



OPEN

Improvement in variance estimation using transformed auxiliary variable under simple random sampling

Hameed Ali¹, Syed Muhammad Asim¹, Muhammad Ijaz², Tolga Zaman^{3✉} & Soofia Iftikhar⁴

This paper offers a novel approach to formulate efficient ratio estimator of the population variance using a transformed auxiliary variable. The impact of transformation on auxiliary information has also been discussed. It is observed that incorporating a transformed auxiliary variable result in a high gain in efficiency. Theoretical properties of the newly developed estimators have been derived. The empirical and simulation studies show that the suggested estimators outperformed the existing estimators.

Keywords Auxiliary variable, Mean square error, Population variance, Percentage relative efficiency

Sampling is a crucial aspect of making well-informed decisions in various real-life domains. Inferences about statistical populations or data are drawn from samples, and it is imperative that a sample accurately represents every characteristic of the population of interest. Data is characterized by specific parameters, and estimating population parameters from a sample poses a challenging task. Two essential measures for specifying data are the measure of location and scale. This article focuses on the latter, specifically the estimation of variance, and a frequently used measure of data scale. Estimating variance is vital in processes where precise quantification of data variation is necessary. From economics and business to physical science, biological science, and environmental sciences, sophisticated tools are required to measure variation for making informed decisions. For instance, economists analyze the variation in commodity prices, manufacturers assess taste preferences for customer satisfaction, and agriculturists study variability in climate factors to optimize yields and minimize costs.

Extensive research has sought to enhance the efficiency of ratio and product estimators for finite population variance, considering the correlation between survey variables and auxiliary variables. Auxiliary variables are those correlated with the main study variable, either positively or negatively. In environmental studies, for example, auxiliary variables like wind speed or temperature are considered when estimating air quality variance. Economic surveys may involve estimating household income variance using employment rates as an auxiliary variable. In healthcare planning, patient demographics or medical histories serve as auxiliary variables when estimating variance in patient recovery times.

Integrating auxiliary variables with the main survey variable to estimate variance provides additional information, refining the accuracy of estimates. This additional information includes contextual data, environmental factors, or supplementary metrics correlated with main study variable. The use of auxiliary information often leads to more efficient and robust variance estimators. Similarly, the transformative role is notable in enhancing estimate efficacy. Transformations make estimators flexible, allowing them to better utilize additional information from auxiliary variables, such as mean, variance, skewness, kurtosis, quantiles, etc. This flexibility, along with generalization and optimization constants, enhances the robustness of estimates against variations in the sample.

That is why various sample survey statisticians preferred transformed auxiliary variables instead of considering them in their original form. Keeping in view the above, numerous researchers developed many efficient estimators of population variance. Some of the pioneer work on estimating variance of finite population using auxiliary variable are due to¹ who proposed an unbiased estimator of variance and² compares variance estimators under various sample designs with available auxiliary information. Illustrates improved bias and mean squared error over common estimators. Similarly³, introduces chain estimators for finite population variance using double

¹Department of Statistics, University of Peshawar, Peshawar, Pakistan. ²Department of Mathematics and Statistics, University of Haripur, Haripur, Pakistan. ³Department of Statistics, Cankiri Karatekin University, Çankırı, Turkey. ⁴Department of Statistics, Shaheed Benazir Bhutto Women University, Peshawar, Pakistan. ✉email: zamantolga@gmail.com

sampling and two auxiliary variables. Compares estimators based on mean square estimator (MSE) criterion. Proposes a ratio-type exponential estimator for population variance, consistently more efficient than previous estimators⁴. Conducts efficiency comparisons mathematically and numerically. Introduces a family of estimators based on adaptations of previous work, showing efficiency through mean square error comparisons⁵. A generalized modified ratio-type estimator for population variance using known parameters of the auxiliary variable was suggested by⁶. Compares with existing estimators for simulated and real data to show the performance of the developed estimator against the competing estimators. Suggests a generalized class of finite population variance, deriving large sample bias and mean square error. Considers special cases and provides numerical examples for comparison⁷. Suggested estimator of population variance utilizing information on two auxiliary variables under SRSWOR scheme⁸. A generalized exponential estimator for estimating population variance using two auxiliary variables was proposed by⁹. Demonstrates efficiency through empirical and simulated studies using real and simulated data. Has suggested efficient formulation of population variance in simple random sampling using supplementary variable¹⁰. Using sear's constants, develop an efficient estimator to estimate the population variance¹¹. The bias and mean squared error of the proposed estimator is obtained up to the first degree of approximation. Suggested a chain ratio type and chain ratio type exponential estimator of variance of finite population using auxiliary information Improved version of the suggested class of estimators is also given along with its properties¹². An empirical study is carried out in support of the findings of the study. Addresses estimation of current population variance in the presence of random non-response¹³. Examines proposed estimators through empirical studies, comparing with estimators for complete response situations. Suggested a class of estimators for finite population variance using an auxiliary attribute¹⁴. Developed variance estimator using the tri-mean and third quartile of the auxiliary variable, demonstrating its superior performance over various competing estimators based on sampling properties, bias, and mean squared error¹⁵. Suggested finite population variance estimator using unconventional measures¹⁶. Demonstrates efficacy and robustness through empirical and simulation studies considering real data sets form various domain of life. Similarly,¹⁷ explores the effect of distribution on suggested variance estimators. Compares twelve estimators across eight distributions through simulation studies. Formulate a Sear's ratio-type estimator using tri-mean and third quartile of the auxiliary variable¹⁸. Demonstrates superiority through bias, mean squared error through theoretical comparison and empirical studies. A hybrid-type estimators of population variance developed by¹⁹, and demonstrated the efficiency over competing estimators through theoretical and empirical comparisons. A generalized family of estimators of population variance is formulated by²⁰ and demonstrated the performance of the estimators through empirical and simulation study. A robust ratio-type estimator for finite population variance suggested by²¹, considering robust covariance matrices. Derive conditions for efficiency against competing estimators and demonstrated the performance of the suggested estimators through empirical and simulation study.

Novelty and significance

This work introduces innovative contributions in the field of survey sampling, with several key aspects. The paper puts forward three novel ratio estimators designed for finite population variance. These estimators consider a transformed auxiliary variable under simple random sampling without replacement. Demonstrated to outperform existing methods, these suggested estimators exhibit superior efficacy when there is a positive correlation between the survey variable and auxiliary variable. In addition to theoretical comparisons of mean squared errors (MSEs), the paper includes an empirical analysis using real data sets. Through a simulation study, it substantiates that the newly proposed estimators consistently outperform competing estimators across various scenarios, such as different correlation and sample size, affirming their high efficiency. The versatility of the proposed estimators is highlighted as they can seamlessly adapted into other sampling methodologies. This includes applications in stratified random sampling, non-response sampling, and adaptive cluster sampling, leading to the derivation of efficient versions of the estimators. The applicability of the proposed estimators extends to diverse fields such as environmental studies, agriculture, and economics, especially in situations where a positive correlation between the study and auxiliary variable exists. The heightened efficiency of these estimators can significantly enhance the accuracy of population variance estimates, offering practical implications for decision-making processes in real-world applications. The proposed estimator can significantly contribute in the estimation of parameters other than variance, such as mean, median, coefficient of variation etc.

In conclusion, this paper significantly contributes to the field of survey sampling by introducing novel estimators for finite population variance. The adaptability of these estimators to various sampling schemes and their practical implications underscores their potential impact on enhancing accuracy in real-world decision-making.

Methodology

Consider a population $\Omega = \{(y_i, x_i)\}, i = 1, 2, \dots, N$. of size N . Suppose a random sample (y_i, z_i) of size n is taken from a population under simple random sampling without replacement case, i-e (SRSWOR). Let (y_i, x_i) be the value of i th unit of the main study and auxiliary variable and $z_i = t(x_i)$ is the transformed auxiliary variable observed on the sample. The supplementary variate (x) is supposed to be correlated positively with the main study variable (y). It is to be noted that the correlation between (y_i, z_i) and between (y_i, x_i) is same. Let $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$, $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ and $\bar{z} = \frac{1}{n} \sum_{i=1}^n z_i$ represents the sample mean of study variable, ancillary variable and transformed ancillary variable.

$\bar{Y} = \frac{1}{N} \sum_{i=1}^N y_i$ and $\bar{X} = \frac{1}{N} \sum_{i=1}^N x_i$ and $\bar{Z} = \frac{1}{N} \sum_{i=1}^N z_i$ be the population mean of variate (y), (x) and (z).

Let us define the random error due to sampling by $\varepsilon_0 = \frac{y - \bar{y}}{S_y^2}$ and $\varepsilon_1 = \frac{x - \bar{x}}{S_x^2}$, such that

$$\left. \begin{aligned}
 &E(\varepsilon_0) = E(\varepsilon_1) = 0 \text{ and} \\
 &E(\varepsilon_0^2) = \lambda(\phi_{40} - 1) = V_{40} \\
 &E(\varepsilon_1^2) = \lambda(\phi_{04} - 1) = V_{04} \\
 &E(\varepsilon_0\varepsilon_1) = \lambda(\phi_{22} - 1) = V_{22} \\
 &\text{where} \\
 &\lambda = 1/n, \phi_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2} \mu_{02}^{s/2}} \\
 &\mu_{rs} = \frac{1}{N} \sum_{i=1}^N (y_i - \bar{Y})^r (x_i - \bar{X})^s
 \end{aligned} \right\} \tag{1}$$

where r and s be the non-negative integer and μ_{rs}, μ_{20} and μ_{02} are the second order moments and ϕ_{rs} is the moment's ratio.

$$\left. \begin{aligned}
 S_y^2 &= \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})^2, s_y^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2 \\
 S_x^2 &= \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{X})^2, S_y^2 = \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2
 \end{aligned} \right\} \tag{2}$$

where C_y^2, C_x^2 are the coefficient of variation of the survey variable and auxiliary variables Y and X respectively, ρ_{yx} is the correlation coefficient between main study variable Y and auxiliary variable X , and $\beta_{(1)x}, \beta_{(2)x}$ are the coefficient of skewness and the coefficient of kurtosis of the auxiliary variables respectively.

