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Soliton solution, breather solution and rational wave solution for a generalized nonlinear Schrödinger equation with Darboux transformation

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In this paper, the exact solutions of generalized nonlinear Schrödinger (GNLS) equation are obtained by using Darboux transformation(DT). We derive some expressions of the 1-solitons, 2-solitons and *n*-soliton solutions of the GNLS equation via constructing special Lax pairs. And we choose different seed solutions and solve the GNLS equation to obtain the soliton solutions, breather solutions and rational wave solutions. Based on these obtained solutions, we consider the elastic interactions and dynamics between two solitons.

The generalized nonlinear Schrödinger(GNLS) equation is an important nonlinear evolution equation, which can describe physical models and phenomena, such as: the Bose–Einstein condensation, nonlinear optics, plasma physics condensed matter physics, fluid mechanics, and so on. Latchio Tiofack, Mohamadou and Kofané considered the nonuniform $1+1$ dimensional coupled nonlinear schrödinger equations[1], and presented some exact solutions by using the transformation. Vijayalekshmi, Mahalingam and Mani–Rajan studied the propagation of optical solitons in the nonautonomous nonlinear Schrödinger equation with a generalized external potential[2]. The nonlinear Schrödinger equation has been extended to various soliton models[3] including variable coefficient, complex coefficient, high dimensional, high order, nonlocal and fractional order equations[4–6]. Some solitary wave solutions[7], rogue wave solutions[8], bright and dark solitons[9] are derived in nonlinear Schrödinger equation.

There are many methods to solve soliton equation, such as Hirota bilinear method[10,11], inverse scattering method[12,13], homogeneous balance method[14,15], Darboux transform (DT) method[16,17] and so on. Some solutions are successfully solved in different types of partial differential equations via these above methods. Some higher-order wave solutions and discrete rogue wave solutions of KE equation were constructed by using DT and Taylor expansion in[18,19]. Ablowitz and Musslimani proposed the nonlocal modified Korteweg–de Vries (mKdV) equation and the nonlocal Sine–Gordon (SG) equation, and proved the integrability of these equations in[20]. Ji and Zhu obtained a series of different types of exact analytical solutions of nonlocal mKdV equations through constructing DT[21], including complexiton solutions, rogue wave solutions, kink soliton solutions and anti-kink soliton solutions. Some bright soliton solutions, dark soliton solutions and breather solutions of the super integrable equation are presented with DT[22]. The non-autonomous multi-rogue wave solutions of the spin-1 coupled nonlinear Gross–Pitaevskii equation with different dispersion, higher-order nonlinear terms, gain (or loss) and external potential are considered in[23–25]. The multiple breather solutions and mixed solutions of the Kundu equation are constructed with generalized Darboux transformation method, which have the Lax pair of Kaup–Newell system in[26].

The paper is organized as follows: in "Results", we successfully solve the GNLS equation with DT, and obtain several new sets of exact solutions, including 1-soliton solutions, 2-soliton solutions and *n*-soliton solutions. In "Conclusions", we select the non-zero seed solution and solve the GNLS equation by using the DT, and obtain the breather solutions of the GNLS equation. In "Methods", we also use the DT and Taylor expansion to derive the rational wave solutions of the GNLS equation. Finally, we give some conclusions in "Rational wave solutions for GNLS Eq. (4)".

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Results

Soliton solutions of GNLS equation. It is well known that the standard nonlinear Schrödinger(NLS) equation

$$iu_t + \gamma u_{xx} + \sigma u|u|^2 = 0, \qquad \sigma = \pm 1, \tag{1}$$

is one of the most important integrable system among many branches of applied mathematics and physics, especially in optics, water wave and so on. The $u = u(x, t)$ is a complex smooth function of x and t , the subscripts denote partial derivatives and the parameter γ is real constant in Eq. (1).

Fokas studied an integrable generalized nonlinear Schrödinger (GNLS) equation by means of bi-Hamiltonian operators

$$iu_t - \nu u_{tx} + \gamma u_{xx} + \sigma|u|^2(u + i\nu u_x) = 0, \qquad \sigma = \pm 1, \tag{2}$$

where γ and ν are real constants. In fact, Eq. (2) can be transformed into Eq. (1) when the parameter $\nu = 0$. Lenells investigated Eq. (2) by the dressing method, and presented a new form of Eq. (2) as following

$$u_{tx} + \alpha\beta^2 u - 2i\alpha\beta u_x - \alpha u_{xx} + \sigma i\alpha\beta^2|u|^2 u_x = 0, \qquad \sigma = \pm 1, \tag{3}$$

under the transformation of $u \to \beta\sqrt{\alpha}e^{i\beta x}u, \sigma = -\sigma$, where $\alpha = \frac{\gamma}{\nu} > 0, \beta = \frac{1}{\nu}$.

Without losing generality, let $\sigma = -1$, then Eq. (3) will become the following form[27]:

$$u_{tx} + \alpha\beta^2 u - 2i\alpha\beta u_x - \alpha u_{xx} - i\alpha\beta^2|u|^2 u_x = 0, \tag{4}$$

and the Lax pair of Eq. (4) is as following

$$\begin{aligned} \varphi_x =& U\varphi, \ \ \varphi_t = V\varphi, \\ U =& \begin{pmatrix} -i\lambda^2 & \lambda u_x \\ \lambda r_x & i\lambda^2 \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{i\alpha\beta^2}{2}ur - i\eta^2 & \frac{i\alpha\beta^2}{2\lambda}u + \alpha\lambda u_x \\ -\frac{i\alpha\beta^2}{2\lambda}r + \alpha\lambda r_x & \frac{i\alpha\beta^2}{2}ur + i\eta^2 \end{pmatrix}, \end{aligned} \tag{5}$$

where $\eta = \sqrt{\alpha}(\lambda - \frac{\beta}{2\lambda}), r = -u^*$, the "$*$" denotes the complex conjugate and the vector $\varphi = (\varphi_1, \varphi_2)^T$ is an eigenfunction associated with λ and potential u, which consists of two complex functions $\varphi_1 = \varphi_1(x, t)$ and $\varphi_2 = \varphi_2(x, t)$. Trough direct calculations, we can verify that the integrability condition $U_t - V_x + [U, V] = 0$ exactly can be derived from Eq. (4), where $[U, V] = UV - VU$.

