



# COMMUNICATIONS PHYSICS

ARTICLE




<https://doi.org/10.1038/s42005-020-00396-0>

OPEN

## Time-reversal of an unknown quantum state

A. V. Lebedev<sup>1</sup> & V. M. Vinokur<sup>2,3</sup>  

For decades, researchers have sought to understand how the irreversibility of the surrounding world emerges from the seemingly time-symmetric, fundamental laws of physics. Quantum mechanics conjectured a clue that final irreversibility is set by the measurement procedure and that the time-reversal requires complex conjugation of the wave function, which is overly complex to spontaneously appear in nature. Building on this Landau-Wigner conjecture, it became possible to demonstrate that time-reversal is exponentially improbable in a virgin nature and to design an algorithm artificially reversing a time arrow for a given quantum state on the IBM quantum computer. However, the implemented arrow-of-time reversal embraced only the known states initially disentangled from the thermodynamic reservoir. Here we develop a procedure for reversing the temporal evolution of an arbitrary unknown quantum state. This opens the route for general universal algorithms sending temporal evolution of an arbitrary system backward in time.

<sup>1</sup>Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudny, Moscow District 141700, Russia. <sup>2</sup>Materials Science Division, Argonne National Laboratory, 9700 S. Cass Avenue, Argonne, IL 60439, USA. <sup>3</sup>Consortium for Advanced Science and Engineering (CASE) University of Chicago, 5801 S Ellis Ave, Chicago, IL 60637, USA. email: [vinokour@anl.gov](mailto:vinokour@anl.gov)

An origin of the arrow of time, the concept coined for expressing one-way direction of time, is inextricably associated with the Second Law of Thermodynamics<sup>1</sup>, which declares that entropy growth stems from the system's energy dissipation to the environment<sup>2–6</sup>. Thermodynamic considerations<sup>7–17</sup>, combined with the quantum mechanical hypothesis that irreversibility of the evolution of the physical system is related to measurement procedure<sup>18,19</sup>, and to the necessity of the anti-unitary complex conjugation of the wave function of the system for time reversal<sup>20</sup>, led to understanding that the energy dissipation can be treated in terms of the system's entanglement with the environment<sup>1,21–24</sup>. The quantum mechanical approach to the origin of the entropy growth problem was crowned by finding that in a quantum system initially not correlated with an environment, the local violation of the second law can occur<sup>25</sup>. Extending then the solely quantum viewpoint on the arrow of time and elaborating on the implications of the Landau–Neumann–Wigner hypothesis<sup>18–20</sup>, enabled to quantify the complexity of reversing the evolution of the known quantum state and realize the reversal of the arrow of time on the IBM quantum computer<sup>26</sup>.

In all these past studies, a thermodynamic reservoir at finite temperatures has been appearing as a high-entropy stochastic bath thermalizing a given quantum system and increasing thus its thermal disorder, hence entropy. We find that most unexpectedly, it is exactly the presence of the reservoir that makes it possible to prepare the high-temperature thermal states of an auxiliary quantum system governed by the same Hamiltonian  $\hat{H}$  as the Hamiltonian of a given system. This enables us to devise the operator of the backward-time evolution  $\hat{U} = \exp(i\hat{H}t)$  reversing the temporal dynamics of the given quantum system. The necessary requirement is that the dynamics of the both, auxiliary and given, systems were governed by the same Hamiltonian  $\hat{H}$ . The time-reversal protocol comprises the cyclic sequential process of quantum computation on the combined auxiliary and the given systems and the thermalization process of the auxiliary system. A universal time-reversal procedure of an unknown quantum state defined through the density matrix  $\hat{\rho}(t)$  of a quantum system  $\mathcal{S}$  will be described as a reversal of the temporal system evolution  $\hat{\rho}(t) \rightarrow \hat{\rho}(0) = \exp(i\hat{H}t/\hbar)\hat{\rho}(t)\exp(-i\hat{H}t/\hbar)$  returning it to system's original state  $\hat{\rho}(0)$ . Importantly, we need not know the quantum state of this system in order to implement the arrow of time reversal. A dramatic qualitative advance of the new protocol is that it eliminates the need of keeping an exponentially huge record of classical information about the values of the state amplitudes. Moreover, the crucial step compared with the protocol of time reversal of the known quantum state<sup>26</sup> is that we now lift the requirement that initially the evolving quantum system must be a pure uncorrelated state. Here, we develop a procedure where the initial state can be a mixed state and, therefore, include correlations due to system's past interaction with the environment.