In literature, some estimators of the population variance are given as

1. The usual classical estimator of population variance is given by

$$\tau_1 = s_y^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2, \tag{3}$$

where τ_1 is an unbiased estimator. Its variance is as under

$$Var(\tau_1) = S_y^4 V_{40}. \tag{4}$$

2. Developed the estimator of population variance, it is given by²

$$\tau_2 = s_y^2 \left(\frac{S_x^2}{s_x^2} \right), \tag{5}$$

The MSE of τ_2 is as below.

$$MSE(\tau_2) \cong S_y^4 \{V_{40} + V_{04} - 2V_{22}\} \tag{6}$$

3. Provide the estimator of population variance, it is given below²²

$$\tau_3 = s_y^2 \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right), \tag{7}$$

The MSE of τ_3 is given by

$$MSE(\tau_3) \approx S_y^4 \left\{ V_{40} + \frac{V_{04}}{4} - V_{22} \right\}. \tag{8}$$

4. For ratio estimator of population variance, the linear regression estimator developed by² is given by

$$\tau_4 = s_y^2 + b(s_x^2 - s_x^2), \tag{9}$$

here $b = \frac{s_y^2 V_{22}}{s_x^2 V_{40}}$ represents the sample regression coefficient.

The variance of τ_4 is as below

$$Var(\tau_4) = S_y^4 V_{40} \left(1 - \frac{V_{22}^2}{V_{40} V_{04}} \right). \quad (10)$$

5. The transformed estimator by²³ is as follows:

$$\tau_5 = \left\{ k_1 s_y^2 + k_2 (S_x^2 - s_x^2) \right\} \left\{ \theta \left(\frac{aS_x^2 + b}{as_x^2 + b} \right) + (1 - \theta) \exp \left(\frac{a(S_x^2 - s_x^2)}{a(S_x^2 + s_x^2) + 2b} \right) \right\}, \quad (11)$$

here k_1 and k_2 are optimization constants, θ is generalization constants which can takes value between 0 and 1 and “a” and “b” some function of auxiliary variable. The optimum value of k_1 and k_1 are as below

$$k_{1(opt)} = \left(\frac{1 - \frac{1}{8}q^2(1 + 3\theta + 4\theta^2)V_{04}}{1 - \frac{1}{4}q^2\theta(1 + 3\theta)V_{04}^2 + V_{40} \left(1 - \frac{V_{22}^2}{V_{04}V_{40}} \right)} \right)$$

and

$$k_{2(opt)} = \frac{S_y^2}{S_x^2} \left\{ 1 + q(1 + \theta) + k_{1(opt)} \left(\frac{V_{22}}{V_{04}} - q(1 + \theta) \right) \right\},$$

The minimum MSE at $k_{1(opt)}$ and $k_{2(opt)}$ is given by

$$MSE(\tau_5) \simeq S_y^4 \left[\left\{ 1 - \frac{1}{4}q^2(1 + \theta)^2 V_{04} \right\} - \frac{\left\{ 1 - \frac{1}{8}q^2(1 + 3\theta + 4\theta^2)V_{04} \right\}^2}{1 - \frac{1}{4}q^2\theta(1 + 3\theta)V_{04}^2 + V_{40} \left(1 - \frac{V_{22}^2}{V_{04}V_{40}} \right)} \right]. \quad (12)$$

where $q = \frac{aS_x^2}{aS_x^2 + b}$.

The MSE of τ is minimum at $(\theta, a, b) = (1, 1, 0)$.

6. Introduced the following difference-cum-exponential estimator of population variance⁷,

$$\tau_6 = \left[c_{11} s_y^2 + c_{22} (S_x^2 - s_x^2) \right] \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right), \quad (13)$$

where c_{11} and c_{11} are optimization constants. The optimum MSE of τ_6 is given by

$$MSE(\tau_6) = MSE(\tau_4) - \frac{S_y^4 \left[(V_{40} - 1) + 8V_{04} \left(1 - \frac{V_{22}^2}{V_{40}V_{04}} \right) \right]}{64 \left\{ 1 + \left\{ V_{40} - \frac{V_{22}^2}{V_{04}} \right\} \right\}}. \quad (14)$$

7. Developed the following ratio estimator of population variance²⁴

$$\tau_7 = s_y^2 \left[k_{11} \left(\frac{S_x^2}{s_x^2} \right) + k_{12} \left(\frac{s_x^2}{S_x^2} \right) \right] \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \quad (15)$$

The optimum value of k_{11} and k_{12} are given by

$$k_{11OPT} = -\frac{1}{8} \frac{A_1 V_{40} + V_{04} + A_2 + 16V_{22}}{A_3}$$

$$k_{12OPT} = \frac{1}{8} \frac{A_1 V_{40} + 21V_{04} - 3A_2 + 16V_{22}}{A_3},$$

where

$$A_1 = 16(V_{04} - V_{22}),$$

$$A_2 = 8(3V_{04}V_{22} - 2V_{22}^2 + V_{04})$$

and

$$A_3 = V_{04}^2 - 4V_{04}V_{22} - 16V_{04}V_{22} + 16V_{22}^2 - 4V_{04}.$$

The minimum MSE at $k_{11(opt)}$ and $k_{12(opt)}$ is given by

$$MSE(\tau_7)_{\min} \cong \frac{S_y^4}{A_3} \left[\frac{1}{16} \left\{ 16V_{22} \left\{ V_{22}(4V_{40}-3V_{04}+4) + (3V_{04}^2-8V_{40}V_{04}) \right\} + 9V_{04}^3 + 64V_{04}(V_{04}-1)V_{40} \right\} \right]. \tag{16}$$

8. Advocated the following difference-cum-exponential type exponential estimators of population variance given by²⁵

$$\tau_8 = \left[\frac{s_y^2}{2} \left\{ \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) + \exp \left(\frac{s_x^2 - S_x^2}{S_x^2 + s_x^2} \right) \right\} + k_1(S_x^2 - s_x^2) + k_2s_y^2 \right] \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right), \tag{17}$$

$$\tau_9 = \left[s_y^2 \left\{ \alpha \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) + (1 - \alpha) \exp \left(\frac{s_x^2 - S_x^2}{S_x^2 + s_x^2} \right) \right\} \right] \left[\begin{array}{l} \delta \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \\ + (1 - \delta) \exp \left(\frac{s_x^2 - S_x^2}{S_x^2 + s_x^2} \right) \end{array} \right] + k_{11}(S_x^2 - s_x^2) + k_{22}s_y^2 \tag{18}$$

where k_1 and k_2 are optimization constants that minimize the MSE of τ_8 , which is given by

$$MSE(\tau_8) \cong MSE(\tau_4) - \frac{S_y^4 \left\{ \left(V_{40} - \frac{V_{22}^2}{V_{04}} \right) + \frac{1}{4} V_{04}^2 \right\}}{\left\{ 1 + \left(V_{40} - \frac{V_{22}^2}{V_{04}} \right) \right\}}. \tag{19}$$

$$MSE(\tau_9) \cong S_y^4 \left[\left\{ V_{40} + \Omega(V_{04} + 2V_{22}) \right\} - \frac{D_1D_5^2 + D_2D_4^2 - 2D_3D_4D_5}{D_1D_2 - D_3^2} \right]. \tag{20}$$

$$\begin{aligned} D_1 &= R^2V_{04}, \\ D_2 &= 1 + \{V_{04} + 2V_{22}(1 - 2\omega) + \omega^2V_{04}\}, \\ D_3 &= R[V_{22} + (1 - 2\omega)V_{04}], \\ D_4 &= R[V_{22} + \Omega V_{04}] \end{aligned}$$

and

$$D_5 = \left[V_{40} + \left\{ \left\{ 2\Omega + \frac{1 - 2\omega}{2} \right\} V_{22} + \left\{ \alpha\omega + \frac{1 - 2\omega}{2} \right\} V_{04} \right\} \right].$$

9. Suggested a general type of estimator of population variance given by²⁰

$$\tau_{(a,b)} = \left[(\hat{t}_{(a,b)} + k(S_x^2 - s_x^2)) \exp \left(\frac{w_1(\bar{X} - \bar{x})}{\bar{X} + (\omega_1 - 1)\bar{x}} + \frac{w_2(S_x^2 - s_x^2)}{S_x^2 + (\omega_2 - 1)s_x^2} \right) \right], \tag{21}$$

The particular case of estimator $\tau_9^{a,b}$ for estimating variance is obtained by putting $a = 0$, $b = 2, w_1 = 0$ and $w_2 = 1$ and $\omega_2 = 2$ as following

$$\tau_9 = \left[(s_y^2 + k(S_x^2 - s_x^2)) \exp \left(\frac{(S_x^2 - s_x^2)}{S_x^2 + s_x^2} \right) \right], \tag{22}$$

With MSE given by

$$\begin{aligned} MSE(t)_{\min} &= MSE(\hat{t}_{(a,b)}) \\ &= \frac{t_{(a,b)}^2}{n} \frac{\left[\{f_3(a, b)\}^2 - 2f_2(a, b)f_3(a, b)\delta_{03} + (\delta_{04} - 1)\{f_2(a, b)\}^2 \right]}{(\delta_{04} - \delta_{03}^2 - 1)} \end{aligned} \tag{23}$$

where

$$\begin{aligned} f_2(a, b) &= \left\{ a\rho_{XY}C_\gamma + \left(\frac{b}{2} \right) \delta_{21} \right\} \\ f_3(a, b) &= \left\{ a\rho_{12}C_\gamma + \left(\frac{b}{2} \right) (\delta_{22} - 1) \right\}. \end{aligned}$$

Which in case of variance estimator, the MSE τ will induce the following particular case:

$$MSE(\tau_9) = MSE(\tau_1) - \frac{[\{f_3(0, 2)\}^2 - 2f_2(0, 2)V_{40} + (V_{04} - 1)\{f_2(0, 2)\}^2]}{(V_{04} - V_{40}^2 - 1)} \quad (24)$$

Proposed estimators

The first proposed estimator

Motivated by²⁴ and using transformed auxiliary variable, the following class of transformed ratio-product type exponential estimator is suggested

$$\tau_{P1} = s_y^2 \left[c_1 \left(\frac{Z}{z} \right) + c_2 \left(\frac{z}{Z} \right) \right] \exp \left(\frac{Z - z}{Z + z} \right)$$

$$\tau_{P1} = s_y^2 \left[c_1 \left(\frac{\gamma_1 S_x^2 - \gamma_2}{\gamma_1 s_x^2 - \gamma_2} \right) + c_2 \left(\frac{\gamma_1 s_x^2 - \gamma_2}{\gamma_1 S_x^2 - \gamma_2} \right) \right] \exp \left(\frac{\gamma_1 (S_x^2 - s_x^2)}{\gamma_1 (S_x^2 + s_x^2) - 2\gamma_2} \right) \quad (25)$$

where c_1 and c_2 are optimization constants and γ_2 and γ_2 are suitable constants or some function of auxiliary variables.

