed period length.

The continuous-time model as the limit of a discrete-time model as the length of the period becomes shorter

Let ϵ be a mean-zero risk with a finite support and a variance of 1. Let h be the length of a period. In order for a discrete-time model to limit into the continuous-time model above, it is important to multiply flows by h and to multiply stochastic shocks by $\sqrt{h}$. The Bellman equation for the appropriate discrete-time model is:

$$V(k, \omega; h) = \max_x hU(k, x) + h\omega v(k, x) + e^{-\rho h} EV\left(k + hA(k, x) + h\omega\alpha(k, x) + \sqrt{h\omega}\sigma(k, x)\epsilon, \omega; h\right). \quad (73)$$

Now, subtract $V(k, \omega; h)$ from both sides of the equation, divide by h and take the limit as $h \rightarrow 0$. Assuming uniform convergence for all needed derivatives, the limit can be interchanged with the maximisation operator. Thus,

$$0 = \max_x \left\{ U(k, x) + \omega v(k, x) + \lim_{h \rightarrow 0} \left[\frac{e^{-\rho h} EV\left(k + hA(k, x) + h\omega\alpha(k, x) + \sqrt{h\omega}\sigma(k, x)\epsilon, \omega; h\right) - V(k, \omega; h)}{h} \right] \right\}. \quad (74)$$

Using L'Hopital's rule—that is, taking the derivative of both numerator and denominator with respect to h and then taking the limit as $h \rightarrow 0$,

$$0 = \max_x \left\{ -\rho V(k, \omega; 0) + U(k, x) + \omega v(k, x) + V_k(k, \omega; 0)[A(k, x) + \omega\alpha(k, x)] + \lim_{h \rightarrow 0} E \left[\frac{1}{2} \sqrt{\frac{\omega}{h}} \sigma(k, x)\epsilon \cdot V_k(k + hA(k, x) + h\omega\alpha(k, x) + \sqrt{h\omega}\sigma(k, x)\epsilon, \omega; h) \right] \right\}. \quad (75)$$

There is a cancellation between the two terms $V_h(k, \omega; 0) - V_h(k, \omega, 0)$ on the way to the result shown. The hard part is the piece still involving a limit. Give this piece the

label Ξ . Then

$$\begin{aligned}
 \Xi &= \lim_{h \rightarrow 0} E \left[\frac{1}{2} \sqrt{\frac{\omega}{h}} \sigma(k, x) \epsilon \cdot V_k \left(k + hA(k, x) + h\omega\alpha(k, x) + \sqrt{h\omega} \sigma(k, x) \epsilon, \omega; h \right) \right] \\
 &= \lim_{h \rightarrow 0} E \left[\frac{E \frac{1}{2} \sqrt{\omega h} \sigma(k, x) \epsilon \cdot V_k \left(k + hA(k, x) + h\omega\alpha(k, x) + \sqrt{h\omega} \sigma(k, x) \epsilon, \omega; h \right)}{h} \right] \\
 &= E \left[\frac{\omega \sigma^2(k, x) \epsilon^2 V_{kk}(k, \omega; 0)}{4} \right] \\
 &\quad + \lim_{h \rightarrow 0} E \left[\frac{1}{4} \sqrt{\frac{\omega}{h}} \sigma(k, x) \epsilon \cdot V_k \left(k + hA(k, x) + h\omega\alpha(k, x) + \sqrt{h\omega} \sigma(k, x) \epsilon, \omega; h \right) \right] \\
 &= E \left[\frac{\omega \sigma^2(k, x) \epsilon^2 V_{kk}(k, \omega; 0)}{4} \right] + \frac{1}{2} \Xi. \tag{76}
 \end{aligned}$$

Many terms vanished in the limit to yield the result in (76) because they were multiplied by $\sqrt{h}$. But the last term remained because it had $\sqrt{\omega/h}$. That term does not look easy to evaluate, but fortunately, it is simply half of the original expression for Ξ , and it is straightforward to solve (76) for Ξ :

$$\Xi = E \left[\frac{\omega \sigma^2(k, x) \epsilon^2 V_{kk}(k, \omega; 0)}{2} \right] = \omega V_{kk}(k, \omega; 0) \frac{\sigma^2(k, x)}{2}. \tag{77}$$

Substituting from (77) into (75) and rearranging yields Eq. (2), with added notation that makes the fact $h=0$ is zero in continuous time explicit:

$$\begin{aligned}
 \rho V(k, \omega; 0) &= \max_x U(k, x) + V_k(k, \omega; 0) A(k, x) \\
 &\quad + \omega \left[v(k, x) + V_k(k, \omega; 0) \alpha(k, x) + V_{kk}(k, \omega; 0) \frac{\sigma^2(k, x)}{2} \right]. \tag{2'}
 \end{aligned}$$