## Results

**Universal procedure.** The calculations are organized as follows. First, we describe how the time reversal of an unknown state can be implemented in a universal manner and estimate its computational complexity. Next, we outline a somewhat more resource-demanding procedure, where, however, one can relax the need of knowing the Hamiltonian  $\hat{H}$ . Then we show that if in addition to the quantum system  $\mathcal{S}$  one is provided by an auxiliary system  $\mathcal{A}$ , so that  $\dim \mathcal{S} = \dim \mathcal{A}$ , whose dynamics is governed by the same Hamiltonian  $\hat{H}$ , one can devise  $\hat{U}^\dagger(t)$  without knowing an exact form of  $\hat{H}$ . Finally, we discuss how the partial knowledge on the

state  $\hat{\rho}(t)$  can reduce and optimize the complexity of the time-reversal procedure.

The starting point of the reversal procedure is drawn from the observation of S. Lloyd et al.<sup>27</sup> that having an ancilla system in a state  $\hat{\sigma}$  one can approximately construct a unitary operation  $\exp(-i\omega\hat{\sigma}\delta t)$  acting on a system  $\mathcal{S}$  simulating its evolution under Hamiltonian  $\hat{H}_a = \hbar\omega\hat{\sigma}$  during the infinitesimal time interval  $\delta t$ . Here,  $\omega$  refers to some arbitrary rate, which for a moment, we leave unspecified. Having  $N$  identical copies of ancillas, one generates a finite time evolution  $\rho(t) \rightarrow \rho(t + \tau) = e^{-i\omega\tau\hat{\sigma}}\hat{\rho}(t)e^{+i\omega\tau\hat{\sigma}}$  over the time interval  $\tau = N\delta t$  with the accuracy  $\propto (\omega\tau)^2/N$  (see “Methods”). The first step of the time-reversal procedure is then constructing the density matrix  $\hat{\sigma}$ . Consider the density operator defined by the given finite-dimensional Hamiltonian  $\hat{H}$  having the maximal eigenvalue  $\epsilon_{\max}$ :

$$\hat{\sigma} = \frac{1}{Z}(\mathbb{1}_{\epsilon_{\max}} - \hat{H}), \quad (1)$$

where  $Z = \epsilon_{\max} \dim \mathcal{S} - \text{Tr} \{\hat{H}\}$  is the normalization factor. Then the Lloyd (LMR) procedure maps the initial density matrix  $\hat{\rho}(t)$  to

$$\hat{\rho}(t) \rightarrow \exp\left(\frac{i\omega}{Z}\hat{H}\tau\right)\hat{\rho}(t)\exp\left(-\frac{i\omega}{Z}\hat{H}\tau\right). \quad (2)$$

One sees that application of the LMR procedure with the specific density matrix  $\hat{\sigma}$  realizes approximately the time-reversed evolution of the system

$$\hat{\rho}(t) \rightarrow \hat{\rho}\left(t - \frac{\hbar\omega}{Z}\tau\right) + \delta\hat{\rho}(\tau) \quad (3)$$

to a backward delay  $\tau_R = (\hbar\omega/Z)\tau$ . The accuracy  $\delta\hat{\rho}(\tau)$  of such a time-reversal procedure is given by (see “Methods”),

$$\|\delta\hat{\rho}(\tau)\| \leq \frac{(\omega\tau)^2}{N} (\|\hat{\sigma}\| + \|\hat{\rho}(t)\| + 2\|\hat{\rho}(t)\|\|\hat{\sigma}\|^2), \quad (4)$$

where  $\|\hat{A}\|$  is the operator norm:  $\|\hat{A}\| = \sup_{|\psi\rangle} \sqrt{\langle\psi|\hat{A}|\psi\rangle/\langle\psi|\psi\rangle}$ .

From Eqs. (3) and (4), one draws two important conclusions. First, the above time-reversal procedure for a backward delay  $\tau_R$  requires time  $\tau$  to be completed. Therefore, while exercising the reversal, the system still maintain the forward evolution governed by its own Hamiltonian. Taking this into account, one has to modify Eq. (3) to

$$\hat{\rho}(t) \rightarrow \hat{\rho}\left(t - \tau\frac{\hbar\omega}{Z} + \tau\right), \quad (5)$$