The second proposed estimator

Motivated by², we can write the proposed estimator as a linear combination of usual ratio and exponential estimators as following

$$\tau_{P2} = \psi_1 \tau_2 + \psi_2 \tau_3$$

$$\tau_{P2} = s_y^2 \left\{ \psi_1 \left(\frac{S_x^2}{s_x^2} \right) + \psi_2 \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2} \right) \right\} \quad (26)$$

ψ_1 , and ψ_2 are optimization constants whose value is to be obtained so that the MSE of τ_{P2} is minimum.

The third proposed estimator

Applying transformation to the auxiliary variable in (26), we can write the third proposed estimator as following

$$\tau_{P3} = s_y^2 \left\{ \psi_3 \left(\frac{Z}{z} \right) + \psi_4 \exp \left(\frac{Z - z}{Z + z} \right) \right\} \quad (27)$$

Or

$$\tau_{P3} = s_x^2 \left\{ \psi_3 \left(\frac{\gamma_1 S_x^2 - \gamma_2}{\gamma_1 s_x^2 - \gamma_2} \right) + \psi_4 \exp \left(\frac{\gamma_1 (S_x^2 - s_x^2)}{\gamma_1 (S_x^2 + s_x^2) - 2\gamma_2} \right) \right\}$$

ψ_3 and ψ_4 are optimization constants whose value is to be determined so that the MSE of τ_{P3} is minimum. It is to be noted that for $\gamma_1 = 1$ and $\gamma_2 = 0$.

The third proposed estimator τ_{P3} given by (26) become equivalent to the second proposed estimator τ_{P2} given by (26).

Similarly, the first proposed estimator τ_{P1} given by (25) become equivalent to τ_{P7} as suggested by Muneer et al.⁹ given by (13).

Theoretical properties of the proposed estimators

This unit aims at, deriving the theoretical properties of the new estimators using the notations given in (1) and (2). Rewriting (25), (26) and (27) respectively in term of error terms, as following

$$\tau_{P1} = S_y^2(1 + \varepsilon_0) [c_1(1 - \pi_k \varepsilon_1 + \pi_k^2 \varepsilon_1^2 + \dots) + c_2(1 + \pi_k \varepsilon_1)] \left(1 - \frac{1}{2} \pi_k \varepsilon_1 + \frac{3}{8} \pi_k^2 \varepsilon_1^2 + \dots \right),$$

$$\tau_{P2} = S_y^2(1 + \varepsilon_0) \left\{ \psi_1(1 - \varepsilon_1 + \varepsilon_1^2 + \dots) + \psi_2 \left(1 - \frac{1}{2} \varepsilon_1 + \frac{3}{8} \varepsilon_1^2 + \dots \right) \right\},$$

and

$$\tau_{P3} = S_y^2(1 + \varepsilon_0) \left\{ \psi_3(1 - \pi_k \varepsilon_1 + \pi_k^2 \varepsilon_1^2 + \dots) + \psi_4 \left(1 - \frac{1}{2} \pi_k \varepsilon_1 + \frac{3}{8} \pi_k^2 \varepsilon_1^2 + \dots \right) \right\},$$

where $\pi_k = \frac{\gamma_1 S_x^2}{\gamma_1 S_x^2 - \gamma_2}$.or

$$\tau_{P1} - S_y^2 \cong S_y^2 \left[c_1 \left(1 + \varepsilon_0 - \pi_k \varepsilon_1 - \frac{3}{2} \pi_k \varepsilon_1 \varepsilon_0 + \frac{15}{8} \pi_k^2 \varepsilon_1^2 \right) + c_2 \left(\frac{1 + \varepsilon_0 + \frac{1}{2} \pi_k \varepsilon_1 +}{\frac{1}{2} \pi_k \varepsilon_1 \varepsilon_0 - \frac{1}{8} \pi_k^2 \varepsilon_1^2} \right) - 1 \right] \tag{28}$$

$$\tau_{P2} - S_y^2 \cong S_y^2 \left[\begin{aligned} &(\psi_1 + \psi_2 - 1) + (\psi_1 + \psi_2) \varepsilon_0 - \left(\psi_1 + \frac{\psi_2}{2} \right) \varepsilon_1 + \left(\psi_1 + \frac{3\psi_2}{8} \right) \varepsilon_1^2 \\ &-\left(\psi_1 + \frac{\psi_2}{2} \right) \varepsilon_0 \varepsilon_1 \end{aligned} \right], \tag{29}$$

and

$$\tau_{P3} - S_y^2 \cong S_y^2 \left[\begin{aligned} &(\psi_3 + \psi_4 - 1) + (\psi_3 + \psi_4) \varepsilon_0 - \pi_k \left(\psi_3 + \frac{\psi_4}{2} \right) \varepsilon_1 + \left(\psi_3 + \frac{3\psi_4}{8} \right) \pi_k^2 \varepsilon_1^2 \\ &-\pi_k \left(\psi_3 + \frac{\psi_4}{2} \right) \varepsilon_0 \varepsilon_1 \end{aligned} \right], \tag{30}$$

Taking expectation of both sides of (28), (29) and (30) respectively, and after simplification we get

$$Bias(\tau_{P1}) \cong S_y^2 \left[\begin{aligned} &(c_1 + c_2 - 1) + c_1 \lambda \left\{ \frac{15}{8} \pi_k^2 (\phi_{04} - 1) - \frac{1}{2} \pi_k (\phi_{02} - 1) \right\} \\ &c_1 \lambda \pi_k \left\{ \frac{1}{2} \pi_k (\phi_{02} - 1) - \frac{1}{8} \pi_k (\phi_{04} - 1) \right\} \end{aligned} \right], \tag{31}$$

$$Bias(\tau_{P2}) \cong S_y^2 \left[(\psi_1 + \psi_2 - 1) + \left(\psi_1 + \frac{3\psi_2}{8} \right) \lambda (\phi_{04} - 1) - \lambda \left(\psi_1 + \frac{\psi_2}{2} \right) (\phi_{22} - 1) \right], \tag{32}$$

$$Bias(\tau_{P3}) \cong S_y^2 \left[(\psi_3 + \psi_4 - 1) + \left(\psi_3 + \frac{3\psi_4}{8} \right) \pi_k^2 \lambda (\phi_{04} - 1) - \pi_k \lambda \left(\psi_3 + \frac{\psi_4}{2} \right) (\phi_{22} - 1) \right], \tag{33}$$

Squaring both sides of (31), (32) and (33) respectively and applying expectation to get the MSE of τ_{P1} , τ_{P2} and τ_{P3} , as following.

$$E(\tau_{P1} - S_y^2)^2 \cong S_y^4 E \left[\begin{aligned} &c_1 \left(1 + \varepsilon_0 - \pi_k \varepsilon_1 - \frac{3}{2} \pi_k \varepsilon_1 \varepsilon_0 + \frac{15}{8} \pi_k^2 \varepsilon_1^2 \right) + \\ &c_2 \left(1 + \varepsilon_0 + \frac{1}{2} \pi_k \varepsilon_1 + \frac{1}{2} \pi_k \varepsilon_1 \varepsilon_0 - \frac{1}{8} \pi_k^2 \varepsilon_1^2 \right) - 1 \end{aligned} \right]^2, \tag{34}$$

$$E(\tau_{P2} - S_y^2)^2 \cong S_y^4 E \left[\begin{aligned} &(\psi_1 + \psi_2 - 1) + (\psi_1 + \psi_2) \varepsilon_0 - \left(\psi_1 + \frac{\psi_2}{2} \right) \varepsilon_1 + \\ &\left(\psi_1 + \frac{3\psi_2}{8} \right) \varepsilon_1^2 - \left(\psi_1 + \frac{\psi_2}{2} \right) \varepsilon_0 \varepsilon_1 \end{aligned} \right]^2, \tag{35}$$

or

$$E(\tau_{P3} - S_y^2)^2 \cong S_y^4 E \left[\begin{aligned} &(\psi_3 + \psi_4 - 1) + (\psi_3 + \psi_4) \varepsilon_0 - \pi_k \left(\psi_3 + \frac{\psi_4}{2} \right) \varepsilon_1 + \\ &\left(\psi_3 + \frac{3\psi_4}{8} \right) \pi_k^2 \varepsilon_1^2 - \pi_k \left(\psi_3 + \frac{\psi_4}{2} \right) \varepsilon_0 \varepsilon_1 \end{aligned} \right]^2, \tag{36}$$

or

$$MSE(\tau_{P1}) \cong S_y^4 \left[\begin{aligned} &(c_1 + c_2 - 1)^2 + (c_1 + c_2)^2 V_{40} + \left\{ \left(\frac{1}{2} (3c_1 - c_2) \right)^2 + \right. \\ &\left. 2(c_1 + c_2 - 1) \left(\frac{15}{8} c_1 - \frac{1}{8} c_2 \right) \right\} \pi_k^2 V_{04} \\ &- (2(c_1 + c_2) - 1)(3c_1 - c_2) \pi_k V_{22} \end{aligned} \right], \tag{37}$$

$$MSE(\tau_{P2}) \cong S_y^4 \left[\begin{array}{l} (\psi_1 + \psi_2 - 1)^2 + (\psi_1 + \psi_2)^2 V_{40} + \left\{ \left(\psi_1 + \frac{\psi_2}{2} \right)^2 + \right. \\ \left. 2(\psi_1 + \psi_2 - 1) \left(\psi_1 + \frac{3\psi_2}{8} \right) \right\} V_{04} \\ -2 \left(\psi_1 + \frac{\psi_2}{2} \right) \{2(\psi_1 + \psi_2) - 1\} V_{22} \end{array} \right] V_{04} \quad (38)$$

and

$$MSE(\tau_{P3}) \cong S_y^4 \left[\begin{array}{l} (\psi_3 + \psi_4 - 1)^2 + (\psi_3 + \psi_4)^2 V_{40} + \left\{ \left(\psi_3 + \frac{\psi_4}{2} \right)^2 + \right. \\ \left. 2(\psi_3 + \psi_4 - 1) \left(\psi_3 + \frac{3\psi_4}{8} \right) \right\} \pi_k^2 V_{04} \\ -2 \left(\psi_3 + \frac{\psi_4}{2} \right) \{2(\psi_3 + \psi_4) - 1\} \pi_k V_{22} \end{array} \right] \cdot \quad (39)$$