From the above analysis, we could construct a N-fold Darboux matrix T for the GNLS equation (4), as follows

$$\widetilde{\varphi}_n = T\varphi_n, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \tag{6}$$

The lower forms are obtained by compatibility

$$\varphi_x = \widetilde{U}\varphi, \ \ \widetilde{U} = (T_x + TU)T^{-1}, \tag{7}$$

$$\varphi_t = \widetilde{V}\varphi, \ \ \widetilde{V} = (T_t + TV)T^{-1}. \tag{8}$$

If the $\widetilde{U}, \widetilde{V}$ and U, V have the same types, the system (6) is called Darboux transformation of the GNLS equation. Let $\psi = (\psi_1, \psi_2)^T, \phi = (\phi_1, \phi_2)^T$ are two basic solutions of the systems (5), then we give the following linear algebraic systems:

$$\begin{cases} \sum_{k=1}^N A_k\lambda_j^{2k} + \sum_{k=1}^N B_k\lambda_j^{2k-1}M_j^{(1)} = -1, \\ \sum_{k=1}^N -B_k^*\lambda_j^{2k-1} + \sum_{k=1}^N A_k^*\lambda_j^{2k}M_j^{(1)} = -M_j^{(1)}, \end{cases} \tag{9}$$

with

$$M_j^{(1)} = \frac{\psi_2 + \nu_j^{(1)}\phi_2}{\psi_1 + \nu_j^{(1)}\phi_1}, \quad (1 \le j \le 2N), \quad A_k^* = D_k, \quad -B_k^* = C_k, \tag{10}$$

where λ_j and $\nu_j^{(k)}$ should choose appropriate parameters, thus the determinants of coefficients for Eq. (9) are nonzero. Hereby, we take a 2×2 matrix T as

$$\begin{cases} T_{11} = 1 + \sum_{k=1}^N A_k\lambda^{2k}, & T_{12} = \sum_{k=1}^N B_k\lambda^{2k-1}, \\ T_{21} = \sum_{k=1}^N -B_k^*\lambda^{2k-1}, & T_{22} = 1 + \sum_{k=1}^N A_k^*\lambda^{2k}, \end{cases} \tag{11}$$

where N is a natural number, the $A_{mn}^{(i)}(m, n = 1, 2, i \ge 0)$ are some functions of x and t. Through calculations, we can obtain ΔT as following

$$\Delta T = \Pi_{j=1}^{2N}(\lambda - \lambda_j), \tag{12}$$

which proves that λ_j $(\lambda_j \neq 0)(j = 1, 2, 3, \ldots, 2N)$ are $2N$ roots of ΔT. Based on these conditions, we will proof that the $\widetilde{U}$ and $\widetilde{V}$ have the same structures as U and V respectively.

The matrix $\widetilde{U}$ defined by (7) has the same type as U, that is,

$$\widetilde{U} = \begin{pmatrix} -i\lambda^2 & \lambda\widetilde{u}_x \\ \lambda\widetilde{r}_x & i\lambda^2 \end{pmatrix}, \tag{13}$$

in which the transformation formula between old and new potentials are defined by

$$\begin{cases} \widetilde{u}_x = u_x + B_{1x}, \\ \widetilde{r}_x = r_x + C_{1x}, \end{cases} \tag{14}$$

the transformations (14) are used to get a Darboux transformation of the spectral problem (7).

Let $T^{-1} = \frac{T^*}{\Delta T}$ and

$$(T_x + TU)T^* = \begin{pmatrix} B_{11}(\lambda) & B_{12}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) \end{pmatrix}, \tag{15}$$

it is easy to verify that $B_{sl}(1 \leq s, l \leq 2)$ is $2N$-order or $2N+1$-order polynomial of λ.

Through some accurate calculations, $\lambda_j(1 \leq j \leq 2,)$ is the root of $B_{sl}(1 \leq s, l \leq 2)$. Thus, Eq. (15) has the following structure

$$(T_x + TU)T^* = (\Delta T)E(\lambda), \tag{16}$$

where

$$E(\lambda) = \begin{pmatrix} E_{11}^{(2)}\lambda^2 + E_{11}^{(1)}\lambda + E_{11}^{(0)} & E_{12}^{(1)}\lambda + E_{12}^{(0)} \\ E_{21}^{(1)}\lambda + E_{21}^{(0)} & E_{22}^{(2)}\lambda^2 + E_{22}^{(1)}\lambda + E_{22}^{(0)} \end{pmatrix}, \tag{17}$$

and $E_{mn}^{(k)}(m, n = 1, 2, k = 0, 1)$ satisfy the functions without λ . Equation (16) can be rewritten as

$$(T_x + TU) = E(\lambda)T. \tag{18}$$

Through comparing the coefficients of λ in Eq. (18), we can obtain

$$\begin{cases} E_{11}^{(0)} = 0, E_{11}^{(1)} = 0, E_{11}^{(2)} = i, E_{12}^{(0)} = 0, E_{12}^{(1)} = u_x + B_{1x} = \widetilde{u}_x, \\ E_{21}^{(0)} = 0, E_{21}^{(1)} = r_x + C_{1x} = \widetilde{r}_x, E_{22}^{(0)} = 0, E_{22}^{(1)} = 0, E_{22}^{(2)} = i. \end{cases} \tag{19}$$

In this section, we assume that the new matrix $\widetilde{U}$ has the same type with U, which means that they have the same structures only $u(x, t)$, $r(x, t)$ of U transformed into $\widetilde{u}(x, t)$, $\widetilde{r}(x, t)$ of $\widetilde{U}$. After careful calculation, we compare the ranks of λ^N, and get the objective equations as following:

$$\begin{cases} \widetilde{u}_x = u_x + B_{1x}, \\ \widetilde{r}_x = r_x + C_{1x}, \end{cases} \tag{20}$$

from Eqs. (13) and (14), we know that $\widetilde{U} = E(\lambda)$. The proof is completed.