Equations for the local effects of uncertainty in the discrete-time model

To address the discrete-time model itself, leave h fixed. With the dependence of V on h suppressed, Eq. (73) is

$$\begin{aligned}
 V(k, \omega) &= \max_x hU(k, x) + h\omega v(k, x) \\
 &\quad + e^{-\rho h} EV \left(k + hA(k, x) + h\omega\alpha(k, x) + \sqrt{h\omega} \sigma(k, x) \epsilon, \omega \right).
 \end{aligned}$$

Taking the derivative of both sides with respect to ω yields

$$\begin{aligned} V_\omega(k, \omega) = & h v(k, x) + e^{-\rho h} \left\{ E h \alpha(k, x) V_k \left(k + h A(k, x) + h \omega \alpha(k, x) + \sqrt{h \omega} \sigma(k, x) \epsilon, \omega \right) \right. \\ & + E \frac{1}{2} \sqrt{\frac{h}{\omega}} \sigma(k, x) \epsilon V_k \left(k + h A(k, x) + h \omega \alpha(k, x) + \sqrt{h \omega} \sigma(k, x) \epsilon, \omega \right) \\ & \left. + E V_\omega \left(k + h A(k, x) + h \omega \alpha(k, x) + \sqrt{h \omega} \sigma(k, x) \epsilon, \omega \right) \right\}. \end{aligned} \quad (78)$$

In Eq. (78), x represents the optimising value of x , and is a function of k and ω . The envelope theorem makes it unnecessary to consider changes in x induced by changes in ω when deriving (78), but *induced changes in x need to be accounted for in any further derivatives*. In order to take induced changes in x into account, given any function that depends on k and x , I will use the notation

$$\frac{d}{dk} = \frac{\partial}{\partial k} + \frac{\partial x}{\partial k} \frac{\partial}{\partial x}.$$

Then

$$\begin{aligned} V_{k\omega}(k, \omega) = & h \frac{dv}{dk} + e^{-\rho h} \left\{ h \frac{d\alpha}{dk} E V_k \left(k + h A + h \omega \alpha + \sqrt{h \omega} \sigma \epsilon, \omega \right) \right. \\ & + h \alpha E \left[\left\{ 1 + h \frac{dA}{dk} + h \omega \frac{d\alpha}{dk} + \sqrt{h \omega} \frac{d\sigma}{dk} \epsilon \right\} V_{kk} \left(k + h A + h \omega \alpha + \sqrt{h \omega} \sigma \epsilon, \omega \right) \right] \\ & + \frac{1}{2} \sqrt{\frac{h}{\omega}} \frac{d\sigma}{dk} E \epsilon V_k \left(k + h A + h \omega \alpha + \sqrt{h \omega} \sigma \epsilon, \omega \right) \\ & + \frac{1}{2} \sqrt{\frac{h}{\omega}} \sigma E \left[\epsilon \left\{ 1 + h \frac{dA}{dk} + h \omega \frac{d\alpha}{dk} + \sqrt{h \omega} \frac{d\sigma}{dk} \epsilon \right\} \right. \\ & \quad \left. V_{kk} \left(k + h A + h \omega \alpha + \sqrt{h \omega} \sigma \epsilon, \omega \right) \right] \\ & \left. + E \left[\left\{ 1 + h \frac{dA}{dk} + h \omega \frac{d\alpha}{dk} + \sqrt{h \omega} \frac{d\sigma}{dk} \epsilon \right\} V_{k\omega} \left(k + h A + h \omega \alpha + \sqrt{h \omega} \sigma \epsilon, \omega \right) \right] \right\}. \end{aligned} \quad (79)$$

It is easier to take the derivative with respect to ω first, and only then with respect to k in calculating $V_{k\omega} = V_{\omega k}$, because using the envelope theorem to avoid having to know $\partial x / \partial \omega$ is more valuable than using it to avoid having to know $\partial x / \partial k$. The derivative of x with respect to k at $(k, \omega) = (k^*, 0)$ can be calculated from the certainty model; the derivative with respect to ω cannot.

Evaluating $V_\omega(k^, 0)$ in discrete time*

The discrete-time equations corresponding to (5) and (8) in the “General analysis for a one-state-variable optimising model in continuous time” above come from the

limits of (78) and (79) as $\omega \rightarrow 0$. That limit for (78) yields

$$V_\omega(k, 0) = hv + e^{-\rho h} \{h\alpha V_k(k + hA, 0) + \Phi + V_\omega(k + hA, 0)\}, \quad (80)$$

where, using the same trick as in (76),

$$\begin{aligned} \Phi &= \lim_{\omega \rightarrow 0} \left\{ E \frac{1}{2} \sqrt{\frac{h}{\omega}} \sigma \epsilon V_k \left(k + hA + h\omega\alpha + \sqrt{h\omega} \sigma \epsilon, \omega \right) \right\} \\ &= \lim_{\omega \rightarrow 0} \left\{ \frac{E \frac{1}{2} \sqrt{h\omega} \sigma \epsilon V_k \left(k + hA + h\omega\alpha + \sqrt{h\omega} \sigma \epsilon, \omega \right)}{\omega} \right\} \\ &= \frac{1}{2} \Phi + \lim_{\omega \rightarrow 0} \left\{ E \left[\frac{1}{2} \sqrt{h\omega} \sigma \epsilon \left(h\alpha + \frac{1}{2} \sqrt{\frac{h}{\omega}} \sigma \epsilon \right) V_{kk} \left(k + hA + h\omega\alpha + \sqrt{h\omega} \sigma \epsilon, \omega \right) \right] \right\} \\ &= \frac{1}{2} \Phi + E \frac{h}{4} \sigma^2 \epsilon^2 V_{kk}(k + hA, 0). \end{aligned} \quad (81)$$