The optimum value of c_1 and c_2 is obtain by differentiating (37) w.r.t c_1 and c_2 respectively and equating to zero, as following

$$\frac{\partial}{\partial c_1} MSE(\tau_{P1}) = 0 \Rightarrow c_{1OPT} = -\frac{1}{8} \frac{A_{1,k} V_{40} + V_{04,k} + A_{2,k} + 16V_{22,k}}{A_{3,k}},$$

$$\frac{\partial}{\partial c_2} MSE(\tau_{P1}) = 0 \Rightarrow c_{2OPT} = \frac{1}{8} \frac{A_{1,k} V_{40} + 21V_{04,k} - 3A_{2,k} + 16V_{22,k}}{A_{3,k}},$$

where,

$$A_{1,k} = 16(V_{04,k} - V_{22,k}),$$

$$A_{2,k} = 8(3V_{04,k} V_{22,k} - 2V_{22,k}^2 + V_{04,k}),$$

and

$$A_{3,k} = V_{04,k}^2 - 4V_{04,k} V_{22,k} - 16V_{04,k} V_{22,k} + 16V_{22,k}^2 - 4V_{04,k}.$$

putting the optimum value of c_1 and c_2 in (37), we get

$$MSE(\tau_{P1})_{\min} \cong \frac{S_y^4}{A_{3,k}} \left[\frac{1}{16} \left\{ \frac{16V_{22,k} \{ V_{22,k} (4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40} V_{04,k}) \}}{9V_{04,k}^3 + 64V_{04,k} (V_{04,k} - 1) V_{40}} \right\} + \right] \quad (40)$$

where

$$V_{04,k} = \pi_k^2 V_{04} \text{ and } V_{22,k} = \pi_k V_{22}$$

Similarly, using calculus rule, the optimum value of ψ_1, ψ_2, ψ_3 and ψ_4 can also be obtain by differentiating MSE of τ_{P2} and τ_{P3} w.r.to ψ_1, ψ_2, ψ_3 and ψ_4 and equating to zero. Hence, we obtain after simplification

$$\psi_{1(opt)} = -\frac{19V_{04}^4 + 40V_{04}^2 V_{40}^2 - 60V_{04}^2 V_{22} - 32V_{40}^2 V_{22} + 32V_{22}^2 - 16V_{04}^2 + 32V_{22}}{D},$$

$$\psi_{2(opt)} = \frac{8(6V_{04}^4 + 5V_{04}^2 V_{40}^2 - 15V_{04}^2 V_{22} - 4V_{40}^2 V_{22} + 8\lambda V_{22}^2 - 4V_{04}^2 + 4V_{22})}{D},$$

$$\psi_{3(opt)} = -\frac{19V_{04,k}^4 + 40V_{04,k}^2 V_{40}^2 - 60V_{04,k}^2 V_{22,k} - 32V_{40}^2 V_{22,k} + 32V_{22,k}^2 - 16V_{04,k}^2 + 32V_{22,k}}{D_k}$$

and

$$\psi_{4(opt)} = \frac{8(6V_{04,k}^4 + 5V_{04,k}^2 V_{40}^2 - 15V_{04,k}^2 V_{22,k} - 4V_{40}^2 V_{22,k} + 8\lambda V_{22,k}^2 - 4V_{04,k}^2 + 4V_{22,k})}{D_k}$$

where

$$D = 33V_{04}^4 - 16V_{04}^2 V_{40}^2 - 80V_{04}^2 V_{22} + 64V_{22}^2 - 16V_{04}^2$$

$$D_k = 33V_{04,k}^4 - 16V_{04,k}^2 V_{40}^2 - 80V_{04,k}^2 V_{22,k} + 64V_{22,k}^2 - 16V_{04,k}^2, \quad k = 1, 2, \dots, 6.$$

The MSE is given by

$$MSE(t_{P2}) = \frac{S_y^4}{D} \left(\left(V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2 \right) - 16V_{04}^2V_{40}^2 + 16V_{22}^2 \right) \tag{41}$$

$$MSE(\tau_{P3}) = \frac{S_y^4}{D_k} \left(\left(V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2 \right) - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 \right) \tag{42}$$

Special Cases of τ_{P1} and τ_{P3} in response to the transformation introduced

1. For $k=1$, τ_{P1} and τ_{P3} takes the following form

$$\tau_{P1,1} = s_y^2 \left[c_1 \left(\frac{S_x^2 - \rho_{yx}}{s_x^2 - \rho_{yx}} \right) + c_2 \left(\frac{s_x^2 - \rho_{yx}}{S_x^2 - \rho_{yx}} \right) \right] \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2 - 2\rho_{yx}} \right)$$

$$\tau_{P3,1} = s_y^2 \left\{ \psi_3 \left(\frac{S_x^2 - \rho_{yx}}{s_x^2 - \rho_{yx}} \right) + \psi_4 \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2 - 2\rho_{yx}} \right) \right\}$$

where $\gamma_1 = 1$ and $\gamma_2 = \rho_{yx}$

The bias and MSEs are given by

$$Bias(\tau_{P1,1}) = S_{y(h)}^2 \left[(c_1 + c_2 - 1) + \left(\frac{3}{2}c_1 - \frac{1}{2}c_2 \right) V_{22,1} + \left(\frac{15}{8}c_1 - \frac{1}{8}c_2 \right) V_{04,1} \right], \tag{43}$$

$$Bias(\tau_{P3,1}) \cong S_y^2 \left[(\psi_3 + \psi_4 - 1) + \left(\psi_3 + \frac{3\psi_4}{8} \right) V_{04,1} - \left(\psi_1 + \frac{\psi_2}{2} \right) V_{22,1} \right] \tag{44}$$

$$MSE(\tau_{P1,1})_{\min} \cong \frac{S_y^4}{A_{3,1}} \left[\frac{1}{16} \left\{ 16V_{22,1} \{ V_{22,1} (4V_{40} - 3V_{04,1} + 4) + (3V_{04,1}^2 - 8V_{40}V_{04,1}) \} + \right\} \right] \tag{45}$$

and

$$MSE(\tau_{P3,1}) = \frac{S_y^4}{D_1} \left(\left(V_{04,1}^6 + (25V_{40}^2 - 10V_{22,1}^2)V_{04,1}^4 + V_{04,1}^2(8V_{22,1} - 40V_{22,1}V_{40}^2) + 16V_{22,1}^2V_{40}^2 \right) - 16V_{04,1}^2V_{40}^2 + 16V_{22,1}^2 \right). \tag{46}$$

$$V_{04,1} = \pi_1^2 V_{04} \text{ and } V_{22,1} = \pi_1 V_{22}, \pi_1 = \frac{S_x^2}{S_x^2 - \rho_{yx}}.$$

2. For $k=2$, $\gamma_1 = 1$ and $\gamma_2 = C_x$, τ_{P1} and τ_{P3} will take the following form

$$\tau_{P1,2} = s_y^2 \left[c_1 \left(\frac{S_x^2 - C_x}{s_x^2 - C_x} \right) + c_2 \left(\frac{s_x^2 - C_x}{S_x^2 - C_x} \right) \right] \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2 - 2C_x} \right).$$

$$\tau_{P3,2} = s_y^2 \left\{ \psi_3 \left(\frac{S_x^2 - C_x}{s_x^2 - C_x} \right) + \psi_4 \exp \left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2 - 2C_x} \right) \right\}$$

The bias and MSEs are given by

$$Bias(\tau_{P1,2}) = S_{y(h)}^2 \left[(c_1 + c_2 - 1) + \left(\frac{3}{2}c_1 - \frac{1}{2}c_2 \right) V_{22,2} + \left(\frac{15}{8}c_1 - \frac{1}{8}c_2 \right) V_{04,2} \right], \tag{47}$$

$$Bias(\tau_{P3,2}) \cong S_y^2 \left[(\psi_3 + \psi_4 - 1) + \left(\psi_3 + \frac{3\psi_4}{8} \right) V_{04,2} - \left(\psi_1 + \frac{\psi_2}{2} \right) V_{22,2} \right]. \tag{48}$$

$$MSE(\tau_{P1,2})_{\min} \cong \frac{S_y^4}{A_{3,2}} \left[\frac{1}{16} \left\{ 16V_{22,2} \{ V_{22,2} (4V_{40} - 3V_{04,2} + 4) + (3V_{04,2}^2 - 8V_{40}V_{04,2}) \} + \right\} \right] \tag{49}$$

and

$$MSE(\tau_{P3,2}) = \frac{S_y^4}{D_2} \left(\left(V_{04,2}^6 + (25V_{40}^2 - 10V_{22,2}^2)V_{04,2}^4 + V_{04,2}^2(8V_{22,2} - 40V_{22,2}V_{40}^2) + 16V_{22,2}^2V_{40}^2 \right) - 16V_{04,2}^2V_{40}^2 + 16V_{22,2}^2 \right). \quad (50)$$

where $V_{04,2} = \pi_2^2 V_{04}$ and $V_{22,2} = \pi_2 V_{22}, \pi_2 = \frac{S_x^2}{S_x^2 - C_x}$.