The matrix $\widetilde{V}$ defined by the second expression of (8) has the same form as V, in which the old potentials u and r are mapped into $\widetilde{u}$ and $\widetilde{r}$, that is,

$$\widetilde{V} = \begin{pmatrix} -\frac{i\alpha\beta^2}{2}\widetilde{u}\widetilde{r} - i\eta^2 & \frac{i\alpha\beta^2}{2\lambda}\widetilde{u} + \alpha\lambda\widetilde{u}_x \\ -\frac{i\alpha\beta^2}{2\lambda}\widetilde{r} + \alpha\lambda\widetilde{r}_x & \frac{i\alpha\beta^2}{2}\widetilde{u}\widetilde{r} + i\eta^2 \end{pmatrix}. \tag{21}$$

We suppose the new matrix $\widetilde{V}$ also has the same form with V. If we obtain the similar relations between $u(x, t)$, $r(x, t)$ and $\widetilde{u}(x, t)$, $\widetilde{r}(x, t)$ in Eq. (14), we can prove that the gauge transformations under T turn the Lax pairs U, V into new Lax pairs $\widetilde{U}$, $\widetilde{V}$ with the same types.

Let $T^{-1} = \frac{T^*}{\Delta T}$ and

$$(T_t + TV)T^* = \begin{pmatrix} C_{11}(\lambda) & C_{12}(\lambda) \\ C_{21}(\lambda) & C_{22}(\lambda) \end{pmatrix}, \tag{22}$$

it is easy to verify that $C_{sl}(1 \leq s, l \leq 2)$ is $2N$-order or $2N+1$-order polynomial of λ. Through some accurate calculations, $\lambda_j(1 \leq j \leq 2)$ is the root of $C_{sl}(1 \leq s, l \leq 2)$. Thus, Eq. (22) has the following structure

$$(T_t + TV)T^* = (\Delta T)F(\lambda), \tag{23}$$

where

$$F(\lambda) = \begin{pmatrix} F_{11}^{(2)}\lambda^2 + F_{11}^{(0)} + F_{11}^{(-2)}\lambda^{-2} & F_{12}^{(1)}\lambda + F_{12}^{(-1)}\lambda^{-1} \\ F_{21}^{(1)}\lambda + F_{21}^{(-1)}\lambda^{-1} & F_{22}^{(2)}\lambda^2 + F_{22}^{(0)}F_{22}^{(-2)}\lambda^{-2} \end{pmatrix}, \tag{24}$$

and $F_{mn}^{(k)}(m, n = 1, 2, k = 0, 1)$ satisfies the functions without λ. According to Eq. (23), the following equation is obtained

$$(T_t + TV) = F(\lambda)T. \tag{25}$$

Through comparing the coefficients of λ in Eq. (25), we get the objective equations as following:

$$\begin{cases} F_{11}^{(-2)} = -\frac{i\alpha\beta^2}{4},\ F_{11}^{(2)} = -i\alpha,\ F_{11}^{(0)} = i\alpha\beta - \frac{i\alpha\beta^2}{2}\widetilde{u}\widetilde{r}, \\ F_{12}^{(-1)} = \frac{i\alpha\beta^2}{2}\widetilde{u},\ F_{12}^{(1)} = \frac{A_N u_x \alpha + 2i\alpha B_N}{D_N}, \\ F_{21}^{(-1)} = -\frac{i\alpha\beta^2}{2}\widetilde{r},\ F_{21}^{(1)} = \frac{D_N r_x \alpha - 2i\alpha C_N}{A_N}, \\ F_{22}^{(-2)} = \frac{i\alpha\beta^2}{4},\ F_{22}^{(2)} = i\alpha,\ F_{22}^{(0)} = -i\alpha\beta + \frac{i\alpha\beta^2}{2}\widetilde{u}\widetilde{r}. \end{cases} \tag{26}$$

In this section, we assume the new matrix $\widetilde{V}$ has the same type with V, which means they have the same structures only $u(x,t)$, $r(x,t)$ of V transformed into $\widetilde{u}(x,t), \widetilde{r}(x,t)$ of $\widetilde{V}$. From Eqs. (14) and (21), we know that $\widetilde{V} = F(\lambda)$. The proof is completed.

We will give some novel explicit solutions of Eq. (4) by applying N-fold DT. Firstly, we give a seed solution $u = 0$ and substitute the solution into Eq. (5), it is easy to find two basic solutions for these equations:

$$\psi(\lambda) = \begin{pmatrix} e^{-i\lambda^2 x - i\eta^2 t + C_1} \\ 0 \end{pmatrix}, \quad \phi(\lambda) = \begin{pmatrix} 0 \\ e^{i\lambda^2 x + i\eta^2 t + C_2} \end{pmatrix}, \tag{27}$$

by using Eqs. (8) and (25), we obtain

$$M_j^{(1)} = \frac{v_j^{(1)} e^{-i\lambda^2 x - i\eta^2 t + C_1}}{e^{i\lambda^2 x + i\eta^2 t + C_2}} = e^{2(i\lambda_j^2 x + i\eta^2 t + F_j)}, \tag{28}$$

with $v_j^{(i)} = e^{(2iF_{ji})}(1 \le i \le 2, 1 \le j \le 2N)$.

In order to derive the expression of N-order DT of Eq. (4) and obtain the matrix T

$$T = \begin{pmatrix} 1 + \sum_{k=1}^{N} A_k \lambda^{2k} & \sum_{k=1}^{N} B_k \lambda^{2k-1} \\ \sum_{k=1}^{N} -B_k^* \lambda^{2k-1} & 1 + \sum_{k=1}^{N} A_k^* \lambda^{2k} \end{pmatrix}, \tag{29}$$

and

$$\begin{cases} \sum_{k=1}^{N} A_k \lambda_j^{2k} + \sum_{k=1}^{N} B_k \lambda_j^{2k-1} M_j^{(1)} + 1 = 0, \\ \sum_{k=1}^{N} -B_k^* \lambda_j^{2k-1} + \sum_{k=1}^{N} A_k^* \lambda_j^{2k} M_j^{(1)} + M_j^{(1)} = 0. \end{cases} \tag{30}$$