Solving for Φ and evaluating the expectation knowing that ϵ has mean zero and variance 1, while other inputs are nonstochastic, yields

$$\Phi = \frac{h\sigma^2}{2} V_{kk}(k + hA, 0). \quad (82)$$

At the steady state, $A = 0$. Setting $k = k^*$ and $A = 0$, substituting for Φ in (80) using (82), and solving for V_ω yields

$$V_\omega(k^*, 0) = \left(\frac{h}{1 - e^{-\rho h}} \right) \left[v + \alpha V_k(k^*, 0) + \frac{\sigma^2}{2} V_{kk}(k^*, 0) \right]. \quad (83)$$

It is easy to use L'Hopital's rule to see that Eq. (5) is the continuous-time limit of Eq. (83) as $h \rightarrow 0$. But the standard discrete-time equation arises instead from setting $h = 1$ and using the notation β for the discount factor:

$$\beta = e^{-\rho h} = e^{-\rho}. \quad (84)$$

That yields

$$V_\omega(k^*, 0) = \frac{v + \alpha V_k(k^*, 0) + \frac{\sigma^2}{2} V_{kk}(k^*, 0)}{1 - \beta}. \quad (85)$$

Evaluating $V_{k\omega}(k^, 0)$ in discrete time*

The limit of Eq. (79) as $\omega \rightarrow 0$ is

$$\begin{aligned} V_{\omega k}(k, 0) &= h \frac{dv}{dk} + e^{-\rho h} \left\{ h \frac{d\alpha}{dk} V_k(k + hA, 0) + h\alpha \left[1 + h \frac{dA}{dk} \right] V_{kk}(k + hA, 0) \right. \\ &\quad \left. + \Psi + \left[1 + h \frac{dA}{dk} \right] V_{\omega k}(k + hA, 0) \right\}, \end{aligned} \quad (86)$$

where

$$\begin{aligned}
 \Psi &= \lim_{\omega \rightarrow 0} \left(\left[\sqrt{h\omega} \frac{d\sigma}{dk} E\epsilon V_k \left(k + hA + h\omega\alpha + \sqrt{h\omega}\sigma\epsilon, \omega \right) \right. \right. \\
 &\quad \left. \left. + \sigma E\epsilon \left\{ 1 + h \frac{dA}{dk} + h\omega \frac{d\alpha}{dk} + \sqrt{h\omega} \frac{d\sigma}{dk} \epsilon \right\} V_{kk} \left(k + hA + h\omega\alpha + \sqrt{h\omega}\sigma\epsilon, \omega \right) \right] / \omega \right) \\
 &= \frac{1}{2} \Psi + \frac{1}{2} \lim_{\omega \rightarrow 0} \left(\sqrt{h\omega} \left[\frac{d\sigma}{dk} E\epsilon \left\{ h\alpha + \frac{1}{2} \sqrt{\frac{h}{\omega}} \sigma\epsilon \right\} V_{kk} \left(k + hA + h\omega\alpha + \sqrt{h\omega}\sigma\epsilon, \omega \right) \right. \right. \\
 &\quad \left. \left. + \sigma E\epsilon \left\{ h \frac{d\alpha}{dk} + \sqrt{\frac{h}{\omega}} \frac{d\sigma}{dk} \epsilon \right\} V_{kk} \left(k + hA + h\omega\alpha + \sqrt{h\omega}\sigma\epsilon, \omega \right) \right. \right. \\
 &\quad \left. \left. + \sigma E\epsilon \left\{ 1 + h \frac{dA}{dk} + h\omega \frac{d\alpha}{dk} + \sqrt{h\omega} \frac{d\sigma}{dk} \epsilon \right\} \left\{ h\alpha + \frac{1}{2} \sqrt{\frac{h}{\omega}} \sigma\epsilon \right\} \right. \right. \\
 &\quad \left. \left. \times V_{kkk} \left(k + hA + h\omega\alpha + \sqrt{h\omega}\sigma\epsilon, \omega \right) \right. \right. \\
 &\quad \left. \left. + \frac{d\sigma}{dk} E\epsilon V_{\omega k} \left(k + hA + h\omega\alpha + \sqrt{h\omega}\sigma\epsilon, \omega \right) \right. \right. \\
 &\quad \left. \left. + \sigma E\epsilon \left\{ 1 + h \frac{dA}{dk} + h\omega \frac{d\alpha}{dk} + \sqrt{h\omega} \frac{d\sigma}{dk} \epsilon \right\} \right. \right. \\
 &\quad \left. \left. \times V_{\omega kk} \left(k + hA + h\omega\alpha + \sqrt{h\omega}\sigma\epsilon, \omega \right) \right] \right) \quad (87)
 \end{aligned}$$