3. For $k=3, \gamma_1 = \rho_{yx}$ and $\gamma_2 = C_x, \tau_{P1}$ and τ_{P3} will take the following form

$$\tau_{P1,3} = s_y^2 \left[c_1 \left(\frac{\rho_{yx} S_x^2 - C_x}{\rho_{yx} S_x^2 - C_x} \right) + c_2 \left(\frac{\rho_{yx} S_x^2 - C_x}{\rho_{yx} S_x^2 - C_x} \right) \right] \exp \left(\frac{\rho_{yx} (S_x^2 - s_x^2)}{\rho_{yx} (S_x^2 + s_x^2) - 2C_x} \right)$$

$$\tau_{P3,3} = s_y^2 \left\{ \psi_3 \left(\frac{\rho_{yx} S_x^2 - C_x}{\rho_{yx} S_x^2 - C_x} \right) + \psi_4 \exp \left(\frac{\rho_{yx} (S_x^2 - s_x^2)}{\rho_{yx} (S_x^2 + s_x^2) - 2C_x} \right) \right\}.$$

The bias along with the mean square error (MSE) is given by

$$Bias(\tau_{P1,3}) = S_y^2 \left[(c_1 + c_2 - 1) + \left(\frac{3}{2}c_1 - \frac{1}{2}c_2 \right) V_{22,3} + \left(\frac{15}{8}c_1 - \frac{1}{8}c_2 \right) V_{04,3} \right], \quad (51)$$

$$Bias(\tau_{P3,3}) \cong S_y^2 \left[(\psi_3 + \psi_4 - 1) + \left(\psi_3 + \frac{3\psi_4}{8} \right) V_{04,3} - \left(\psi_1 + \frac{\psi_2}{2} \right) V_{22,3} \right] \quad (52)$$

$$MSE(\tau_{P1,3})_{\min} \cong \frac{S_y^4}{A_{3,3}} \left[\frac{1}{16} \left\{ 16V_{22,3} \{ V_{22,3} (4V_{40} - 3V_{04,3} + 4) + (3V_{04,3}^2 - 8V_{40}V_{04,3}) \} + \right\} \right] \quad (53)$$

and

$$MSE(\tau_{P3,3}) = \frac{S_y^4}{D_3} \left(\left(V_{04,3}^6 + (25V_{40}^2 - 10V_{22,3}^2)V_{04,3}^4 + V_{04,3}^2(8V_{22,3} - 40V_{22,3}V_{40}^2) + 16V_{22,3}^2V_{40}^2 \right) - 16V_{04,3}^2V_{40}^2 + 16V_{22,3}^2 \right). \quad (54)$$

where $V_{04,3} = \pi_3^2 V_{04}$ and $V_{22,3} = \pi_3 V_{22}, \pi_3 = \frac{\rho_{yx} S_x^2}{\rho_{yx} S_x^2 - C_x}$.

4. For $k=4, \gamma_1 = C_x$ and $\gamma_2 = \rho_{yx}$. the estimators τ_{P1} and τ_{P3} will take the following form

$$\tau_{P1,4} = s_y^2 \left[c_1 \left(\frac{C_x S_x^2 - \rho_{yx}}{C_x S_x^2 - \rho_{yx}} \right) + c_2 \left(\frac{C_x S_x^2 - \rho_{yx}}{C_x S_x^2 - \rho_{yx}} \right) \right] \exp \left(\frac{C_x (S_x^2 - s_x^2)}{C_x (S_x^2 + s_x^2) - 2\rho_{yx}} \right)$$

$$\tau_{P3,4} = s_y^2 \left\{ \psi_3 \left(\frac{C_x S_x^2 - \rho_{yx}}{C_x S_x^2 - \rho_{yx}} \right) + \psi_4 \exp \left(\frac{C_x (S_x^2 - s_x^2)}{C_x (S_x^2 + s_x^2) - 2\rho_{yx}} \right) \right\}.$$

The bias and MSEs are obtained as

$$Bias(\tau_{P1,4}) = S_y^2 \left[(c_1 + c_2 - 1) + \left(\frac{3}{2}c_1 - \frac{1}{2}c_2 \right) V_{22,4} + \left(\frac{15}{8}c_1 - \frac{1}{8}c_2 \right) V_{04,4} \right], \quad (55)$$

$$Bias(\tau_{P3,4}) \cong S_y^2 \left[(\psi_3 + \psi_4 - 1) + \left(\psi_3 + \frac{3\psi_4}{8} \right) V_{04,4} - \left(\psi_1 + \frac{\psi_2}{2} \right) V_{22,4} \right] \quad (56)$$

$$MSE(\tau_{P1,4})_{\min} \cong \frac{S_y^4}{A_{3,4}} \left[\frac{1}{16} \left\{ 16V_{22,4} \{ V_{22,4} (4V_{40} - 3V_{04,4} + 4) + (3V_{04,4}^2 - 8V_{40}V_{04,4}) \} + \right\} \right] \quad (57)$$

and

$$MSE(\tau_{P3,4}) = \frac{S_y^4}{D_4} \left(\left(V_{04,4}^6 + (25V_{40}^2 - 10V_{22,4}^2)V_{04,4}^4 + V_{04,4}^2(8V_{22,4} - 40V_{22,4}V_{40}^2) + 16V_{22,4}^2V_{40}^2 \right) - 16V_{04,4}^2V_{40}^2 + 16V_{22,4}^2 \right). \quad (58)$$

where $V_{04,1} = \pi_1^2 V_{04}$ and $V_{22,1} = \pi_1 V_{22}, \pi_4 = \frac{C_x S_x^2}{C_x S_x^2 - \rho_{yx}}$.

5. For $k=5, \gamma_1 = \bar{X}$ and $\gamma_2 = \rho_{yx}$, the estimators τ_{P1} and τ_{P3} will take the following form

$$\begin{aligned} \tau_{P1,5} &= s_y^2 \left[c_1 \left(\frac{\bar{X}S_x^2 - \rho_{yx}}{\bar{X}S_x^2 - \rho_{yx}} \right) + c_2 \left(\frac{\bar{X}s_x^2 - \rho_{yx}}{\bar{X}S_x^2 - \rho_{yx}} \right) \right] \exp \left(\frac{\bar{X}(S_x^2 - s_x^2)}{\bar{X}(S_x^2 + s_x^2) - 2\rho_{yx}} \right) \\ \tau_{P3,5} &= s_y^2 \left\{ \psi_3 \left(\frac{\bar{X}S_x^2 - \rho_{yx}}{\bar{X}S_x^2 - \rho_{yx}} \right) + \psi_4 \exp \left(\frac{\bar{X}(S_x^2 - s_x^2)}{\bar{X}(S_x^2 + s_x^2) - 2\rho_{yx}} \right) \right\}. \end{aligned}$$

The bias and MSEs are given by

$$Bias(\tau_{P1,5}) = S_y^2 \left[(c_1 + c_2 - 1) + \left(\frac{3}{2}c_1 - \frac{1}{2}c_2 \right) V_{22,5} + \left(\frac{15}{8}c_1 - \frac{1}{8}c_2 \right) V_{04,5} \right], \tag{59}$$

$$Bias(\tau_{P3,5}) \cong S_y^2 \left[(\psi_3 + \psi_4 - 1) + \left(\psi_3 + \frac{3\psi_4}{8} \right) V_{04,5} - \left(\psi_1 + \frac{\psi_2}{2} \right) V_{22,5} \right] \tag{60}$$

$$MSE(\tau_{P1,5})_{\min} \cong \frac{S_y^4}{A_{3,5}} \left[\frac{1}{16} \left\{ 16V_{22,5} \{ V_{22,5} (4V_{40} - 3V_{04,5} + 4) + (3V_{04,5}^2 - 8V_{40}V_{04,5}) \} + \right\} \right] \tag{61}$$

and

$$MSE(\tau_{P3,5}) = \frac{S_y^4}{D_5} \left(\left(V_{04,5}^6 + (25V_{40}^2 - 10V_{22,5}^2) V_{04,5}^4 + V_{04,5}^2 (8V_{22,5} - 40V_{22,5}V_{40}^2) + 16V_{22,5}^2 V_{40}^2 \right) - 16V_{04,5}^2 V_{40}^2 + 16V_{22,5}^2 \right). \tag{62}$$

where $V_{04,5} = \pi_5^2 V_{04}$ and $V_{22,5} = \pi_5 V_{22}$ and $\pi_5 = \frac{\bar{X}S_x^2}{\bar{X}S_x^2 - \rho_{yx}}$.

6. For $k=6, \gamma_1 = \rho_{yx}$ and $\gamma_2 = 1$

$$\begin{aligned} \tau_{P1,6} &= s_y^2 \left[c_1 \left(\frac{\rho_{yx}S_x^2 - 1}{\rho_{yx}S_x^2 - 1} \right) + c_2 \left(\frac{\rho_{yx}s_x^2 - 1}{\rho_{yx}S_x^2 - 1} \right) \right] \exp \left(\frac{\rho_{yx}(S_x^2 - s_x^2)}{\rho_{yx}(S_x^2 + s_x^2) - 2} \right) \\ \tau_{P3,6} &= s_y^2 \left\{ \psi_3 \left(\frac{\rho_{yx}S_x^2 - 1}{\rho_{yx}S_x^2 - 1} \right) + \psi_4 \exp \left(\frac{\rho_{yx}(S_x^2 - s_x^2)}{\rho_{yx}(S_x^2 + s_x^2) - 2} \right) \right\}. \end{aligned}$$

The bias and MSEs are given by

$$Bias(\tau_{P1,6}) = S_y^2 \left[(c_1 + c_2 - 1) + \left(\frac{3}{2}c_1 - \frac{1}{2}c_2 \right) V_{22,6} + \left(\frac{15}{8}c_1 - \frac{1}{8}c_2 \right) V_{04,6} \right], \tag{63}$$

$$Bias(\tau_{P3,6}) \cong S_y^2 \left[(\psi_3 + \psi_4 - 1) + \left(\psi_3 + \frac{3\psi_4}{8} \right) V_{04,6} - \left(\psi_1 + \frac{\psi_2}{2} \right) V_{22,6} \right] \tag{64}$$

$$MSE(\tau_{P1,6})_{\min} \cong \frac{S_y^4}{A_{3,6}} \left[\frac{1}{16} \left\{ 16V_{22,6} \{ V_{22,6} (4V_{40} - 3V_{04,6} + 4) + (3V_{04,6}^2 - 8V_{40}V_{04,6}) \} + \right\} \right] \tag{65}$$

and

$$MSE(\tau_{P3,6}) = \frac{S_y^4}{D_6} \left(\left(V_{04,6}^6 + (25V_{40}^2 - 10V_{22,6}^2) V_{04,6}^4 + V_{04,6}^2 (8V_{22,6} - 40V_{22,6}V_{40}^2) + 16V_{22,6}^2 V_{40}^2 \right) - 16V_{04,6}^2 V_{40}^2 + 16V_{22,6}^2 \right). \tag{66}$$

where $V_{04,6} = \pi_6^2 V_{04}$ and $V_{22,6} = \pi_6 V_{22}$ and $\pi_{6(h)} = \frac{\rho_{(h)}S_{w_x(h)}^2}{\rho_{(h)}S_{w_x(h)}^2 - 1}$. where $V_{04,6} = \pi_6^2 V_{04}$ and $V_{22,6} = \pi_6 V_{22}$. $\pi_6 = \frac{S_x^2}{S_x^2 - 1}$.

Efficiency comparisons

This section aims to compare the MSEs of the newly developed estimators with the competing estimators discussed in the literature.