Solving Eq. (30) via the Gramer's rule, we have

$$B_N = \frac{\Delta B_N}{\Delta}, C_N = \frac{\Delta C_N}{\Delta} \tag{31}$$

with

$$\Delta = \begin{vmatrix} \lambda_1^2 & \lambda_1^4 & \lambda_1^6 & \dots & \lambda_1^{2N} & M_1\lambda_1 & M_1\lambda_1^3 & \dots & M_1\lambda_1^{(2N-1)} \\ \lambda_2^2 & \lambda_2^4 & \lambda_2^6 & \dots & \lambda_2^{2N} & M_2\lambda_2 & M_2\lambda_2^3 & \dots & M_2\lambda_2^{(2N-1)} \\ \lambda_3^2 & \lambda_3^4 & \lambda_3^6 & \dots & \lambda_3^{2N} & M_3\lambda_3 & M_3\lambda_3^3 & \dots & M_3\lambda_3^{(2N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ \lambda_{2N}^2 & \lambda_{2N}^4 & \lambda_{2N}^6 & \dots & \lambda_{2N}^{2N} & M_{2N}\lambda_{2N} & M_{2N}\lambda_{2N}^3 & \dots & M_{2N}\lambda_{2N}^{(2N-1)} \end{vmatrix},$$

$$\Delta B_N = \begin{vmatrix} \lambda_1^2 & \lambda_1^4 & \lambda_1^6 & \dots & \lambda_1^{2N} & M_1\lambda_1 & M_1\lambda_1^3 & \dots & -1 \\ \lambda_2^2 & \lambda_2^4 & \lambda_2^6 & \dots & \lambda_2^{2N} & M_2\lambda_2 & M_2\lambda_2^3 & \dots & -1 \\ \lambda_3^2 & \lambda_3^4 & \lambda_3^6 & \dots & \lambda_3^{2N} & M_3\lambda_3 & M_3\lambda_3^3 & \dots & -1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ \lambda_{2N}^2 & \lambda_{2N}^4 & \lambda_{2N}^6 & \dots & \lambda_{2N}^{2N} & M_{2N}\lambda_{2N} & M_{2N}\lambda_{2N}^3 & \dots & -1 \end{vmatrix}, \tag{32}$$

$$\Delta C_N = \begin{vmatrix} \lambda_1 & \lambda_1^3 & \lambda_1^5 & \dots & -M_1 & M_1\lambda_1^2 & M_1\lambda_1^4 & \dots & M_1\lambda_1^{2N} \\ \lambda_2 & \lambda_2^3 & \lambda_2^5 & \dots & -M_2 & M_2\lambda_2^2 & M_2\lambda_2^4 & \dots & M_2\lambda_2^{2N} \\ \lambda_3 & \lambda_3^3 & \lambda_3^5 & \dots & -M_3 & M_3\lambda_3^2 & M_3\lambda_3^4 & \dots & M_3\lambda_3^{2N} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ \lambda_{2N} & \lambda_{2N}^3 & \lambda_{2N}^5 & \dots & -M_{2N} & M_{2N}\lambda_{2N}^2 & M_{2N}\lambda_{2N}^4 & \dots & M_{2N}\lambda_{2N}^{2N} \end{vmatrix}.$$

Using Eqs. (6), (20) and (31), we can derive the new formulas of N-soliton solutions for GNLS equation