Taking the limits, collecting terms, solving for Ψ , and using the fact that $E\epsilon^2 = 1$ yields

$$\Psi = h\sigma \frac{d\sigma}{dk} V_{kk}(k + hA, 0) + h \left\{ 1 + h \frac{dA}{dk} \right\} \frac{\sigma^2}{2} V_{kkk}(k + hA, 0). \quad (88)$$

Using (88) to substitute in for Ψ in (86), setting $k = k^*$ and $A = 0$, and solving for $V_{\omega k}$ yields the discrete-time counterpart to Eq. (8):

$$\begin{aligned}
 V_{\omega k}(k^*, 0) &= \left(\frac{h}{1 - e^{-\rho h} - h e^{-\rho h} \frac{dA}{dk}} \right) \frac{dv}{dk} \\
 &\quad + \left(\frac{h e^{-\rho h}}{1 - e^{-\rho h} - h e^{-\rho h} \frac{dA}{dk}} \right) \left\{ \frac{d\alpha}{dk} V_k(k^*, 0) + \left[1 + h \frac{dA}{dk} \right] \alpha V_{kk}(k^*, 0) \right. \\
 &\quad \left. + \sigma \frac{d\sigma}{dk} V_{kk}(k^*, 0) + \left[1 + h \frac{dA}{dk} \right] \frac{\sigma^2}{2} V_{kkk}(k^*, 0) \right\}. \quad (89)
 \end{aligned}$$

The algebra behind this equation can be cross-checked by noting that the limit of (89) as $h \rightarrow 0$ is Eq. (8). For the discrete-time equation, set $h = 1$, use the definition of the utility discount factor $\beta = e^{-\rho h} = e^{-\rho}$ and define the discrete-time convergence

factor ζ by

$$\zeta = 1 + h \frac{dA}{dk} = 1 + \frac{dA}{dk}. \quad (90)$$

Then, Eq. (89) yields

$$\begin{aligned} V_{\omega k}(k^*, 0) = & \left(\frac{1}{1 - \beta\zeta} \right) \frac{dv}{dk} + \left(\frac{\beta}{1 - \beta\zeta} \right) \left\{ \frac{d\alpha}{dk} V_k(k^*, 0) + \zeta \alpha V_{kk}(k^*, 0) \right. \\ & \left. + \sigma \frac{d\sigma}{dk} V_{kk}(k^*, 0) + \zeta \frac{\sigma^2}{2} V_{kkk}(k^*, 0) \right\}. \end{aligned} \quad (91)$$

Note that by stability of the steady state, ζ must be less than 1 and greater than -1 . Therefore, $1 - \beta\zeta$ is positive.

Finding k^ , $V_k(k^*, 0) = \lambda^*$ and x^* in the discrete-time certainty model*

In most applications, relevant quantity data actually give values for k^* and x^* . When V is in “utils” that have no other natural metric, it is often convenient to normalise the marginal value of k to one: $V_k(k^*, 0) = \lambda^* = 1$. In many other cases, price data indicate an appropriate steady-state value for the marginal value of k . These values of k^* , λ^* and x^* can then be used to determine some of the parameters of the functions U and A that govern the discrete-time certainty model. Nevertheless, it is entertaining to see how, in principle, one could find k^* , $V_k(k^*, 0) = \lambda^*$ and x^* if one really did know the exact functional forms for U and A , but not the steady-state values of k , λ and x . The solution just below takes its clue from what one would do in analysing a phase diagram.

In addition to the definition $\beta = e^{-\rho h}$, a notational convention that greatly simplifies formulas for the discrete-time model (but obscures a bit the relationship to the continuous-time model) is to write

$$\Gamma(k, x) = k + hA(k, x). \quad (92)$$

Given this notation, the discrete-time convergence factor ζ is

$$\zeta = \Gamma_k(k^*, x^*) + \Gamma_x^T(k^*, x^*) \frac{dx}{dk}. \quad (93)$$

Also, if $h \neq 1$, in the mind’s eye rename the utility function everywhere above $\mathcal{U}$ and define

$$U(k, x) = h\mathcal{U}(k, x). \quad (94)$$

(Those who always set h equal to 1 are spared any mind’s-eye gymnastics.)