which immediately poses the constraint on the operation rate  $\omega$  of the LMR procedure: the actual time reversal occurs only for  $\hbar\omega > Z$ . If this constraint is not satisfied, the time-reversal procedure only slows down the forward time evolution of the system. For a quantum system  $\mathcal{S}$ , the threshold rate  $Z/\hbar$  is proportional to the Hilbert space dimension  $\dim \mathcal{S}$ :  $Z = \hbar\tilde{\omega} \dim \mathcal{S}$  with  $\tilde{\omega} = (\epsilon_{\max} - \text{Tr} \{\hat{H}\})/\dim \mathcal{S}$ , which is typically an exponentially large number. In particular, in order to make the time reversal with the same rate as the forward time evolution, one has to demand  $\omega > 2Z/\hbar$ . This brings straightforwardly the second conclusion: as far as  $\omega$  is large, the infinitesimal time step  $\delta t$  of the procedure has to be small so that  $\omega\delta t \ll 1$ , therefore the number  $N$  has to be large. Indeed, fixing the backward delay  $\tau_R$ , the operation rate  $\omega = 2Z/\hbar$ , and setting the reversal accuracy  $\epsilon$ :  $\|\delta\hat{\rho}(\tau = \tau_R)\| \leq \epsilon$  one finds from Eq. (4):

$$N_\epsilon = \frac{\|\hat{\rho}(t)\|}{\epsilon} \left(\dim \mathcal{S} \frac{\tau_R}{\tilde{\tau}}\right)^2, \quad (6)$$

where  $\tilde{\tau} = \tilde{\omega}^{-1}$  is the typical timescale of the system dynamics, and  $\|\hat{\sigma}\| \propto (\dim \mathcal{S})^{-1} \ll \|\hat{\rho}(t)\|$  is assumed. Equation (6) implies that the computational complexity of the time-reversal procedure

for an unknown quantum state is proportional to the square of the system’s Hilbert space dimension. In contrast, the time reversal of a known pure quantum state  $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$  is proportional to the dimension of the Hilbert space, which is swept by the system in the course of its forward time evolution  $|\psi(0)\rangle \rightarrow |\psi(t)\rangle$ <sup>26</sup>. As follows from Eq. (6), the time-reversal computational cost of an unknown pure state is maximal as long as  $\|\hat{\rho}\| = 1$  in this case. For a mixed high-entropy state  $\hat{\rho}$ , the reversal complexity is reduced: given a state  $\hat{\rho}$  with the entropy  $S_\rho = \ln(\dim S) - k \ln(2)$ , where only  $k \ll \log_2(\dim S)$  bits of information is known, the upper estimate for the reversal complexity is given by (see “Methods”)

$$N_\epsilon \leq \frac{k}{\epsilon \log_2(\dim S)} \left( \dim S \frac{\tau_R}{\tau} \right)^2. \quad (7)$$

Having complete information about the Hamiltonian  $\hat{H}$  allows one to construct a corresponding quantum circuit realizing the forward time evolution operator  $\hat{U} = \exp(-i\hat{H}t/\hbar)$  through a specific fixed set  $\mathcal{G}$  of universal quantum gates:  $\hat{U} = \hat{U}_1 \cdots \hat{U}_N$ ,  $\hat{U}_i \in \mathcal{G}$ . As far as  $\mathcal{G}$  is an universal set, for every  $\hat{U}_i \in \mathcal{G}$  one can construct the inverse gate  $\hat{U}_i^\dagger$ . Therefore, the time-reversed evolution operator  $\hat{U}^\dagger$  can be constructed in a purely algorithmic way given the gate decomposition of  $\hat{U}$ . Thus, the above procedure may appear extremely ineffective for a practical time-reversal task. However, the situation turns completely different if we relax the requirement of the exact knowledge of  $\hat{H}$  and assume that one, instead, is provided by the equivalent copy of the system  $S$  governed by the same Hamiltonian  $\hat{H}$ .

**Auxiliary system.** Let one be equipped with the thermodynamic bath at the temperature  $T = \beta^{-1}$  in addition to the ancilla. One can then thermalize the ancilla and prepare it in the equilibrium state  $\sigma_\beta = Z_\beta^{-1} \exp(-\beta\hat{H})$  with  $Z_\beta = \text{Tr}\{\exp(-\beta\hat{H})\}$  being a statistical sum. For high-enough temperature  $\beta \rightarrow 0$ , one has  $\beta\epsilon_{\max} \sim 1$  and, therefore,  $\sigma_\beta \approx Z_\beta^{-1}(1 - \beta\hat{H})$  which gives the desired state of the ancilla to implement the reverse evolution through the LMR procedure. In this case,

$$\hat{\rho}(t) \rightarrow \hat{\rho}\left(t - \frac{\hbar\omega\beta}{Z_\beta} \tau + \tau\right) + \delta\hat{\rho}(\tau). \quad (8)$$