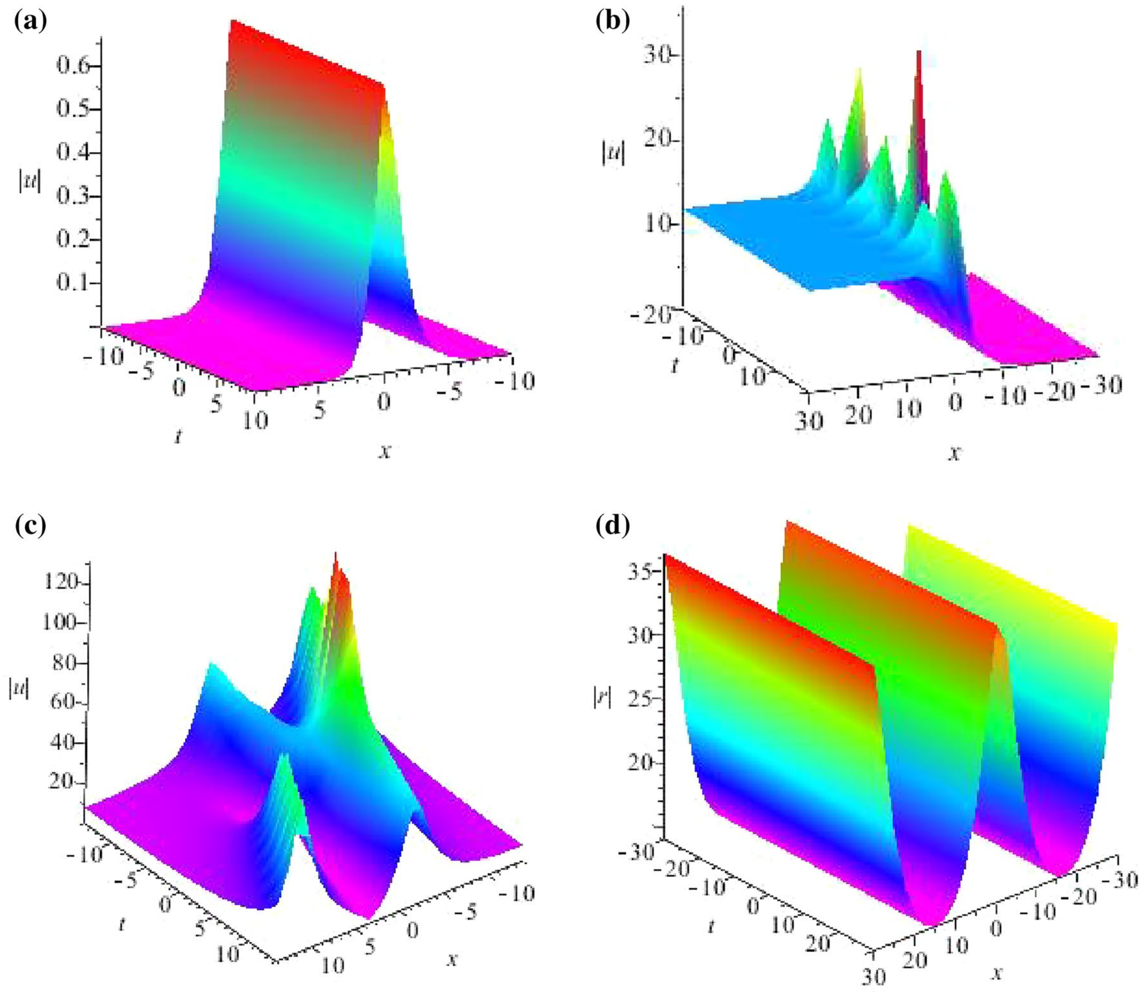

Figure 1. Profiles of intensity distribution (**a**) $|\widetilde{u}(x,t)|$ of Eq. (34) with parameters $\lambda_1 = 1 - 0.8i, \lambda_2 = 0.6 + 0.4i, \alpha = 0.0004, \beta = 1, F_1 = 0.4 + i, F_2 = 0.3 + 0.6i$; (**b**) $|\widetilde{u}(x,t)|$ of Eq. (34) with parameters $\lambda_1 = 0.2i, \lambda_2 = 0.1, \alpha = 0.4, \beta = 0.2, F_1 = 0.01, F_2 = 0.02$; (**c**) $|\widetilde{u}(x,t)|$ of Eq. (36) with parameters $\lambda_1 = 0.2, \lambda_2 = 0.3{+}0.2i, \lambda_3 = 0.3, \lambda_4 = 0.3{-}0.2i, \alpha = 0.2, \beta = 0.3, F_1 = 0.2{+}0.2i, F_2 = 0.3{-}0.2i, F_3 = 0.3 + 0.2i, F_4 = 0.3 - 0.2i$; (**d**) $|\widetilde{r}(x,t)|$ of Eq. (36) with parameters $\lambda_1 = 0.5, \lambda_2 = 0.2, \lambda_3 = 0.5, \lambda_4 = 0.3, \alpha = 0.004, \beta = 0.2, F_1 = 0.5 + 0.2i, F_2 = 0.5 - 0.2i, F_3 = 0.3 + 0.1i, F_4 = 0.3 - 0.1i$.

$$\begin{cases} \widetilde{u}(x,t) = \frac{\Delta B_N}{\Delta}, \\ \widetilde{r}(x,t) = \frac{\Delta C_N}{\Delta}, \end{cases} \tag{33}$$

in order to understand solutions (33), we consider $N = 1, 2$ separately and plot their structure figures in Fig. 1a,b.

(I) We take $N = 1$ with $\lambda = \lambda_j (j = 1, 2)$. Solving Eq. (9), we can yield the 1-soliton solutions of the GNLS equation (4) as following:

$$\widetilde{u}(x,t) = \frac{\Delta B_1}{\Delta}, \quad \widetilde{r}(x,t) = -\widetilde{u}^*(x,t), \tag{34}$$

with

$$\Delta = \begin{vmatrix} \lambda_1^2 & e^{2(i\lambda_1 x + i\eta^2 t + F_1)}\lambda_1 \\ \lambda_2^2 & e^{2(i\lambda_2 x + i\eta^2 t + F_2)}\lambda_2 \end{vmatrix}, \quad \Delta B_1 = \begin{vmatrix} \lambda_1^2 & -1 \\ \lambda_2^2 & -1 \end{vmatrix},$$
$$\Delta C_1 = \begin{vmatrix} -e^{2(i\lambda_1 x + i\eta^2 t + F_1)} & \lambda_1^2 e^{2(i\lambda_1 x + i\eta^2 t + F_1)} \\ -e^{2(i\lambda_2 x + i\eta^2 t + F_2)} & \lambda_2^2 e^{2(i\lambda_2 x + i\eta^2 t + F_2)} \end{vmatrix}. \tag{35}$$

(II) We take $N = 2$ in the N-times DT with $\lambda = \lambda_j (j = 1, 2, 3, 4)$. The linear algebraic system (9) leads to the 2-soliton solutions of GNLS (4) as following:

$$\widetilde{u}(x,t)=\frac{\Delta B_2}{\Delta},\quad \widetilde{r}(x,t)=-\widetilde{u}^*(x,t), \tag{36}$$

with

$$\Delta=\begin{vmatrix} \lambda_1^2 & \lambda_1^4 & e^{2(i\lambda_1 x+i\eta^2 t+F_1)}\lambda_1 & e^{2(i\lambda_1 x+i\eta^2 t+F_1)}\lambda_1^3 \\ \lambda_2^2 & \lambda_2^4 & e^{2(i\lambda_2 x+i\eta^2 t+F_2)}\lambda_2 & e^{2(i\lambda_2 x+i\eta^2 t+F_2)}\lambda_2^3 \\ \lambda_3^2 & \lambda_3^4 & e^{2(i\lambda_3 x+i\eta^2 t+F_3)}\lambda_3 & e^{2(i\lambda_3 x+i\eta^2 t+F_3)}\lambda_3^3 \\ \lambda_4^2 & \lambda_4^4 & e^{2(i\lambda_4 x+i\eta^2 t+F_4)}\lambda_4 & e^{2(i\lambda_4 x+i\eta^2 t+F_4)}\lambda_4^3 \end{vmatrix},\quad \Delta B_2=\begin{vmatrix} \lambda_1^2 & \lambda_1^4 & e^{2(i\lambda_1 x+i\eta^2 t+F_1)}\lambda_1 & -1 \\ \lambda_2^2 & \lambda_2^4 & e^{2(i\lambda_2 x+i\eta^2 t+F_2)}\lambda_2 & -1 \\ \lambda_3^2 & \lambda_3^4 & e^{2(i\lambda_3 x+i\eta^2 t+F_3)}\lambda_3 & -1 \\ \lambda_4^2 & \lambda_4^4 & e^{2(i\lambda_4 x+i\eta^2 t+F_4)}\lambda_4 & -1 \end{vmatrix},$$

$$\Delta C_2=\begin{vmatrix} \lambda_1 & -e^{2(i\lambda_1 x+i\eta^2 t+F_1)} & \lambda_1^2 e^{2(i\lambda_1 x+i\eta^2 t+F_1)} & \lambda_1^4 e^{2(i\lambda_1 x+i\eta^2 t+F_1)} \\ \lambda_2 & -e^{2(i\lambda_2 x+i\eta^2 t+F_2)} & \lambda_2^2 e^{2(i\lambda_2 x+i\eta^2 t+F_2)} & \lambda_2^4 e^{2(i\lambda_2 x+i\eta^2 t+F_2)} \\ \lambda_3 & -e^{2(i\lambda_3 x+i\eta^2 t+F_3)} & \lambda_3^2 e^{2(i\lambda_3 x+i\eta^2 t+F_3)} & \lambda_3^4 e^{2(i\lambda_3 x+i\eta^2 t+F_3)} \\ \lambda_4 & -e^{2(i\lambda_4 x+i\eta^2 t+F_4)} & \lambda_4^2 e^{2(i\lambda_4 x+i\eta^2 t+F_4)} & \lambda_4^4 e^{2(i\lambda_4 x+i\eta^2 t+F_4)} \end{vmatrix}. \tag{37}$$

In order to understand solutions (36), we consider $N=2$ and plot their structure figures in Fig. 1c,d.

