Introduction: A Shortcut to “MM” (derivative) Asset Pricing**

by Eric Briys*

Introduction

A fairly large body of academic literature has been devoted to the topic of derivative asset pricing. Indeed, since the seminal contribution by Black and Scholes (1973), many scholars have tried to generalize the risk neutral option pricing technique and to relate it to some well known results of the probability field (see for instance Duffie for more details).

The purpose of the present paper is not to propose a “brand new” derivative asset pricing technique even if its title may seem rather puzzling. This paper has basically two objectives. First, it serves as an introduction to this special issue of the Geneva Papers whose content is centered around the Twelfth Annual Lecture of the Geneva Association which Professor Merton delivered at the Centre HEC ISA in June 1988. Secondly, it aims at giving the reader a short pedagogical overview of Professor Bob Merton’s many contributions. This survey focuses on what might be considered a narrow although important topic: derivative asset pricing and risk neutral valuation relationships. The material presented here is not new as such. It rather establishes some links between Bob Merton’s various articles, especially those concerning optimum consumption and investment rules and those dealing with derivative asset pricing.

The first section of this introductory paper will briefly summarize Bob Merton’s main breakthroughs. The second section will examine the internal consistency between Bob Merton’s paper of 1969 and the risk neutral pricing technique. We will see that provided we introduce some assumptions (both distributional and utility related) a martingale pricing is implementable. A short conclusion will sketch the main arguments which are put forward in Bob Merton’s Annual Lecture which follows this paper.

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** Although MM is often used for Modigliani-Miller, this tradition does not apply here. MM stands for a “financial hybrid”, namely Merton-Martingale! I hope Bob Merton will excuse this rather strange addition to his family name! I would also like to take this opportunity to thank him for many enlightning discussions during the June 1988 Conference at the Centre HEC ISA. Many thanks also to the Geneva Association, and especially to Orio Giarini, for making it possible. The usual caveat applies.

1. Overview of Bob Merton's main Breakthroughs

It is always a challenge to try to capture in a few words a work which extends over several years and which is furthermore multifaceted. Although this division can be questioned, three main axis of research emerge from Bob Merton's rich bibliography.

The first one deals with the intertemporal portfolio and consumption choices. No need to say that Bob Merton's innovation was to introduce a continuous time stochastic setting. Using stochastic dynamic programming, Bob Merton was able to enrich our understanding of the optimizing behavior of individuals confronted with various sources of uncertainty (see Merton [1969, 1971]). This contribution triggered a huge interest in stochastic techniques and their current success in today's finance owes certainly much to Bob's seminal contribution.

The second topic covers equilibrium asset pricing and is a natural extension to the previous one (see Merton [1973]). By aggregating a portfolio first order condition over investors, Bob gave us new insights into the CAPM world. The so called continuous time CAPM was extended to incorporate stochastic opportunity sets and to show the crucial role of hedge portfolios. Once again the "continuous time stochastic technology" was used to derive results which would have been painful to achieve under traditional discrete time models.

Last but not least, Bob Merton has hallmarked the domain of option pricing (Merton [1973, 1976, 1977]). In his 1973 paper, he has extended Black and Scholes' pathbreaking paper. Numerous questions such as boundary conditions, early exercise of american options, stochastic interest rate option pricing are given an answer in this reference paper. In later researches, the impact of jumps in the underlying asset prices was also investigated.

From this short overview, one could be tempted to believe that the only backbone of Bob's work is the continuous time technology. Such an interpretation is misleading as witnessed for instance by the close connection between individual choices and equilibrium asset pricing. It is true however that the relationship between individual consumption and portfolio optimizing behavior and derivative asset risk neutral pricing is less immediate and deserves some clarification.

This is precisely what we have called MM derivative asset pricing.

2. Merton-Martingale Asset Pricing

People involved in the option pricing field know that Black and Scholes' breakthrough was to propose a risk neutral pricing technique by using a continuous time dynamic hedging argument. This pricing technique is very powerful since it is preference free and does not require the construction of an equilibrium asset pricing model to assess the risk premium embedded in any underlying risky asset. One can incidentally note that probability scientists use the same kind of approach when they use the so-called Girsanov theorem (see Duffie [1988]).

Basically, this risk neutral pricing technique (or martingale pricing) can be summarized as follows. Consider a one period risky asset with random period one cash flow $\tilde{y}$. Using standard financial arguments its present value can be computed as:

$$V = E(\tilde{y})/k$$

where k is one plus the risk adjusted discount rate. To assess k , we obviously need an equi-

brium asset pricing model which links a relevant measure of the asset riskiness to its expected return (or at least an arbitrage pricing model like the APT).

What martingale pricing does is to change the probability measure with respect to which the expectation operator is computed, i. e.

$$V = \hat{E}(\bar{y}) / R$$

where $\hat{E}$ is now measured with respect to risk neutral probabilities (i. e. martingale probabilities) and R is one plus the (risk neutralized) risk free rate (or the riskless return if a riskless asset does exist). This martingale pricing technique is also known as a “certainty equivalent approach” to asset pricing.

This change of probability measure may seem purely cosmetic. It is nonetheless the robust basis of modern option pricing technique. Indeed, continuous time trading is not the only way for Black and Scholes risk neutral option pricing formula to obtain. As shown for example by Brennan [1979], the joint assumptions of lognormality of asset prices and constant relative risk aversion (or normality and constant absolute risk aversion) give rise to the B/S formula in a discrete time economy. More precisely and to put it in Ingersoll’s words (1987, p. 344), a risk neutral valuation will occur (i. e. the martingale density will correspond to a lognormal density) if “any investor’s consumption has a lognormal distribution and his utility is isoelastic or if some investor’s utility is exponential and his consumption has a normal distribution”.