Condition 1. By using (37), (38), (39) and (4) we can write

$$MSE(\tau_{Pi}) - Var(\tau_1) < 0, \quad i = 1, 2, 3.$$

$$\text{For } i = 1 \frac{S_y^4}{A_{3,k}} \left[\frac{1}{16} \left\{ 16V_{22,k} \{ V_{22,k} (4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k}) \} + \right\} \right] < 0$$

$$\text{For } i = 2 \frac{S_y^4}{D} \left(\left(\frac{V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2}{V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2} \right) - 16V_{04}^2V_{40}^2 + 16V_{22}^2 - DV_{40} \right) < 0.$$

$$\text{For } i = 3 \frac{S_y^4}{D_k} \left(\left(\frac{V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2}{V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2} \right) - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 - D_kV_{40} \right).$$

Condition 2. By using (37), (38), (39) and (6) we can write $MSE(\tau_{Pi}) - MSE(\tau_2) < 0, i = 1, 2, 3.$

$$\text{For } i = 1 \frac{S_y^4}{A_{3,k}} \left[\frac{1}{16} \left\{ 16V_{22,k} \left\{ V_{22,k} (4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k}) \right\} + 9V_{04,k}^3 + 64V_{04,k}(V_{04,k} - 1)V_{40} - 16A_{3,k}(V_{40} + V_{04} - 2V_{22}) \right\} \right] < 0.$$

$$\text{For } i = 2 \frac{S_y^4}{D} \left(\frac{(V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2) - 16V_{04}^2V_{40}^2 + 16V_{22}^2 - D(V_{40} + V_{04} - 2V_{22})}{16V_{04}^2V_{40}^2 + 16V_{22}^2 - D(V_{40} + V_{04} - 2V_{22})} \right) < 0.$$

$$\text{For } i = 3 \frac{S_y^4}{D_k} \left(\frac{(V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2) - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 - D_k(V_{40} + V_{04} - 2V_{22})}{+16V_{22,k}^2V_{40}^2} \right) < 0.$$

Condition 3. By using (37), (38), (39) and (8) we can write. $MSE(\tau_{Pi}) - MSE(\tau_3) < 0, i = 1, 2, 3.$

$$\text{For } i = 1 \frac{S_y^4}{A_{3,k}} \left[\frac{1}{16} \left\{ 16V_{22,k} \left\{ V_{22,k} (4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k}) \right\} + 9V_{04,k}^3 + 64V_{04,k}(V_{04,k} - 1)V_{40} - 16A_{3,k}(V_{40} + \frac{V_{04}}{4} - V_{22}) \right\} \right] < 0.$$

$$\text{For } i = 2 \frac{S_y^4}{D} \left(\frac{(V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2) - 16V_{04}^2V_{40}^2 + 16V_{22}^2 - D(V_{40} + \frac{V_{04}}{4} - V_{22})}{(V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2)} \right) < 0.$$

$$\text{For } i = 3 \frac{S_y^4}{D_k} \left(\frac{(V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2) - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 - D_k(V_{40} + \frac{V_{04}}{4} - V_{22})}{+16V_{22,k}^2V_{40}^2} \right) < 0.$$

Condition 4. By using (37), (38), (39) and (10) we can write. $MSE(\tau_{Pi}) - MSE(\tau_4) < 0, i = 1, 2, 3.$

$$\text{For } i = 1 \frac{S_y^4}{A_{3,k}} \left[\frac{1}{16} \left\{ 16V_{22,k} \left\{ V_{22,k} (4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k}) \right\} + 9V_{04,k}^3 + 64V_{04,k}(V_{04,k} - 1)V_{40} - 16A_{3,k}V_{40} \left(1 - \frac{V_{22}^2}{V_{40}V_{04}} \right) \right\} \right] < 0.$$

$$\text{For } i = 2 \frac{S_y^4}{D} \left(\frac{(V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2) - 16V_{04}^2V_{40}^2 + 16V_{22}^2 - DV_{40} \left(1 - \frac{V_{22}^2}{V_{40}V_{04}} \right)}{(V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2)} \right) < 0.$$

$$\text{For } i = 3 \frac{S_y^4}{D_k} \left(\frac{(V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2) - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 - D_kV_{40} \left(1 - \frac{V_{22}^2}{V_{40}V_{04}} \right)}{+16V_{22,k}^2V_{40}^2} \right) < 0.$$

Condition 5. By using (37), (38), (39) and (12) we can write $MSE(\tau_{Pi}) - MSE(\tau_5) < 0, i = 1, 2, 3.$

$$\text{For } i = 1 \frac{S_y^4}{A_{3,k}} \left[\frac{1}{16} \left\{ 16V_{22,k} \left\{ V_{22,k} (4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k}) \right\} + 9V_{04,k}^3 + 64V_{04,k}(V_{04,k} - 1)V_{40} - 16A_{3,k} \left(\frac{\left\{ 1 - \frac{1}{4}q^2(1+\theta)^2V_{04} \right\} - \left\{ 1 - \frac{1}{8}q^2(1+3\theta+4\theta^2)V_{04} \right\}^2}{1 - \frac{1}{4}q^2\theta(1+3\theta)V_{04}^2 + V_{40} \left(1 - \frac{V_{22}^2}{V_{04}V_{40}} \right)} \right) \right\} \right] < 0.$$

$$\begin{aligned} \text{For } i=2 \frac{S_y^4}{D} & \left(\left(\frac{V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2}{V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2} - 16V_{04}^2V_{40}^2 + 16V_{22}^2 - \right. \right. \\ & \left. \left. D \left(\left\{ 1 - \frac{1}{4}q^2(1 + \theta)^2V_{04} \right\} - \frac{\left\{ 1 - \frac{1}{8}q^2(1 + 3\theta + 4\theta^2)V_{04} \right\}^2}{1 - \frac{1}{4}q^2\theta(1 + 3\theta)V_{04}^2 + V_{40}\left(1 - \frac{V_{22}^2}{V_{04}V_{40}}\right)} \right) \right) < 0. \\ \text{For } i=3 \frac{S_y^4}{D_k} & \left(\left(\frac{V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2}{V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2} - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 \right. \right. \\ & \left. \left. -D_k \left\{ 1 - \frac{1}{4}q^2(1 + \theta)^2V_{04} \right\} - \frac{\left\{ 1 - \frac{1}{8}q^2(1 + 3\theta + 4\theta^2)V_{04} \right\}^2}{1 - \frac{1}{4}q^2\theta(1 + 3\theta)V_{04}^2 + V_{40}\left(1 - \frac{V_{22}^2}{V_{04}V_{40}}\right)} \right) < 0. \end{aligned}$$

Condition 6. By using Eqs. (37), (38), (39) and (14) we can write $MSE(\tau_{pi}) - MSE(\tau_6) < 0, i=1,2,3.$

$$\begin{aligned} & 16V_{22,k}\{V_{22,k}(4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k})\} + \\ & 9V_{04,k}^3 + 64V_{04,k}(V_{04,k} - 1)V_{40} - \\ \text{For } i=1. & A_{3,k} \left(16MSE(\tau_4) - \frac{\left[(V_{40} - 1) + 8V_{04}\left(1 - \frac{V_{22}^2}{V_{40}V_{04}}\right) \right]}{4\left\{ 1 + \left\{ V_{40} - \frac{V_{22}^2}{V_{04}} \right\} \right\}} \right) < 0 \\ & \left(\frac{V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2}{V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2} - 16V_{04}^2V_{40}^2 + 16V_{22}^2 - \right. \\ \text{For } i=2. & D \left(MSE(\tau_4) - \frac{\left[(V_{40} - 1) + 8V_{04}\left(1 - \frac{V_{22}^2}{V_{40}V_{04}}\right) \right]}{64\left\{ 1 + \left\{ V_{40} - \frac{V_{22}^2}{V_{04}} \right\} \right\}} \right) < 0 \\ & \left(\frac{V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2}{V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2} - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 \right. \\ \text{For } i=3 & \left. -D_k \left(MSE(\tau_4) - \frac{\left[(V_{40} - 1) + 8V_{04}\left(1 - \frac{V_{22}^2}{V_{40}V_{04}}\right) \right]}{64\left\{ 1 + \left\{ V_{40} - \frac{V_{22}^2}{V_{04}} \right\} \right\}} \right) < 0. \end{aligned}$$

Condition 7. By using (37), (38), (39) and (16) we can write $MSE(\tau_{pi}) - MSE(\tau_7) < 0, i=1,2,3.$

$$\begin{aligned} & 16V_{22,k}\{V_{22,k}(4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k})\} + \\ & 9V_{04,k}^3 + 64V_{04,k}(V_{04,k} - 1)V_{40} - \\ \text{For } i=1. & -\frac{A_{3,k}}{A_3} \left[\left\{ \frac{16V_{22}\{V_{22}(4V_{40} - 3V_{04} + 4) + (3V_{04}^2 - 8V_{40}V_{04})\}}{+9V_{04}^3 + 64V_{04}(V_{04} - 1)V_{40}} \right\} \right] < 0 \\ & \left(\frac{V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2}{V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2} - 16V_{04}^2V_{40}^2 + 16V_{22}^2 - \right. \\ \text{For } i=2. & \frac{D}{16A_3} \left[\left\{ \frac{16V_{22}\{V_{22}(4V_{40} - 3V_{04} + 4) + (3V_{04}^2 - 8V_{40}V_{04})\}}{+9V_{04}^3 + 64V_{04}(V_{04} - 1)V_{40}} \right\} \right] < 0 \\ & \left(\frac{V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2}{V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2} - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 \right. \\ \text{For } i=3. & \left. -\frac{D_k}{16A_3} \left[\left\{ \frac{16V_{22}\{V_{22}(4V_{40} - 3V_{04} + 4) + (3V_{04}^2 - 8V_{40}V_{04})\}}{+9V_{04}^3 + 64V_{04}(V_{04} - 1)V_{40}} \right\} \right] < 0. \end{aligned}$$

Condition 8. By using (37), (38), (39) and (19), we can write $MSE(\tau_{pi}) - MSE(\tau_8) < 0, i=1,2,3.$

$$\begin{aligned} & 16V_{22,k}\{V_{22,k}(4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k})\} \\ & +9V_{04,k}^3 + 64V_{04,k}(V_{04,k} - 1)V_{40} - \\ \text{For } i=1. & -A_{3,k} \left\{ \frac{MSE(\tau_4)}{S_y^4} - \frac{\left\{ \left(V_{40} - \frac{V_{22}^2}{V_{04}} \right) + \frac{1}{4}V_{04}^2 \right\}}{\left\{ 1 + \left(V_{40} - \frac{V_{22}^2}{V_{04}} \right) \right\}} \right\} < 0 \end{aligned}$$

$$\left(\frac{V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2}{V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2} \right) - 16V_{04}^2V_{40}^2 + 16V_{22}^2$$

For $i=2$.

$$-D \left\{ \frac{MSE(\tau_4)}{S_y^4} - \frac{\left\{ \left(V_{40} - \frac{V_{22}^2}{V_{04}} \right) + \frac{1}{4} V_{04}^2 \right\}}{\left\{ 1 + \left(V_{40} - \frac{V_{22}^2}{V_{04}} \right) \right\}} \right\} < 0$$

$$\left(\frac{V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2}{V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2} \right) - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2$$

For $i=3$

$$-D_k \left\{ \frac{MSE(\tau_4)}{S_y^4} - \frac{\left\{ \left(V_{40} - \frac{V_{22}^2}{V_{04}} \right) + \frac{1}{4} V_{04}^2 \right\}}{\left\{ 1 + \left(V_{40} - \frac{V_{22}^2}{V_{04}} \right) \right\}} \right\} < 0.$$

Condition 9. By using (37), (38), (39) and (20), we can write $MSE(\tau_{pi}) - MSE(\tau_9) < 0, i=1,2,3$.