Conclusions

The integrable GNLS equation can describe the propagation of nonlinear light pulses in optical fibers, the high-order nonlinear effects are taken into consideration. In this paper, we investigate the exact solutions (including soliton solutions, breather solutions, and rational wave solutions) of a GNLS equation via DT method. And the 1-solitons, 2-solitons and N-soliton solutions of the GNLS equation are obtained via constructing special Lax pairs. And we choose different seed solutions and obtain three kinds of solutions. Based on these obtained solutions, we consider the elastic interactions and dynamics between two solitons for the GNLS equation.

Methods

Breather solutions for GNLS equation (4). Now we choose there kinds of seed solutions of (4) as follows:

$$u=c_0e^{i\sigma\gamma_0^2 x},\quad c_0=\frac{\beta+\sigma\gamma_0^2}{\beta\gamma_0}, \tag{38}$$

$$u=\frac{\omega_0}{\beta\gamma_0}e^{-i(\gamma_0^2 x+\delta_0 t)},\quad \delta_0=\alpha[(\beta+\sigma\gamma_0^2)^2-\omega_0^2]\gamma_0^{-2}, \tag{39}$$

and

$$u=e^{i\theta},\quad \theta=ax+bt, \tag{40}$$

where γ_0, ω_0, a and b are arbitrary constants.

Case 1: We give a seed solution $u=c_0e^{i\sigma\gamma_0^2 x}$ with $c_0=\frac{\beta+\sigma\gamma_0^2}{\beta\gamma_0}$. According to Eq. (5), we can yield the following systems

$$\begin{cases} -i\lambda^2\psi_1+i\sigma\gamma_0^2c_0\lambda e^{i\sigma\gamma_0^2 x}\psi_2=\psi_{1x}, \\ i\sigma\gamma_0^2c_0\lambda e^{-i\sigma\gamma_0^2 x}\psi_1+i\lambda^2\psi_2=\psi_{2x}, \end{cases} \tag{41}$$

without loss of generality, we assume that $\sigma=-1$, $\psi_1=\alpha e^{px}$, $\psi_2=\gamma e^{px-i\sigma\gamma_0^2 x}$, then Eq. (41) is solved by

$$\begin{cases} p=\frac{(i\lambda^2\alpha-i\alpha\gamma_0^2-i\lambda^2)\pm\sqrt{(i\alpha\gamma_0^2-i\lambda^2\alpha+i\lambda^2)^2-4\alpha(\lambda^4-\gamma_0^2\lambda^2+\gamma_0^4c_0^2\alpha\lambda^2)}}{2\alpha}, \\ \gamma=\frac{-i\gamma_0^2c_0\lambda\alpha}{\beta+i\gamma_0^2-i\lambda^2}. \end{cases} \tag{42}$$

Based on Eq. (5), we obtain

$$\begin{cases} \left(\frac{i\alpha\beta^2}{2}c_0^2e^{-2i\gamma_0^2 x}-i\eta^2\right)\psi_1+e^{-i\gamma_0^2 x}\left(\frac{i\alpha\beta^2c_0}{2\lambda}-i\alpha c_0\gamma_0^2\lambda\right)\psi_2=\psi_{1t}, \\ e^{-i\gamma_0^2 x}\left(\frac{i\alpha\beta^2c_0}{2\lambda}+i\alpha c_0\gamma_0^2\lambda\right)\psi_1+\left(-\frac{i\alpha\beta^2c_0^2}{2}e^{-2i\gamma_0^2 x}+i\eta^2\right)\psi_2=\psi_{2t}, \end{cases} \tag{43}$$

we can derive the following system form Eq. (43)

$$\begin{cases} \left(\frac{i\alpha\beta^2}{2}c_0^2e^{-2i\gamma_0^2 x}-i\eta^2-\lambda_{11}\right)a+\left(e^{-i\gamma_0^2 x}\frac{i\alpha\beta^2c_0}{2\lambda}-e^{-i\gamma_0^2 x}i\alpha c_0\gamma_0^2\lambda\right)b=0, \\ \left(e^{-i\gamma_0^2 x}\frac{i\alpha\beta^2c_0}{2\lambda}+e^{i\gamma_0^2 x}i\alpha c_0\gamma_0^2\lambda\right)a+\left(-\frac{i\alpha\beta^2c_0^2}{2}e^{-2i\gamma_0^2 x}+i\eta^2-\lambda_{12}\right)b=0. \end{cases} \tag{44}$$

We obtain that

$$\lambda_{11}=\sqrt{\alpha\beta^2c_0^2\eta^2-\eta^4-\frac{\alpha^2\beta^4c_0^2}{4\lambda^2}},\quad \lambda_{12}=-\sqrt{\alpha\beta^2c_0^2\eta^2-\eta^4-\frac{\alpha^2\beta^4c_0^2}{4\lambda^2}},$$

with

$$a=\left(\frac{i\alpha c_0\gamma_0^2\lambda e^{-i\gamma_0^2x}-\frac{i\alpha\beta^2c_0}{2\lambda}e^{-i\gamma_0^2x}}{\frac{i\alpha\beta^2c_0^2}{2}-i\eta^2-\lambda_{11}}\right)b,$$

substituting the above solutions and Eq. (44) into Eq. (5), it is easy to find two basic solutions for these equations:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=C_1e^{\lambda_{11}t}\begin{pmatrix}\frac{i\alpha c_0\gamma_0^2\lambda-\frac{-i\alpha\beta^2c_0^2}{2}e^{-i\gamma_0^2x}}{\frac{i\alpha\beta^2c_0^2}{2}e^{-2i\gamma_0^2x}-i\eta^2-\lambda_{11}}\\ 1\end{pmatrix}+C_2e^{\lambda_{12}t}\begin{pmatrix}1\\ \frac{\frac{i\alpha\beta^2c_0^2}{2}e^{-2i\gamma_0^2x}+i\eta^2+\lambda_{12}}{e^{-2i\gamma_0^2x}\frac{i\alpha\beta^2c_0}{2\lambda}-e^{-i\gamma_0^2x}i\alpha c_0\gamma_0^2\lambda}\end{pmatrix}. \tag{45}$$