As can be seen from the above equation the actual time reversal requires the operation rate of the LMR procedure to exceed the threshold

$$\omega > \omega_{\text{th}} = \frac{T}{\hbar} Z_\beta \approx \frac{T}{\hbar} \dim S. \quad (9)$$

The approximation error  $\delta\hat{\rho}$  splits now to two contributions,  $\delta\hat{\rho} = \delta\hat{\rho}_1 + \delta\hat{\rho}_2$ , where  $\delta\hat{\rho}_1$  is the approximation error resulting from the LMR procedure, see Eq. (4) with  $\hat{\sigma} \rightarrow \hat{\sigma}_\beta$ , while the error  $\delta\hat{\rho}_2$  describes the error due to the  $\beta$  expansion of the thermal state  $\hat{\sigma}_\beta$ . Assuming  $\omega = 2\omega_{\text{th}}$ , i.e. the backward evolution goes with the same rate as the forward time evolution one finds

$$\delta\hat{\rho}_2(\tau) = -i \frac{\tau\beta}{\hbar} \left[ \hat{H}^2, \hat{\rho}(t - \tau) \right]. \quad (10)$$

Then for  $\|\hat{\sigma}_\beta\| \ll \|\hat{\rho}(t)\|$  one can estimate the net error as

$$\|\delta\hat{\rho}(\tau)\| \leq \left( 4 \frac{Z_\beta^2}{N} \left( \frac{\tau}{\tau_\beta} \right)^2 + \frac{\tau}{\tau_\beta} (\beta\epsilon_{\max})^2 \right) \|\hat{\rho}(t - \tau)\|. \quad (11)$$

where  $\tau_\beta = \hbar\beta$ . The temperature dependence of two error contributions in Eq. (11) oppositely depends on the inverse temperature: the error due to thermal expansion (second term)

reduces as  $\beta \rightarrow 0$ , while the error due to LMR dynamics (first term) increases with decreasing  $\beta$ . For a given reverse time delay  $\tau$  and the number of LMR iterations  $N \gg Z_\beta^2 \approx (\dim S)^2$ , one has an optimal temperature

$$\beta\epsilon_{\max} = \left( 8 \frac{Z_\beta^2 \epsilon_{\max} \tau}{N \hbar} \right)^{1/3}, \quad (12)$$

and the corresponding net accuracy of the reversal procedure is then given by

$$\|\delta\hat{\rho}(\tau)\| = \epsilon = 3 \left( \frac{Z_\beta^2}{N} \right)^{1/3} \left( \frac{\epsilon_{\max} \tau}{\hbar} \right)^{4/3} \|\hat{\rho}(t - \tau)\|. \quad (13)$$

Comparing with the case of the known Hamiltonian time-reversal procedure, see Eq. (6), the reversal complexity here is again proportional to the square of the system’s Hilbert space dimension, but, at the same time, has more adverse scaling with the reversal duration and the net accuracy.

The above analysis does not need any prior information about the state  $\hat{\rho}$  which would require very high temperature of the auxiliary thermostat in order to cover all the possible energy states of the system’s Hilbert space that finally results in a tremendously high rate  $\sim \hbar\beta \dim S$  of the LMR procedure. If, however, some information about the energy content of the state  $\hat{\rho}$  is available, one can appreciably reduce the reversal cost. Indeed, let the state  $\hat{\rho}$  have the average energy  $\bar{E} = \text{Tr}\{\hat{\rho}\hat{H}\}$  with an energy variance  $(\delta E)^2 = \text{Tr}\{\hat{\rho}((\hat{H} - \bar{E})^2)\}$ . Then one can present the density matrix as the result of the low-energy contribution,  $\hat{\rho}_< = \hat{P}\hat{\rho}\hat{P}/\text{Tr}\{\hat{P}\hat{\rho}\}$  and the high-energy remainder  $\hat{\rho}_> = (1 - \hat{P})\hat{\rho}(1 - \hat{P})/\text{Tr}\{(1 - \hat{P})\hat{\rho}\}$ , where  $\hat{P} = \sum_{E < E_{\max}} |E\rangle\langle E|$  is a projection operator to the subspace with energies below some cutoff energy  $E_{\max} > \bar{E}$ :  $\hat{\rho} = (1 - \epsilon_E)\hat{\rho}_< + \epsilon_E\hat{\rho}_>$ . The additional error due to truncating the system’s Hilbert space to the low-energy subspace is given by the constant  $\epsilon_E$ , which is a probability for the system to be found in the energy state  $E > E_{\max}$ , and, according to the Chebyshev inequality, is bound by

$$\epsilon_E \leq \left( \frac{E_{\max} - \bar{E}}{\delta E} \right)^{-2}. \quad (14)$$