The interesting thing to note here is that, provided some assumptions are added, Merton’s (1969) economy satisfies these requirements.

Indeed, assume in addition to Merton’s assumptions, that the investors populating the market place exhibit the following:

- a) linear risk tolerance utility functions
- b) identical cautiousness
- c) identical time preference

Assumptions (a) and (b) yield the well known two fund separation property. Assumption (c) added to (a) and (b) ensures that the aggregation property is satisfied so that a composite individual can be manufactured whose characteristics are an average of those for the actual individuals in the economy.

Let us assume, as in Merton, that only two assets are available. The first one is a risky asset (which can be considered as the market portfolio by the separation property) whose price dynamics is given by a geometric brownian motion:

$$(1) \quad dS = \mu S dt + \sigma S dz$$

where μ and σ^2 denote respectively the instantaneous expected return and variance. $(z_t, F_t, t > 0)$ is a standard brownian motion process on a probability space (Ω, F, P) where $(F_t, t > 0)$ is a non decreasing right continuous family of σ -fields and P a probability measure.

The second asset, whose price at time t is $B(t)$ is riskless:

$$(2) \quad dB/B = r dt$$

where r is the risk free rate.

Under these assumptions the stochastic process governing wealth W is given by the following stochastic differential equation:

$$(3) \quad \begin{aligned} dW &= (\alpha(\mu - r)W + rW - c)dt + \varpi Wdz \\ W(0) &= W^0 \end{aligned}$$

where α denote the fraction of wealth invested in the risky asset and c consumption per unit time.

The composite individual's objective is to maximize the expected utility of his lifetime consumption :

$$(4) \quad \underset{c, \alpha}{\text{Max}} \quad E_0 \int_0^{\infty} e^{-\rho s} u(c(s)) ds$$

where $u(c(s)) = c(s)^{\delta} / \delta$ (i. e. isoelastic marginal utility assumption). Note that the optimal $\alpha(t)$ and $c(t)$ have to belong to an admissible set which ensures that wealth $W(t)$ will remain positive at any time.

Using Bellmann dynamic programming principle one can write down the Hamilton-Bellmann-Jacobi equation of optimality :

$$(5) \quad \underset{c, \alpha}{0} = \underset{c, \alpha}{\text{MAX}} (c^{\delta} / \delta + (\alpha(\mu - r)W + rW)J_w + 1/2 \alpha^2 \varpi^2 W^2 J_{ww}) \\ W(t) > 0$$

$$\text{where } J(W, t) = \underset{c, \alpha}{\text{Max}} \quad E_t \int_t^{\infty} e^{-\rho s} c(s) ds$$

From Merton (1969), we known that the optimal controls are :

$$(6) \quad \alpha^* = (\mu - r) / (1 - \delta) \varpi$$

$$(7) \quad c^* = (\delta / (1 - \delta)) (\rho / \delta - r - 1/2 \frac{(\mu - r)^2}{\varpi^2} (1 - \delta)) W(t)$$

The consumption function $c(t)$ can be rewritten as $C(W) = aW$ and we denote its inverse function by $V(.) = 1/a..$

Differentiating (5) with respect to C , differentiating the envelope condition for consumption with respect to W and using (6) yields :

$$0 = ((r - \rho - 2\delta)(\delta - 1) c^{\delta-1} + \delta(\delta - 1)(\delta - 2)c^{\delta-1})1/a + ((\delta - 1)c^{\delta-1})^2(r/ac - c)$$

where $\delta = 1/2((\mu - r)/\varpi)^2$

From this we can derive the stochastic differential equation for $u'[c(t)] = c(t)^{\delta-1}$

Indeed, by using Ito's lemma and (6) and (7) yields :

$$du'[c(t)] = -(r - \rho) u'(c(t)) dt - (\mu - r)/\varpi u'(c(t)) dz$$

which means that marginal utility is log normally distributed.

Note also that consumption $c^*(t)$ is also lognormally distributed : $c^*(t)$ follows a geometric brownian motion.

But from Ingersoll (1987), we know that marginal utility is one valid martingale price process for a complete market (which we do have here). Furthermore, this pricing process is lognormally distributed as shown above. Hence we can use it to price any derivative security. Assume for instance that the derivative asset under consideration is a european call. At maturity T , its payoff is given by $h[S(T)] = \text{Max} [0, S(T) - K]$ where K is the strike.

The value of this call at time t is:

$$V(t) = q E(h(S(t)) c^{s-1}(T)/c^{s-1}(t))$$

which is nothing but Black and Scholes option pricing formula. This is easily shown since both S and c^* are log normally distributed.

Although not really new this shortcut shows that derivative assets can be priced in such a way that their pricing is fully compatible with consumption and portfolio optimizing behavior (for more details in the discrete time case see Huang and Litzenberger (p. 163, 1988).

It also witnesses that continuous time finance, although technical, allows many insights in “seemingly” unrelated topics.

Conclusion

In the following pages, the reader will discover more about Bob Merton's current research interest. Indeed, this lecture covers brand new material which tries to establish a reasonably comprehensive theory of financial intermediation. The main ingredients of that theory are transaction costs. Indeed, if intermediation is to serve an important economic function, one has to show that it may save significant transactions costs. Note that the same argument applies to replicable derivative securities. To be sure, Bob Merton's investigation of the intermediation world will reveal most useful to those many involved in the thorny task of asset and liability management. Once again, Bob Merton shows that the contingent claim pricing theory is most relevant and will undoubtedly find its way into the real world of financial intermediation.

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For the references concerning Robert Merton, see his paper in this issue.

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