For $i=1$

$$\frac{S_y^4}{A_{3,k}} \left\{ \left[\begin{aligned} &16V_{22,k} \{ V_{22,k} (4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k}) \} + \\ &9V_{04,k}^3 + 64V_{04,k} (V_{04,k} - 1) V_{40} \\ &-A_{3,k} \left[\{ V_{40} + \Omega(\Omega V_{04} + 2V_{22}) \} - \frac{D_1D_5^2 + D_2D_4^2 - 2D_3D_4D_5}{D_1D_2 - D_3^2} \right] \end{aligned} \right] \right\} < 0.$$

For $i=2$

$$\frac{S_y^4}{D} \left(\left(\frac{V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2}{V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2} \right) - 16V_{04}^2V_{40}^2 + 16V_{22}^2 \right) - D \left[\{ V_{40} + \Omega(\Omega V_{04} + 2V_{22}) \} - \frac{D_1D_5^2 + D_2D_4^2 - 2D_3D_4D_5}{D_1D_2 - D_3^2} \right] < 0.$$

For $i=3$

$$\frac{S_y^4}{D_k} \left(\left(\frac{V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2}{V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2} \right) - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 \right) - D_k \left[\{ V_{40} + \Omega(\Omega V_{04} + 2V_{22}) \} - \frac{D_1D_5^2 + D_2D_4^2 - 2D_3D_4D_5}{D_1D_2 - D_3^2} \right] < 0.$$

Condition 10. By using (37), (38), (39) and (24), we can write $MSE(\tau_{pi}) - MSE(\tau_{10}) < 0, i=1,2,3$.

For $i=1$

$$\frac{S_y^4}{A_{3,k}} \left\{ \left[\begin{aligned} &16V_{22,k} \{ V_{22,k} (4V_{40} - 3V_{04,k} + 4) + (3V_{04,k}^2 - 8V_{40}V_{04,k}) \} + \\ &9V_{04,k}^3 + 64V_{04,k} (V_{04,k} - 1) V_{40} \\ &-A_{3,k} \left[1 - \frac{\{f_3(0, 2)\}^2 - 2f_2(0, 2)V_{40} + (V_{04} - 1)\{f_2(0, 2)\}^2}{(V_{04} - V_{40}^2 - 1)} \right] \end{aligned} \right] \right\} < 0.$$

Summary statistics	Population I: Source ²⁶ Y: Amount (tons) of recyclable waste collection in Italy in 2003 X: Number of inhabitants in 2003	Population II: Source ²³ Y: The leaf area for the newly developed strain of wheat X: Weight of leaves	Population III: Source ²⁷ Y: weekly expenditure of food X: size of persons	Population IV: Source ²⁸ Y: Total amount of recyclable-waste collection in Italy (2003) X: Total amount of recyclable-waste collection in Italy (2002)
\bar{Y}	51.82	26.84	27.49091	62.62
\bar{X}	11.26	106.20	3.7272	556.55
N	80	39	33	103
n	20	14	9	40
S_y	18.35	6.24	9.976094	91.35
S_x	8.45	11.14	1.500198	610.16
ϕ_{40}	2.26	2.26	5.72	37.12
ϕ_{04}	2.86	2.99	2.380	17.87
ϕ_{22}	2.22	2.40	1.43	17.22
ρ_{yx}	0.94	0.93	0.4237	0.72

Table 1. Summary statistics of various data sets.

$$\text{For } i = 2 \frac{S_y^4}{D} \left(\left(\begin{aligned} &V_{04}^6 + (25V_{40}^2 - 10V_{22}^2)V_{04}^4 + \\ &V_{04}^2(8V_{22} - 40V_{22}V_{40}^2) + 16V_{22}^2V_{40}^2 \end{aligned} \right) - 16V_{04}^2V_{40}^2 + 16V_{22}^2 \right) < 0.$$

$$\text{For } i = 3 \frac{S_y^4}{D_k} \left(\left(\begin{aligned} &V_{04,k}^6 + (25V_{40}^2 - 10V_{22,k}^2)V_{04,k}^4 + \\ &V_{04,k}^2(8V_{22,k} - 40V_{22,k}V_{40}^2) + 16V_{22,k}^2V_{40}^2 \end{aligned} \right) - 16V_{04,k}^2V_{40}^2 + 16V_{22,k}^2 \right) < 0.$$

The condition mentioned above always hold true for all type of real data where the correlation is positive between the main study variable and supplementary variable.

Empirical analysis

This section aims to investigate the performance of the proposed estimators against the competing estimators using data from some real-life situations. Table 1 consists of summary statistics of various datasets.

The percentage relative efficiency (PRE) of all estimators discussed in the literature against τ_1 has been used as performance index. The formula of PRE is given below

$$PRE = \frac{Var(\tau_1)}{MSE(\tau_i) \text{ or } MSE(\tau_{p_j,k})} \times 100 \tag{67}$$

where, $i = 2, 3, \dots, 10$. $J = 1, 2, 3$. and $k = 1, 2, \dots, 6$. are denoted by existing estimators and proposed estimators.

It is obvious from the Table 2 that the newly transformed estimators always perform well than existing estimators for all real data sets. The transformation introduced results high gain in efficiency. The first proposed estimator given by (25) is more efficient than the parent estimator suggested by²⁴, moreover it also outperforms all the competing estimators discussed in the literature. The second proposed estimator given by (26) is more efficient even than our first proposed estimators and all other competing estimators. Further, incorporating the transformed auxiliary variable in the second proposed estimator generates the third proposed estimators given by (27) which are more efficient than all other estimators.

Simulation study

The simulation study of the suggested and existing estimators is conducted to assess the performance of both suggested and existing estimators. Three different populations of size 10,000 have been generated using positive correlation between main study and auxiliary variables. The intensity of correlation between the main study and the supplementary variable is high, moderate, and low in the first, the second and the third population respectively.

Estimator		Population no.			
		1	2	3	4
Conventional estimators	τ_1	100.000	100.000	100.000	100.000
	τ_2	185.2941	280.000	102.4253	175.7390
	τ_3	249.5049	352.4475	101.1806	149.7570
	τ_4	274.0411	458.0562	102.9216	175.9545
	τ_5	195.2023	290.7025	124.2060	183.5658
	τ_6	289.34199	473.2631	153.7084	257.5825
	τ_7	336.50648	663.02559	159.1806	178.7570
	τ_8	302.428041	584.1095	169.9411	196.2725
	τ_9	236.64728	203.75595	102.7552	115.8661
	τ_{10}	269.8926	430.5612	101.5689	171.9047
Proposed estimators	τ_{p11}	339.83799	671.81178	171.4423	278.0721
	τ_{p13}	339.60901	664.96302	174.8473	279.5321
	τ_{p16}	341.674241	674.76219	169.2539	296.2645
	τ_{p2}	455.9280	1312.5474	218.2653	4059.0256
	τ_{p32}	456.2014	1421.83896	220.6593	4066.6241
	τ_{p34}	457.8603	1340.99191	216.2530	4080.5010
	τ_{p35}	278.8679	1314.1116	245.8746	4075.9032

Table 2. PRE of proposed and competing estimators against the usual estimator τ_1 .

Estimators	Population								
	1			2			3		
	n = 50	n = 150	n = 300	n = 50	n = 150	n = 300	n = 50	n = 150	n = 300
τ_1	100.00	100.000	100.00	100.00	100.00	100.00	100.00	100.00	100.00
τ_2	195.36	198.9839	201.47	101.55	102.46	100.37	104.84	102.95	103.30
τ_3	93.38	97.6381	101.63	110.53	104.32	102.56	103.45	101.36	102.49
τ_4	212.54	209.3710	218.63	115.63	108.37	111.35	105.25	108.54	106.26
τ_5	223.58	201.2898	219.35	123.92	119.73	113.27	110.47	107.16	108.57
τ_6	235.64	208.7391	202.63	120.46	123.18	115.27	115.65	107.16	110.75
τ_7	238.29	230.4728	224.38	139.47	130.36	128.38	112.59	110.47	108.73
τ_8	235.73	202.3729	223.24	130.28	121.46	119.37	110.45	108.38	106.39
τ_9	232.27	228.4732	214.73	137.47	128.56	126.18	107.38	106.75	106.95
τ_9	198.76	203.2417	215.76	119.78	108.45	110.56	104.24	107.96	104.22
$\tau_{P1,1}$	241.47	241.66	240.63	141.46	138.86	131.25	119.27	111.45	112.45
$\tau_{P1,4}$	242.45	240.25	238.58	140.48	132.45	131.57	115.26	110.47	110.46
$\tau_{P1,6}$	241.35	235.76	226.65	139.73	131.46	129.45	112.46	11.52	108.98
τ_{P2}	260.85	262.37	275.46	185.36	189.83	190.89	135.85	136.86	141.86
$\tau_{P3,1}$	262.46	265.77	275.85	191.92	195.53	197.04	136.95	139.18	140.47
$\tau_{P3,3}$	265.66	266.75	276.85	195.84	195.95	198.90	140.43	142.65	145.96
$\tau_{P3,5}$	262.65	263.75	275.75	186.53	189.54	190.66	136.06	136.93	142.47
$\tau_{P3,6}$	271.46	270.01	276.66	197.59	197.49	198.45	140.94	142.95	147.88

Table 3. Simulation results for the percent relative efficiencies of the newly developed estimators and conventional estimators with respect to the usual estimator considering different sample sizes and correlation coefficients.

$$\text{Population 1 } \mu = [5.0 \ 10.0] \Sigma = \begin{bmatrix} 9.0 & 2.9 \\ 2.9 & 1.0 \end{bmatrix} \rho = 0.9665.$$

$$\text{Population 2 } \mu = [5.0 \ 10.0] \Sigma = \begin{bmatrix} 10.0 & 3.0 \\ 3.0 & 1.5 \end{bmatrix} \rho = 0.7559.$$

$$\text{Population 3 } \mu = [5.0 \ 10.0] \Sigma = \begin{bmatrix} 10.0 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \rho = 0.3983.$$

We consider sample of sizes $n = 50, 150, 300$ are consider for each population, using simple random sampling without replacement approach. The steps below summarize the whole simulation procedure in R-Studio.