**Single-particle wave packet.** Now, we consider an exemplary time-reversal procedure for a spreading single-particle wave packet with the quadratic spectrum. Let the packet at the time  $t = 0$  be localized at the origin and have the Lorentzian shape with the width  $\xi_0$ :

$$\Psi(x, 0) = \sqrt{\frac{\xi_0}{2\pi}} \frac{2\xi_0}{x^2 + \xi^2} \equiv \sum_p \sqrt{2\pi\xi_0} e^{-|p|\xi_0} e^{ipx}. \quad (15)$$

A subsequent free evolution with quadratic Hamiltonian  $\hat{H} = \hbar^2 \hat{p}^2 / 2m$  during the time interval  $\tau > 0$  broadens the particle’s wave function into

$$\Psi(x, \tau) \approx \frac{e^{-|x|m\xi_0/\hbar\tau}}{\sqrt{2\pi\hbar\tau/m}} \exp\left(\frac{imx^2}{2\hbar\tau}\right), \quad (16)$$

having the typical size  $\xi_\tau = \hbar\tau/m\xi_0$  or, equivalently,  $\xi_\tau/\xi_0 = 4\bar{E}\tau/\hbar$ , where  $\bar{E} = \hbar^2/4m\xi_0^2$  is the average energy carried by the wave packet. The statistical sum  $Z_\beta$  within the volume  $\sim \xi_\tau$  is given by

$$Z_\beta \sim \xi_\tau \int dE \nu_{1D}(E) e^{-\beta E} \sim \frac{\tau}{\hbar} \sqrt{\frac{\bar{E}}{\beta}}, \quad (17)$$

where  $\nu_{1D}(E) = (m/2\pi^2\hbar^2E)^{1/2}$  is one-dimensional density of states. Assuming  $E_{\max} \sim \bar{E}$ , the reversal complexity for the time-reversal procedure with the accuracy  $\epsilon$  is given by (see Eqs. (12) and (13))

$$N_\epsilon \sim \frac{1}{\epsilon^4} \left( \frac{\bar{E}\tau}{\hbar} \right)^7 = \frac{1}{\epsilon^4} \left( \frac{\xi_\tau}{\xi_0} \right)^7. \quad (18)$$

The optimal inverse temperature of the thermostat is then given by

$$\beta \sim \frac{1}{\bar{E}} \frac{\epsilon}{\xi_\tau/\xi_0}. \quad (19)$$

Comparing this with the reversal complexity of a known state of the wave packet,  $N'_\epsilon \sim \epsilon^{-1}(\xi_\tau/\xi_0)$ , see ref. <sup>26</sup>, one finds that the reversal of an unknown wave-packet state is a more laborious computational task.

## Discussion

We have described the time-reversal procedure of an unknown mixed quantum state. The procedure relies on the ability to perform the LMR protocol and on the existence of an ancilla system whose dynamics is governed by the same Hamiltonian as the Hamiltonian of the reversed system, which is not required to be known to us. The reversal procedure is comprised of  $N \gg 1$  sequential applications of the LMR procedure to the joint state of the system and ancilla prepared in a thermal state. In contrast to the known state-reversal procedure, the introduced algorithm does not require to keep an information about all amplitudes of the reversed state. Yet the reversal complexity given by  $N$  scales typically as squared dimension of a Hilbert space spanned the unknown state. Moreover, the operation rate of the LMR procedure has to be sufficiently high to overrun the forward time evolution of the reversed system during the execution of the reversal protocol.