After suppressing the ω argument, the Bellman equation in the discrete-time certainty model is

$$V(k) = \max_x U(k, x) + \beta V(\Gamma(k, x)). \quad (95)$$

The first-order condition for an optimal choice of the control variable vector x is

$$U_x(k, x) + \beta \Gamma_x(k, x) V_k(\Gamma(k, x)) = 0. \quad (96)$$

At the steady state, using the notation $\lambda^* = V_k(k^*)$,

$$U_x(k^*, x^*) + \beta \Gamma_x(k^*, x^*) \lambda^* = 0. \quad (97)$$

With full knowledge of the functional forms for U and Γ , including all their parameters, this can be solved for x^* as a function of k^* and λ^* :

$$x^* = x^*(k^*, \lambda^*).$$

Using the envelope theorem, the first derivative of Bellman equation (94) with respect to k is

$$V_k(k) = U_k(k, x) + \beta \Gamma_k(k, x) V_k(\Gamma(k, x)). \quad (98)$$

This is the Euler equation. At the steady state, the Euler equation becomes

$$\lambda^* = U_k(k^*, x^*(k^*, \lambda^*)) + \beta \lambda^* \Gamma_k(k^*, x^*(k^*, \lambda^*)). \quad (99)$$

(This is effectively the $\Delta\lambda=0$ locus.) Finally, by the definition of the steady state,

$$\Gamma(k^*, x^*(k^*, \lambda^*)) = k^*. \quad (100)$$

(This is effectively the $\Delta k=0$ locus.) Thus, Eqs. (97), (98), (99) and (100) will ordinarily be sufficient to determine the three magnitudes k^* , λ^* and x^* .

Finding dx/dk , $V_{kk}(k^)$ and the convergence factor ζ in the discrete-time certainty model*

Taking the derivative of the first-order condition (96) with respect to k and setting $k = k^*$ yields

$$U_{kx} + \beta \lambda^* \Gamma_{kx} + [U_{xx} + \beta \lambda^* \Gamma_{xx}] \frac{dx}{dk} + \beta \Gamma_x^T \left[\Gamma_k + \Gamma_x \frac{dx}{dk} \right] V_{kk} = 0. \quad (101)$$

Recognising $\Gamma_k + \Gamma_x^T(dx)/(dk)$ as ζ , Eq. (101) can be solved for dx/dk thus:

$$\frac{dx}{dk} = - [U_{xx} + \beta \lambda^* \Gamma_{xx}]^{-1} [U_{kx} + \beta \lambda^* \Gamma_{kx}] - [U_{xx} + \beta \lambda^* \Gamma_{xx}]^{-1} \Gamma_x \beta \zeta V_{kk}. \quad (102)$$

Next, take the derivative of the Euler equations (98) with respect to k and set $k = k^*$ to get

$$\begin{aligned} V_{kk} &= U_{kk} + \beta\lambda^* \Gamma_{kk} + \beta\zeta \Gamma_k V_{kk} + [U_{kx} + \beta\lambda^* \Gamma_{kx}] \frac{dx}{dk} \\ &= U_{kk} + \beta\lambda^* \Gamma_{kk} + \beta\zeta \Gamma_k V_{kk} - [U_{kx} + \beta\lambda^* \Gamma_{kx}] [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} [U_{kx} + \beta\lambda^* \Gamma_{kx}] \\ &\quad - [U_{kx} + \beta\lambda^* \Gamma_{kx}] [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} \Gamma_x \beta \zeta V_{kk}. \end{aligned} \quad (103)$$

Solving for $V_{kk}(k^*)$ yields

$$V_{kk} = \frac{U_{kk} + \beta\lambda^* \Gamma_{kk} - [U_{kx} + \beta\lambda^* \Gamma_{kx}] [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} [U_{kx} + \beta\lambda^* \Gamma_{kx}]}{1 - \beta\zeta \Gamma_k + [U_{kx} + \beta\lambda^* \Gamma_{kx}] [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} \Gamma_x \beta \zeta}. \quad (104)$$

In order to determine the value of ζ , expand out its definition using both (102) and (104):

$$\begin{aligned} \zeta &= \Gamma_k + \Gamma_x^T \frac{dx}{dk} \\ &= \Gamma_k - \Gamma_x^T [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} [U_{kx} + \beta\lambda^* \Gamma_{kx}] - \Gamma_x^T [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} \\ &\quad \times \Gamma_x \beta \zeta \left(\frac{U_{kk} + \beta\lambda^* \Gamma_{kk} - [U_{kx} + \beta\lambda^* \Gamma_{kx}] [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} [U_{kx} + \beta\lambda^* \Gamma_{kx}]}{1 - \beta\zeta \Gamma_k + [U_{kx} + \beta\lambda^* \Gamma_{kx}] [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} \Gamma_x \beta \zeta} \right). \end{aligned} \quad (105)$$

Clearing fractions, this is a quadratic equation in ζ :

$$\begin{aligned} 0 &= \left\{ \beta \Gamma_k - [U_{kx} + \beta\lambda^* \Gamma_{kx}] [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} \Gamma_x \beta \right\} \zeta^2 - \left\{ 1 + \beta \left(\Gamma_x^T [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} \Gamma_x \right) \right. \\ &\quad \times \left. \left(U_{kk} + \beta\lambda^* \Gamma_{kk} - [U_{kx} + \beta\lambda^* \Gamma_{kx}] [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} [U_{kx} + \beta\lambda^* \Gamma_{kx}] \right) \right\} \zeta \\ &\quad \Gamma_k - \Gamma_x^T [U_{xx} + \beta\lambda^* \Gamma_{xx}]^{-1} [U_{kx} + \beta\lambda^* \Gamma_{kx}]. \end{aligned} \quad (106)$$