Step 1: Population is generated using Bivariate normal distribution with mean vector $\mu = c(5.0 \ 10.0)$ and $\Sigma = \begin{bmatrix} 9.0 & 2.9 \\ 2.9 & 1.0 \end{bmatrix}$ and calculated all parameters of auxiliary variable and constant from the data.

Step 2: From each of the target population, samples of $n = 50, 150, 300$ have been drawn using SRSWOR method respectively, allow the loop to 100,000 times, select the sample, and calculated the estimates in each iteration.

Step 3: Utilizing samples generated in step 2, the 100,000 values of $\tau_i, i = 1, 2, \dots, 10$. and $\tau_{pjk}, j = 1, 2, 3$ and $k = 1, 2, \dots, 6$, are obtained separately, using (4–20) and (23), (34) and (35), respectively.

Step 4: The estimates obtained from each iteration are store in a matrix and calculate the percent PRE averaging over all iteration of each estimates by using the formula (67), and report the results in Table 3.

The simulation results indicate that the newly developed estimators are highly efficient as compared to the conventional estimators in all situations. It is obvious that for the high correlation $\rho \geq 0.96$, the estimators are tend to be more efficient than moderate $\rho \geq 0.75$ and low $\rho \geq 0.39$ correlation. Thus, the intensity of relationship between the study and supplementary variable plays a vital role in increasing the efficiency. The newly developed estimator retains its efficiency in all three cases. Further, the transformation also plays a remarkable role in enhancing efficiency. As it can be seen from the simulation in the Table 3, the first transformed proposed estimator is efficient than its parent estimator τ_7 for different choices of γ_1 and γ_2 . Similarly, the third proposed estimator is also efficient not only from the conventional estimators but also from the second proposed estimator for different choices of γ_1 and γ_2 .

Conclusion

The empirical and simulation findings highlight the remarkable efficiency of the newly developed estimators in comparison to conventional counterparts across diverse scenarios. Particularly noteworthy is the enhanced efficiency of these estimators in situations with higher correlation, emphasizing the pivotal role of the strength of the relationship between the study and supplementary variable. This aligns with the expectation that a stronger correlation contributes to increased estimator efficiency.

Furthermore, the robustness of the newly developed estimator is evident across all correlation levels—high, moderate, and low. This consistent performance underscores the versatility and reliability of the proposed estimators, making them applicable in a broad spectrum of scenarios.

The introduced transformation in the estimators emerges as a key contributor to enhanced efficiency. As depicted in Table 3, simulation results affirm that the first transformed proposed estimator consistently outperforms its parent estimator across various choices. Similarly, the third proposed estimator not only demonstrates efficiency over conventional estimators but also surpasses the second proposed estimator for different choices. This emphasizes the significant role played by transformations in elevating the overall performance of the estimators.

Additionally, the proposed estimators exhibit flexibility and effectiveness in various sampling designs, such as stratified random sampling, non-response sampling, and adaptive cluster sampling. The extension of these estimators to non-conventional sampling designs, including adaptive cluster sampling and stratified adaptive cluster sampling, is also under consideration for variance estimation. These estimators display flexibility in exploring potential improvements in formulating estimates of population parameter utilizing two auxiliary variables.

In conclusion, the proposed estimators shows remarkable efficiency in finite population variance estimation under the simple random sampling scheme without replacement. The encouraging findings suggest their applicability in diverse survey scenarios, and future research avenues could further enhance their adaptability, extending their utility to more intricate sampling designs.

Data availability

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Received: 7 April 2023; Accepted: 3 April 2024

Published online: 06 April 2024

References

1. Sankhya, L. T. A general unbiased estimator for the variance of a finite population. *Sankhya C* **36**, 23–32 (1974).
2. Isaki, C. T. Variance estimation using auxiliary information. *J. Am. Stat. Assoc.* **78**, 117–123 (1983).
3. Al-Jararha, J. & Ahmed, M. S. The class of chain estimators for a finite population variance using double sampling. *Int. J. Inf. Manage. Sci.* **13**, 13–18 (2002).
4. Shabbir, J. & Gupta, S. On improvement in variance estimation using auxiliary information. *Commun. Stat. Theory Methods* **36**, 2177–2185 (2007).
5. Singh, R. & Malik, S. Improved estimation of population variance using information on auxiliary attribute in simple random sampling. *Appl. Math. Comput.* **235**, 43–49 (2014).
6. Subramani, J. Generalized modified ratio type estimator for estimation of population variance. *Sri Lankan J. Appl. Stat.* **16**, 69 (2015).
7. Yadav, S. K., Kadilar, C., Shabbir, J. & Gupta, S. Improved family of estimators of population variance in simple random sampling. *J. Stat. Theory Pract.* **9**, 219–226 (2015).
8. Adichwal, N. K., Sharma, P. & Singh, R. Generalized class of estimators for population variance using information on two auxiliary variables. *Int. J. Appl. Comput. Math.* **3**, 651–661 (2017).
9. Sanaullah, A., Asghar, A. & Hanif, M. General class of exponential estimator for estimating finite population variance. *J. Reliab. Stat. Stud.* **2017**, 1–16 (2017).
10. Yadav, S. K., Misra, S., Mishra, S. S. & Khanal, S. P. Variance estimator using tri-mean and inter quartile range of auxiliary variable. *Nepal. J. Stat.* **1**, 83–91 (2017).
11. Lone, H. A. & Tailor, R. Estimation of population variance in simple random sampling. *J. Stat. Manag. Syst.* **20**, 17–38 (2017).
12. Singh, H. P., Pal, S. K. & Yadav, A. A study on the chain ratio-ratio-type exponential estimator for finite population variance. *Commun. Stat. Theory Methods* **47**, 1442–1458 (2018).
13. Singh, G. N. & Khalid, M. Effective estimation strategy of population variance in two-phase successive sampling under random non-response. *J. Stat. Theory Pract.* **13**, 4 (2018).
14. Shahzad, U., Hanif, M., Koyuncu, N. & Sanaullah, A. On the estimation of population variance using auxiliary attribute in absence and presence of non-response. *Electron. J. Appl. Stat. Anal.* **11**, 608–621 (2018).
15. Yadav, S. K., Sharma, D. K. & Mishra, S. S. Searching efficient estimator of population variance using tri-mean and third quartile of auxiliary variable. *Int. J. Business Data Anal.* **1**, 30–40 (2019).
16. Gulzar, M. A., Abid, M., Nazir, H. Z., Zahid, F. M. & Riaz, M. On enhanced estimation of population variance using unconventional measures of an auxiliary variable. *J. Stat. Comput. Simul.* **90**, 2180–2197 (2020).
17. Harrison Oghenekevwe, E., Cecilia Njideka, E. & Chizoba Sylvia, O. Distribution effect on the efficiency of some classes of population variance estimators using information of an auxiliary variable under simple random sampling. *SJAMS* **8**, 27 (2020).
18. Sharma, D. K., Yadav, S. K. & Sharma, H. Improvement in estimation of population variance utilising known auxiliary parameters for a decision-making model. *Int. J. Math. Model. Numer. Optim.* **12**, 15–28 (2022).
19. Niaz, I., Sanaullah, A., Saleem, I. & Shabbir, J. An improved efficient class of estimators for the population variance. *Concurr. Comput.* **34**, e6620 (2022).
20. Kumar-Adichwal, N. *et al.* Estimation of general parameters using auxiliary information in simple random sampling without replacement. *J. King Saud Univ. Sci.* **34**, 101754 (2022).
21. Zaman, T. & Bulut, H. An efficient family of robust-type estimators for the population variance in simple and stratified random sampling. *Commun. Stat. Theory Methods* **52**, 2610–2624 (2023).
22. Singh, M. P. On the estimation of ratio and product of the population parameters. *Sankhyā Indian J. Stat. Ser. B (1960-2002)* **27**, 321–328 (1965).
23. Yadav, S. K. & Kadilar, C. A two parameter variance estimator using auxiliary information. *Appl. Math. Comput.* **226**, 117–122 (2014).
24. Muneer, S., Khalil, A., Shabbir, J. & Narjis, G. A new improved ratio-product type exponential estimator of finite population variance using auxiliary information. *J. Stat. Comput. Simul.* **88**, 3179–3192 (2018).
25. Sanaullah, A., Niaz, I., Shabbir, J. & Ehsan, I. A class of hybrid type estimators for variance of a finite population in simple random sampling. *Commun. Stat. Simul. Comput.* **51**, 5609–5619 (2022).
26. Murthy, M. N. Sampling theory and methods. *Sampling Theory Methods* **1967**, 145 (1967).
27. Cochran, W. G. Sampling theory when the sampling-units are of unequal sizes. *J. Am. Stat. Assoc.* **37**, 199–212 (1942).

28. Subramani, J. & Gnanasegaran, K. Variance estimation using quartiles and their functions of an auxiliary variable. *Int. J. Stat. Appl.* 2(5), 67–72 (2012).

Author contributions

Hameed Ali: conceptualization, validation, investigation, data curation, methodology, writing, review & editing, formal analysis, visualization, original draft preparation. Syed Muhammad Asim: supervision, methodology, conceptualization, validation, review & editing, data curation. Muhammad Ijaz: supervision, validation, review & editing. Tolga Zaman: supervision, validation, review & editing. Soofia Iftikhar: supervision, validation, review & editing.

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to T.Z.

Reprints and permissions information is available at www.nature.com/reprints.

Publisher's note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

© The Author(s) 2024