The experimental realization of such a protocol is a feasible yet challenging task. As a first step, it will require an upgrade of the existing design of quantum chips. In particular, one needs a set of interacting qubits (denoted by  $\mathcal{Q}_A$ ) capable to get thermalized on-demand being connected with the high-temperature environment. For superconducting qubits<sup>28</sup>, this can be implemented by coupling them with a transmission line, where the high-temperature thermal radiation is fed in, once one needs to set the qubits into a high-temperature state. Next, the second set of qubits  $\mathcal{Q}_B$ ,  $\dim \mathcal{Q}_B = \dim \mathcal{Q}_A$  is required, where one can store a quantum state prepared in the set  $\mathcal{Q}_A$ . Then the time-reversal procedure goes as follows. First, one prepares some state  $\psi_A(0)$  of the qubits  $\mathcal{Q}_A$ , and lets it evolve according to an intrinsic Hamiltonian of the qubits  $\mathcal{Q}_A$ :  $\psi_A(0) \rightarrow \psi_A(\tau) = e^{-i\hat{H}_A\tau/\hbar}\psi_A(0)$ . Second, at the end of the evolution, one swaps the states between  $\mathcal{Q}_A$  and  $\mathcal{Q}_B$ ,  $\psi_A \leftrightarrow \psi_B$ . We assume that the set  $\mathcal{Q}_B$  can keep its quantum state untouched for a sufficient time. The procedure then continues as described above: one subsequently thermalizes the set  $\mathcal{Q}_A$  and implements the joint LMR evolution  $e^{-i\omega\delta t\hat{S}_{AB}}\hat{\rho}_A \otimes \hat{\rho}_B e^{i\omega\delta t\hat{S}_{AB}}$ . As a result, the qubits  $\mathcal{Q}_B$  will undergo the time-reversed dynamics under the same Hamiltonian  $\hat{H}_A$ . This procedure is to be implemented on the emergent quantum computers with the on-demand thermalizable qubits.

## Methods

**LMR procedure.** The LMR procedure goes as follows: one considers a combined system of the system in question and an ancilla  $\hat{\rho} \otimes \hat{\sigma}$  and performs the joint unitary evolution over an infinitesimal time instant  $\delta t$  under a Hamiltonian  $\hbar\omega\hat{S}$ ,

$$\hat{\rho} \otimes \hat{\sigma} \rightarrow \exp(-i\omega\delta t\hat{S})[\hat{\rho} \otimes \hat{\sigma}] \exp(+i\omega\delta t\hat{S}), \quad (20)$$

where  $\hat{S}$  is a unitary SWAP operator<sup>29</sup> acting on the system and ancilla:

$\hat{S}(|x\rangle_S \otimes |y\rangle_A) = |y\rangle_S \otimes |x\rangle_A$ . The operator  $\hat{S}$  is itself Hermitian and, therefore, the unitary operator  $\exp(-i\omega\delta t\hat{S})$  can be implemented. Making use of its property  $\hat{S}^2 = \hat{1}$ , one gets  $\exp(-i\omega\delta t\hat{S}) = \hat{1} \cos(\omega\delta t) - i\hat{S} \sin(\omega\delta t)$  and, therefore, its computational complexity is equivalent to the complexity of the unitary swap operator acting on the direct product of Hilbert spaces with the dimensions  $\dim S$ . Next, we trace out the ancilla and get the quantum channel for the system's density matrix

$$\rho \rightarrow \Phi_{\delta t}[\hat{\rho}] = \cos^2(\omega\delta t)\hat{\rho} + \sin^2(\omega\delta t)\hat{\sigma} - i \sin(\omega\delta t) \cos(\omega\delta t)[\hat{\sigma}, \hat{\rho}]. \quad (21)$$

At the infinitesimal time instant  $\omega\delta t \rightarrow 0$ , one gets the channel,

$\Phi_{\delta t}[\hat{\rho}] = \hat{\rho} - i\omega\delta t[\hat{\sigma}, \hat{\rho}] + (\omega\delta t)^2(\hat{\sigma} - \hat{\rho})$ . In this expression, the term  $\hat{\rho} - i\omega\delta t[\hat{\sigma}, \hat{\rho}]$  corresponds to the infinitesimal time evolution of the density matrix  $\hat{\rho}(t)$  under the Hamiltonian  $\hat{H}_\sigma = \hbar\omega\hat{\sigma}$ :  $\hat{\rho}(t) \rightarrow \hat{\rho}(t + \delta t) = e^{-i\hat{H}_\sigma\delta t/\hbar}\hat{\rho}(t)e^{i\hat{H}_\sigma\delta t/\hbar} \approx \hat{\rho}(t) - i\omega\delta t[\hat{\sigma}, \hat{\rho}] - \frac{1}{2}(\omega\delta t)^2[\hat{\sigma}[\hat{\sigma}, \hat{\rho}]]$ . Therefore, up to the  $(\omega\delta t)^2$  terms, Eq. (21) can be transformed into the exponential form

$$\Phi_{\delta t}[\hat{\rho}(t)] \approx e^{-i\omega\delta t\hat{\sigma}}\hat{\rho}(t)e^{+i\omega\delta t\hat{\sigma}} + (\omega\delta t)^2 \left( \hat{\sigma} + \frac{1}{2}[\hat{\sigma}[\hat{\sigma}, \hat{\rho}(t)]] - \hat{\rho}(t) \right). \quad (22)$$