The coefficient of the linear term is unambiguously negative. The other two terms (the quadratic and constant terms) cancel each other out at both $\zeta = -\beta^{-0.5}$ and $\zeta = \beta^{-0.5}$. Thus, the value of the quadratic function is positive at $\zeta = -\beta^{-0.5}$ and negative at $\zeta = \beta^{-0.5}$. Because it is a quadratic function, that means it must cross zero exactly one time between $\zeta = -\beta^{-0.5}$ and $\zeta = \beta^{-0.5}$. So there is exactly one real root between $\zeta = -\beta^{-0.5}$ and $\zeta = \beta^{-0.5}$. This is the root of interest. This value for ζ can be substituted into (104) to find V_{kk} . Substituting the values of both ζ and V_{kk} into (102) gives dx/dk .

One possibly distressing aspect of Eq. (104) is that, structurally, it does not guarantee that $\zeta \in (-1, 1)$ when $\beta < 1$. Indeed, as a (continuous-time) example in section 5.5.2 of Kimball⁹ indicates, even continuous-time concave dynamic control problems sometimes have only an unstable steady state. So the assumption that the

⁹ Kimball (2004).

steady state anchoring the calculations is stable is a substantive assumption with some meat to it.

Finding V_{kkk} in the discrete-time certainty model

In discrete time, the most convenient definition for the prevalue function is

$$F(k, x) = U(k, x) + \beta V(\Gamma(k, x)). \quad (107)$$

Then

$$V(k) = \max_x F(k, x). \quad (108)$$

Thinking of the prevalue function abstractly in this way as a function of k and x , and taking derivatives of (108) without at first using the envelope theorem,

$$V_k = F_k + F_x^T \left[\frac{dx}{dk} \right], \quad (109)$$

$$V_{kk} = F_{kk} + 2F_{kx}^T \left[\frac{dx}{dk} \right] + \left(\left[\frac{dx}{dk} \right]^T F_{xx} \left[\frac{dx}{dk} \right] \right) + F_x^T \left[\frac{d^2x}{dk^2} \right] \quad (110)$$

$$\begin{aligned} V_{kkk} &= F_{kkk} + 3F_{kkx}^T \left[\frac{dx}{dk} \right] + \left(3 \left[\frac{dx}{dk} \right]^T F_{kxx} \left[\frac{dx}{dk} \right] \right) + \left(\sum_i \frac{dx_i}{dk} \cdot \left[\frac{dx}{dk} \right]^T F_{xxx} \left[\frac{dx}{dk} \right] \right) \\ &\quad + 3F_{kx}^T \left[\frac{d^2x}{dk^2} \right] + 3 \left(\left[\frac{dx}{dk} \right]^T F_{xx} \left[\frac{d^2x}{dk^2} \right] \right) + F_x^T \left[\frac{d^3x}{dk^3} \right] \\ &= F_{kkk} + 3F_{kkx}^T \left[\frac{dx}{dk} \right] + \left(3 \left[\frac{dx}{dk} \right]^T F_{kxx} \left[\frac{dx}{dk} \right] \right) + \left(\sum_i \frac{dx_i}{dk} \cdot \left[\frac{dx}{dk} \right]^T F_{xxx} \left[\frac{dx}{dk} \right] \right) \\ &\quad + 3 \left(F_{kx}^T + \left[\frac{dx}{dk} \right]^T F_{xx} \right) \left[\frac{d^2x}{dk^2} \right] + F_x^T \left[\frac{d^3x}{dk^3} \right]. \end{aligned} \quad (111)$$

The first-order condition and its first two derivatives with respect to k are

$$0 = F_x, \quad (112)$$

$$0 = F_{kx} + F_{xx} \left[\frac{dx}{dk} \right], \quad (113)$$

$$0 = F_{kkx} + 2F_{kxx} \left[\frac{dx}{dk} \right] + \left[\frac{dx}{dk} \right]^T F_{xxx} \left[\frac{dx}{dk} \right] + F_{xx} \left[\frac{d^2x}{dk^2} \right]. \quad (114)$$

Eq. (114) would be helpful if one needed to calculate $[d^2x/dk^2]$ —as one would, for example, in order to find the fourth derivative V_{kkkk} . But for the first three derivatives of V , there is no need to find $[d^2x/dk^2]$, and (114) is not needed. In particular,

Eqs. (112) and (113) simplify (109), (110) and (111) as follows:

$$V_k = F_k, \quad (115)$$

$$\begin{aligned} V_{kk} &= F_{kk} + 2F_{kx}^T \left[\frac{dx}{dk} \right] + \left(\left[\frac{dx}{dk} \right]^T F_{xx} \left[\frac{dx}{dk} \right] \right) \\ &= F_{kk} + F_{kx}^T \left[\frac{dx}{dk} \right] + \left(F_{kx}^T + \left[\frac{dx}{dk} \right]^T F_{xx} \right) \left[\frac{dx}{dk} \right] = F_{kk} + F_{kx}^T \left[\frac{dx}{dk} \right], \end{aligned} \quad (116)$$

$$V_{kkk} = F_{kkk} + 3F_{kkx}^T \left[\frac{dx}{dk} \right] + \left(3 \left[\frac{dx}{dk} \right]^T F_{kxx} \left[\frac{dx}{dk} \right] \right) + \left(\sum_i \frac{dx_i}{dk} \cdot \left[\frac{dx}{dk} \right]^T F_{xxx} \left[\frac{dx}{dk} \right] \right). \quad (117)$$