It is easy to find two basic solutions for Eqs. (42) and (45):

$$\begin{pmatrix}\psi\\ \phi\end{pmatrix}=\begin{pmatrix}\alpha C_1e^{\lambda_{11}t+px}\left(\frac{i\alpha c_0\gamma_0^2\lambda e^{-i\gamma_0^2x}-\frac{i\alpha\beta^2c_0}{2\lambda}e^{-i\gamma_0^2x}}{\frac{i\alpha\beta^2c_0^2}{2}-i\eta^2-\lambda_{11}}\right)+\alpha C_2e^{\lambda_{12}t+px}\\ \gamma C_1e^{\lambda_{11}t+px+i\sigma\gamma_0^2x}+\gamma C_2e^{\lambda_{12}t+px+i\sigma\gamma_0^2x}\left(\frac{e^{i\gamma_0^2x}\frac{i\alpha\beta^2c_0}{2\lambda}-e^{i\gamma_0^2x}i\alpha c_0\gamma_0^2\lambda}{\frac{i\alpha\beta^2c_0^2}{2}-i\eta^2+\lambda_{12}}\right)\end{pmatrix}, \tag{46}$$

we can obtain by using Eq. (10),

$$M_j=\frac{C_1\gamma e^{F_j+i\gamma_0^2x+\lambda_{11}t}+\frac{C_2\gamma e^{F_j+2i\gamma_0^2x+\lambda_{12}t}(i\alpha\beta^2c_0-2i\alpha c_0\lambda^2\gamma_0^2)}{\lambda(i\alpha\beta^2c_0^2-2i\eta^2+2\lambda_{12})}}{C_2\alpha e^{\lambda_{12}t}+\frac{C_1\alpha e^{\lambda_{11}t-i\gamma_0^2x}(2i\alpha\lambda^2c_0\gamma_0^2-i\alpha\beta^2c_0)}{\lambda(i\alpha\beta^2c_0^2-2i\eta^2-2\lambda_{11})}},\quad 1\le j\le 2N, \tag{47}$$

with $v_j^{(i)}=e^{F_j}(1\le i\le 2, 1\le j\le 2N)$.

(I) We take $N=1$ with $\lambda=\lambda_j$ (j = 1, 2). We can yield the 1-soliton solutions of the GNLS equation (4) from Eq. (9) as following:

$$\widetilde{u}(x,t)=c_0e^{i\sigma\gamma_0^2x}+\frac{\Delta B_1}{\Delta},\quad \widetilde{r}(x,t)=-\widetilde{u}^*(x,t), \tag{48}$$

with

$$\Delta=\begin{vmatrix}\lambda_1^2 & M_1\lambda_1\\ \lambda_2^2 & M_2\lambda_2\end{vmatrix},\ \Delta B_1=\begin{vmatrix}\lambda_1^2 & -1\\ \lambda_2^2 & -1\end{vmatrix},\ \Delta C_1=\begin{vmatrix}-M_1 & \lambda_1^2M_1\\ -M_2 & \lambda_2^2M_2\end{vmatrix}. \tag{49}$$

(II) We take $N=2$ in the N-times DT with $\lambda=\lambda_j(j=1,2,3,4)$. The linear algebraic system (9) leads to 2-soliton solutions of GNLS Eq. (4) as following:

$$\widetilde{u}(x,t)=c_0e^{i\sigma\gamma_0^2x}+\frac{\Delta B_2}{\Delta},\quad \widetilde{r}(x,t)=-\widetilde{u}^*(x,t), \tag{50}$$

with

$$\Delta=\begin{vmatrix}\lambda_1^2 & \lambda_1^4 & M_1\lambda_1 & M_1\lambda_1^3\\ \lambda_2^2 & \lambda_2^4 & M_2\lambda_2 & M_2\lambda_2^3\\ \lambda_3^2 & \lambda_3^4 & M_3\lambda_3 & M_3\lambda_3^3\\ \lambda_4^2 & \lambda_4^4 & M_4\lambda_4 & M_4\lambda_4^3\end{vmatrix},\ \Delta B_2=\begin{vmatrix}\lambda_1^2 & \lambda_1^4 & M_1\lambda_1 & -1\\ \lambda_2^2 & \lambda_2^4 & M_2\lambda_2 & -1\\ \lambda_3^2 & \lambda_3^4 & M_3\lambda_3 & -1\\ \lambda_4^2 & \lambda_4^4 & M_4\lambda_4 & -1\end{vmatrix},\ \Delta C_2=\begin{vmatrix}\lambda_1 & -M_1 & \lambda_1^2M_1 & \lambda_1^4M_1\\ \lambda_2 & -M_2 & \lambda_2^2M_2 & \lambda_2^4M_2\\ \lambda_3 & -M_3 & \lambda_3^2M_3 & \lambda_3^4M_3\\ \lambda_4 & -M_4 & \lambda_4^2M_4 & \lambda_4^4M_4\end{vmatrix}. \tag{51}$$

Some periodic and breather solutions for GNLS equation (4) are shown, we consider $N=2$ and plot their structure figures in Fig. 2.