Repeating the above procedure  $N$  times one can generate the forward time evolution  $\exp(-i\omega\tau\hat{\sigma})$  over a finite time interval  $\tau = N\delta t$

$$\Phi_{\delta t}^N[\hat{\rho}(t)] \approx \exp(-i\omega\tau\hat{\sigma})\hat{\rho} \exp(+i\omega\tau\hat{\sigma}) + \delta\hat{\rho}(\tau), \quad (23)$$

where the approximate accuracy is given by

$$\delta\hat{\rho}(\tau) = \frac{(\omega\tau)^2}{N} \left( \sigma + \frac{1}{2}[\hat{\sigma}[\hat{\sigma}, \hat{\rho}(t + \tau)]] - \hat{\rho}(t + \tau) \right), \quad (24)$$

with  $\hat{\rho}(t + \tau) = \exp(-i\omega\tau\hat{\sigma})\hat{\rho}(t) \exp(+i\omega\tau\hat{\sigma})$  being the final state of the system.

**High-entropy-state-reversal complexity.** Here, we derive the Eq. (7) for the time-reversal complexity of the state  $\hat{\rho}$  with the entropy  $S = \ln \dim(N) - k \ln(2)$ , where  $N = \dim(S)$  is the Hilbert space dimension of the system. The norm  $\|\hat{\rho}\|$  is given by its maximum eigenvalue  $\|\hat{\rho}\| = p_1 > p_i, i = 2, \dots, N$  of the density operator. The von Neumann entropy can be decomposed into a sum

$$S = H(p_1) + (1 - p_1) \sum_{i=2}^N \tilde{p}_i \ln(\tilde{p}_i), \quad (25)$$

where  $\tilde{p}_i = p_i/(1 - p_1)$  with  $\sum_{i=2}^N \tilde{p}_i = 1$ ,  $H(x) = -x \ln(x) - (1 - x) \ln(1 - x) \leq \ln(2)$ . Let us find a maximal possible  $p_1$  for a given  $S$ . One sees straightforwardly that  $p_1$  is maximal if all  $\tilde{p}_i, i = 2, \dots, N$  are uniform and Eq. (25) is reduced to

$$\ln(N) - k \ln(2) = H(p_1) + (1 - p_1) \ln(N - 1). \quad (26)$$

For  $N \gg 1$ , one can assume  $p_1 \ll 1$  and get the approximate solution  $p_1 \approx k/\log_2(N)$  that results in Eq. (7).

## Data availability

Data sharing is not applicable to this article, as no data sets were generated or analyzed during this study.

Received: 2 March 2020; Accepted: 1 July 2020;

Published online: 21 July 2020

## References

- Lloyd, S. On the spontaneous generation of complexity in the universe. in *Complexity and the Arrow of Time*, (eds Lineweaver, C. H., Davies, P. C. W. & Ruse, M.) 80–112 (Cambridge University Press, 2013).
- Thomson, W. I. X. On the dynamical theory of heat. Part V. Thermo-electric currents. *Trans. R. Soc. Edinb.* **21**, 123–171 (1857).
- Maxwell, J. C. Illustrations of the dynamical theory of gases. Part I. On the motions and collisions of perfectly elastic spheres. *Philos. Mag., 4th Ser.* **19**, 19–32 (1860).
- Boltzmann, L. Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen. *Wien. Ber.* **75**, 62–100 (1872).
- Boltzmann, L. Entgegnung auf die wärme-theoretischen Betrachtungen des Hrn. E. Zermelo. *Ann. der Phys. (Leipzig)* **57**, 773–784 (1896).
- Lebowitz, J. L. Statistical mechanics: a selective review of two central issues. *Rev. Mod. Phys.* **71**, S346–S357 (1999).
- Holster, A. T. The criterion for time symmetry of probabilistic theories and the reversibility of quantum mechanics. *N. J. Phys.* **5**, 1–28 (2013).
- Andrieux, D. et al. Entropy production and time asymmetry in nonequilibrium fluctuations. *Phys. Rev. Lett.* **98**, 150601 (2007).