Using the definition of the discrete-time value function $F(k, x)$ in (107), its derivatives up to the third order are:

$$F_k = U_k + \beta \Gamma_k V_k(\Gamma), \quad (118)$$

$$F_x = U_x + \beta \Gamma_x V_k(\Gamma), \quad (119)$$

$$F_{kk} = U_{kk} + \beta \Gamma_{kk} V_k(\Gamma) + \beta \Gamma_k^2 V_{kk}(\Gamma), \quad (120)$$

$$F_{kx} = U_{kx} + \beta \Gamma_{kx} V_k(\Gamma) + \beta \Gamma_k \Gamma_x V_{kk}(\Gamma), \quad (121)$$

$$F_{xx} = U_{xx} + \beta \Gamma_{xx} V_k(\Gamma) + \beta \Gamma_x \Gamma_x^T V_{kk}(\Gamma), \quad (122)$$

$$F_{kkk} = U_{kkk} + \beta \Gamma_{kkk} V_k(\Gamma) + 3\beta \Gamma_{kk} \Gamma_k V_{kk}(\Gamma) + \beta \Gamma_k^3 V_{kkk}(\Gamma), \quad (123)$$

$$F_{kkx} = U_{kkx} + \beta \Gamma_{kkx} V_k(\Gamma) + 2\beta \Gamma_{kx} \Gamma_k V_{kk}(\Gamma) + \beta \Gamma_{kk} \Gamma_x V_{kk}(\Gamma) + \beta \Gamma_k^2 \Gamma_x V_{kkk}(\Gamma), \quad (124)$$

$$\begin{aligned} F_{kxx} &= U_{kxx} + \beta \Gamma_{kxx} V_k(\Gamma) + \beta \Gamma_{xx} \Gamma_k V_{kk}(\Gamma) + \beta [\Gamma_{kx} \Gamma_x^T + \Gamma_x \Gamma_{kx}^T] V_{kk}(\Gamma) \\ &\quad + \beta \Gamma_x \Gamma_x^T \Gamma_k V_{kkk}(\Gamma), \end{aligned} \quad (125)$$

and multiplying F_{xxx} and other three-dimensional arrays by $[dx/dk]$ in the sense of

$$F_{xxx} \left[\frac{dx}{dk} \right] = \sum_i F_{xxx_i} \left[\frac{dx_i}{dk} \right]$$

to make what remains a two-dimensional matrix,

$$\begin{aligned} F_{xxx} \left[\frac{dx}{dk} \right] &= U_{xxx} \left[\frac{dx}{dk} \right] + \beta \Gamma_{xxx} \left[\frac{dx}{dk} \right] V_k(\Gamma) + 3\beta \Gamma_{xx} \left\{ \Gamma_x^T \left[\frac{dx}{dk} \right] \right\} V_{kk}(\Gamma) \\ &\quad + \beta \Gamma_x \Gamma_x^T \left\{ \Gamma_x^T \left[\frac{dx}{dk} \right] \right\} V_{kkk}(\Gamma). \end{aligned} \quad (126)$$

Note that the quantities in curly brackets in (126) are scalars.

Computing V_{kkk} in the discrete-time certainty model's steady state

From here on, all the equations are for the steady state; I will omit all the arguments for the functions, since they are all k^* and x^* . In combination, Eqs. (115)–(126) make it straightforward to compute the steady-state value of V_{kkk} —and the parallel formulas for V_k and V_{kk} . These formulas for steady state V_k and V_{kk} differ from the following formulas that arise from solving Eqs. (98), (99) for V_k and Eq. (103) for V_{kk} :

$$V_k = \lambda^* = \frac{U_k}{1 - \beta\Gamma_k}, \quad (127)$$

$$V_{kk} = \frac{U_{kk} + U_{kx} \left[\frac{dx}{dk} \right] + \beta\lambda^* \left\{ \Gamma_{kk} + \Gamma_{kx} \left[\frac{dx}{dk} \right] \right\}}{1 - \beta\Gamma_k \zeta}. \quad (128)$$

Eqs. (127) and (128) have the disadvantage of giving no guarantee that the denominator is nonzero. Indeed, there are many situations in which $U_k=0$ and V_k is a positive number for which (127) gives the value 0/0. The equations that arise most naturally from (115)–(126) treat k and x a little more even-handedly, and have denominators that are clearly nonzero whenever $\zeta < 1$.