9. Parrondo, J. M. R., Van den Broeck, C. & Kawai, R. Entropy production and the arrow of time. *N. J. Phys.* **11**, 073008 (2009).
10. Campisi, M., Hänggi, P. & Talkner, P. Colloquium: quantum fluctuation relations: foundations and applications. *Rev. Mod. Phys.* **83**, 771 (2011).
11. Campisi, M. & Hänggi, P. Fluctuation, dissipation and the arrow of time. *Entropy* **13**, 2024–2035 (2011).
12. Deffner, S. & Lutz, E. Nonequilibrium entropy production for open quantum systems. *Phys. Rev. Lett.* **107**, 140404 (2011).
13. Oreshkov, O. & Cerf, N. J. Operational formulation of time reversal in quantum theory. *Nat. Phys.* **11**, 853 (2015).
14. Manzano, G., Horowitz, J. M. & Parrondo, J. M. R. Quantum fluctuation theorems for arbitrary environments: adiabatic and non-adiabatic entropy production. *Phys. Rev. X* **8**, 031037 (2018).
15. Santos, J. P., Landi, G. T. & Paternostro, M. Wigner entropy production rate. *Phys. Rev. Lett.* **118**, 220601 (2017).
16. Batalhão, T. B., Gherardini, S., Santos, J. P., Landi, G. T. & Paternostro, M. Characterizing irreversibility in open quantum systems. *Thermodynamics in the Quantum Regime. Fundamental Theories of Physics*, vol 195 (eds Binder, F., Correa, L., Gogolin, C., Anders, J. & Adesso, G.) (Springer, Cham, 2019).
17. Gherardini, S., Müller, M. M., Trombettoni, A., Ruffo, S. & Ncaruso, F. Reconstructing quantum entropy production to probe irreversibility and correlations. *Quantum Sci. Technol.* **3**, 035013 (2018).
18. Landau, L. Das Dämpfungsproblem in der Wellenmechanik. *Z. Phys.* **45**, 430 (1927).
19. VonNeuman, J. Beweis des Ergodensatzes und des H-Theorems in der neuen Mechanik. *Z. für. Phys.* **57**, 30–70 (1929).
20. Wigner, E. Ueber die Operation der Zeitumkehr in der Quantenmechanik. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1932**, 546–559 (1932).
21. Batalhão, T. B. et al. Irreversibility and the arrow of time in a quenched quantum system. *Phys. Rev. Lett.* **115**, 190601 (2015).
22. Camati, P. A. et al. Experimental rectification of entropy production by Maxwell's Demon in a quantum system. *Phys. Rev. Lett.* **117**, 240502 (2016).
23. Partovi, M. H. Entanglement versus Stosszahlansatz: disappearance of the thermodynamic arrow in a high-correlation environment. *Phys. Rev. E* **77**, 021110 (2008).
24. Jennings, D. & Rudolph, T. Entanglement and the thermodynamic arrow of time. *Phys. Rev. E* **81**, 061130 (2010).
25. Lesovik, G. B., Lebedev, A. V., Sadovskyy, I. A., Suslov, M. V. & Vinokur, V. M. H-theorem in quantum physics. *Sci. Rep.* **6**, 32815 (2016).
26. Lesovik, G. B., Sadovskyy, I. A., Suslov, M. V., Lebedev, A. V. & Vinokur, V. M. Arrow of time and its reversal on the IBM quantum computer. *Sci. Rep.* **9**, 4396 (2019).
27. Lloyd, S., Mohseni, M. & Rebentrost, P. Quantum principal component analysis. *Nat. Phys.* **10**, 631 (2014).
28. Krantz, P. et al. A quantum engineer's guide to superconducting qubits. *Appl. Phys. Rev.* **6**, 021318 (2019).
29. Garcia-Escartin, J. C. & Chamorro-Posada, P. A SWAP gate for qudits. *Quant. Inf. Process.* **12**, 3625 (2013).

## Acknowledgements

The work was supported by the U.S. Department of Energy, Office of Science, Basic Energy Sciences, Materials Sciences and Engineering Division (V.M.V. and partly A.V.L.) and by the RFBR Grant No. 18-02-00642A (A.V.L.). A.V.L. acknowledges the support from the Ministry of the Education and Science of the Russian Federation 16.7162.2017/8.9 and from the Government of the Russian Federation (Agreement 05.Y09.21.0018).

## Author contributions

A.V.L. and V.M.V. conceived the work, A.V.L. carried out the calculations, both authors analyzed the results and wrote the paper.

## Competing interests

The authors declare no competing interests.

## Additional information

**Supplementary information** is available for this paper at <https://doi.org/10.1038/s42005-020-00396-0>.

**Correspondence** and requests for materials should be addressed to V.M.V.

**Reprints and permission information** is available at <http://www.nature.com/reprints>

**Publisher's note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this license, visit <http://creativecommons.org/licenses/by/4.0/>.

This is a U.S. government work and not under copyright protection in the U.S.; foreign copyright protection may apply 2020