After substituting from (118)–(126) into (115), (116—first line) and (117), a key to simplifying the right-hand sides is to recognise that for each appearance of Γ_k , there is a corresponding appearance of $\Gamma_x^T [dx/dk]$ that allows the right-hand side to be expressed in terms of $\zeta = \Gamma_x + \Gamma_x^T [dx/dk]$. One element of this is recognising that the coefficients of V_k , V_{kk} and V_{kkk} on the right are respectively $\beta\zeta$, $\beta\zeta^2$ and $\beta\zeta^3$. Solving for the highest derivative of V in each equation yields the following equations:

$$V_k = \lambda^* = \frac{U_k + U_x^T \left[\frac{dx}{dk} \right]}{1 - \beta\zeta}, \quad (129)$$

$$V_{kk} = \frac{U_{kk} + 2U_{kx}^T \left[\frac{dx}{dk} \right] + \left(\left[\frac{dx}{dk} \right]^T U_{xx} \left[\frac{dx}{dk} \right] \right) + \beta\lambda^* \left\{ \Gamma_{kk} + 2\Gamma_{kx}^T \left[\frac{dx}{dk} \right] + \left(\left[\frac{dx}{dk} \right]^T \Gamma_{xx} \left[\frac{dx}{dk} \right] \right) \right\}}{1 - \beta\zeta^2}, \quad (130)$$

$$\begin{aligned} V_{kkk} = & \left(\frac{1}{1 - \beta\zeta^3} \right) \left\{ 3\beta\zeta V_{kk} \left[\Gamma_{kk} + 2\Gamma_{kx}^T \left[\frac{dx}{dk} \right] + \left(\left[\frac{dx}{dk} \right]^T \Gamma_{xx} \left[\frac{dx}{dk} \right] \right) \right] \right. \\ & + U_{kkk} + 3U_{kkx}^T \left[\frac{dx}{dk} \right] + \left(3 \left[\frac{dx}{dk} \right]^T U_{kxx} \left[\frac{dx}{dk} \right] \right) + \left(\sum_i \frac{dx_i}{dk} \cdot \left[\frac{dx}{dk} \right]^T U_{xxx} \left[\frac{dx}{dk} \right] \right) \\ & \left. + \beta\lambda^* \left[\Gamma_{kkk} + 3\Gamma_{kkx}^T \left[\frac{dx}{dk} \right] + \left(3 \left[\frac{dx}{dk} \right]^T \Gamma_{kxx} \left[\frac{dx}{dk} \right] \right) + \left(\sum_i \frac{dx_i}{dk} \cdot \left[\frac{dx}{dk} \right]^T \Gamma_{xxx} \left[\frac{dx}{dk} \right] \right) \right] \right\}. \quad (131) \end{aligned}$$

There is a certain elegance to Eqs. (129), (130) and (131). The remarkable thing is that they depend only on a linear approximation of the behaviour of the control variable vector along the saddle path, as indicated by dx/dk .

It is a good exercise (which in this case I leave to the reader) to show that if β is replaced by $e^{-\rho h}$, ζ replaced by $1+h(A_k+A_x^T[dx/dk])$ and U replaced by $h\mathcal{U}$, then the limits as $h \rightarrow 0$ for Eqs. (127), (129) and (131) yield the corresponding continuous-time unnumbered equations in the Introduction and Eq. (40) in the “General analysis for a one-state-variable optimising model in continuous time” (remembering that U in those early equations must be replaced by $\mathcal{U}$). The limits as $h \rightarrow 0$ of (128) and (130) are new continuous-time equations not found above.

Conclusion

This paper is only a taste of what the method here can do. The working paper version of this paper (Kimball, 2014) includes many more applications, with associated extensions of the techniques above:

1. The examples of the sections “Continuous-time application: Choosing how much to use a risky technology in general equilibrium” and “Continuous-time application: When the only technology available is risky” extended to allow labour to be endogenous and non-constant along the saddle path, with King *et al.*¹⁰ preferences.
2. The examples of those sections, extended to also allow for Kreps-Porteus preferences that allow risk tolerance and the elasticity of intertemporal substitution to be distinguished. This requires extending the techniques above to allow for Epstein-Zin-Weil preferences.¹¹
3. The effect of the variance of permanent technology shocks in a real business cycle model that allows for steady-state growth. After rewriting the model in terms of the ratio of the capital stock to the labour-augmenting technology, this example requires introducing what looks formally like a dependence of the utility discount rate on ω . In addition to the importance of that particular model in its own right, this illustrates the technique for extending the reach of the method to 2-state-variable models that have a strong scale symmetry.
4. The effect of the variance of permanent technology shocks in a real business cycle model with Q-theory-type adjustment costs.

Substantively, the key message of this paper is the importance of paying attention to the persistence of the effects of shocks on a model; the convergence rate is central to many of the calculations above. Pedagogically, the key message is the tight

¹⁰ King *et al.* (1988).

¹¹ Epstein and Zin (1989) and Philippe Weil (1990).

connection between continuous-time and discrete-time analysis. Methodologically, the key message is that, even in a dynamic stochastic model, the phase diagram is a key to understanding.

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