Case 2: We consider a solution $u=\frac{\omega_0}{\beta\gamma_0}e^{-i(\gamma_0^2x+\delta_0t)}$ with $\delta_0=\alpha[(\beta+\sigma\gamma_0^2)^2-\omega_0^2]\gamma_0^{-2}$. Based on Eq. (5), we can yield the following systems

$$\begin{cases}-i\lambda\psi_1-\frac{i\omega_0\gamma_0^2}{\beta\gamma_0}e^{-i(\gamma_0^2x+\delta_0t)}\psi_2=\psi_{1x},\\ -\frac{i\omega_0\gamma_0^2}{\beta\gamma_0}e^{i(\gamma_0x+\delta_0t)}\psi_1+i\lambda\psi_2=\psi_{2x},\end{cases} \tag{52}$$

without loss of generality, we assume that $\sigma=-1,\psi_1=\alpha_1e^{\beta_1x},\psi_2=\gamma_1e^{\beta_1x+i(\gamma_0^2x+\delta_0t)}$, then Eq. (52) is solved by

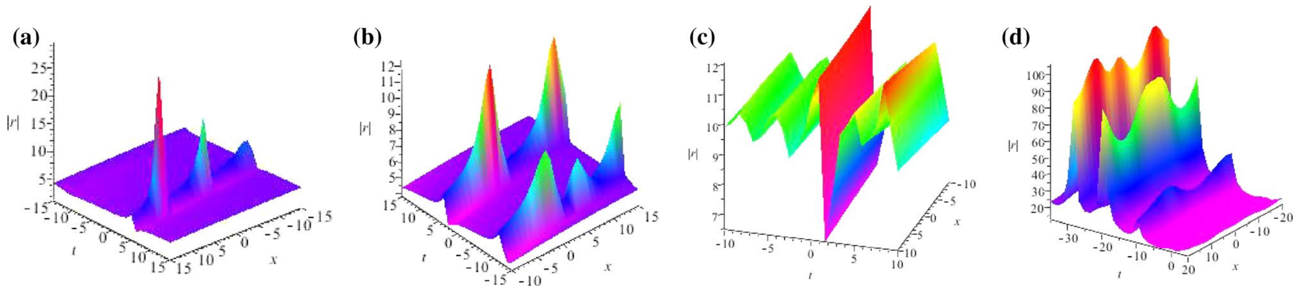

Figure 2. Profiles of intensity distribution (**a**) $|\widetilde{r}(x,t)|$ of Eq. (48) with parameters $\lambda_1 = 0.2 + 0.3i, \lambda_2 = 0.2 - 0.3i, \alpha = 0.2, \beta = 0.3, \gamma_0 = 0.2, \sigma = -1, F_1 = 0.3, F_2 = 0.2$; (**b**) $|\widetilde{r}(x,t)|$ of Eq. (48) with parameters $\lambda_1 = 0.3i, \lambda_2 = 0.2 - 0.4i, \alpha = 0.2, \beta = 0.3, \gamma_0 = 0.2, \sigma = -1, F_1 = 0.3, F_2 = 0.2$; (**c**) $|\widetilde{r}(x,t)|$ of Eq. (50) with parameters $\lambda_1 = -0.3i, \lambda_2 = 0.2 + 0.3i, \lambda_3 = 0.1 - 0.3i, \lambda_4 = 0.4i, \alpha = 0.6, \beta = 0.2, \gamma_0 = 0.1, \sigma = -1, F_1 = 0.3, F_2 = 0.2, F_3 = 0.4, F_4 = 0.1$; (**d**) $|\widetilde{r}(x,t)|$ of Eq. (50) with parameters $\lambda_1 = 0.1i, \lambda_2 = 0.2 - 0.4i, \lambda_3 = 0.3i, \lambda_4 = 0.2i, \alpha = 0.2, \beta = 0.3, \gamma_0 = 0.5, \sigma = -1, F_1 = 0.3, F_2 = 0.2, F_3 = 0.4, F_4 = 0.1$.

$$\begin{cases} \beta_1 = \frac{-i\gamma_0^2 \pm \sqrt{\Delta_1}}{2\beta^2}, \\ \alpha_1 = \frac{-i\omega_0\gamma_0\gamma_1}{\beta(\beta_1 + i\lambda)}, \end{cases} \tag{53}$$

we can obtain $\Delta_1 = -\gamma_0^4\beta^4 - 4\beta^2(\omega_0^2\gamma_0^2 + \lambda^2\beta^2 - \lambda\gamma_0^2\beta^2)$. By using Eq. (5), we obtain

$$\begin{cases} \left(\frac{i\alpha\omega_0^2}{2\gamma_0^2} - i\eta^2\right)\psi_1 + \left(\frac{i\alpha\beta\omega_0}{2\lambda\gamma_0}e^{-i(\gamma_0^2 x + \delta_0 t)} - \frac{i\alpha\lambda\gamma_0\omega_0}{\beta}e^{-i(\gamma_0^2 x + \delta_0 t)}\right)\psi_2 = \psi_{1t}, \\ \left(\frac{i\alpha\beta\omega_0}{2\lambda\gamma_0} - \frac{i\alpha\lambda\gamma_0\omega_0}{\beta}\right)e^{i(\gamma_0^2 x + \delta_0 t)}\psi_1 + \left(-\frac{i\alpha\omega_0^2}{2\gamma_0^2} + i\eta^2\right)\psi_2 = \psi_{2t}, \end{cases} \tag{54}$$

without loss of generality, we assume that $\psi_1 = a_1 e^{ct}, \psi_2 = b_1 e^{ct + i(\gamma_0^2 x + \delta_0 t)}$, then Eq. (54) is solved by

$$\begin{cases} c = \frac{i\alpha\lambda\beta\omega_0^2 - 2i\lambda\beta\eta^2\gamma_0^2 + i\alpha\beta^2\omega_0\gamma_0 - 2i\alpha\lambda^2\gamma_0^3\omega_0}{2\lambda\beta\gamma_0^2 a_1}, \\ b_1 = \frac{a_1(i\alpha\beta^2\gamma_0\omega_0 - 2i\alpha\lambda^2\gamma_0^3\omega_0 - i\alpha\lambda\beta\omega_0^2 + 2i\eta^2\lambda\beta\gamma_0^2)}{i\alpha\lambda\beta\omega_0^2 - 2i\lambda\beta\eta^2\gamma_0^2 + i\alpha\beta^2\omega_0\gamma_0 - 2i\alpha\lambda^2\gamma_0^3\omega_0 + 2i\lambda\beta\gamma_0^2\delta_0 a_1}. \end{cases} \tag{55}$$

It is easy to find two basic solutions for Eqs. (53) and (55) as following

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} C_3 e^{\beta_1 x + ct} \\ C_4 e^{\beta_1 x + dt + 2i(\gamma_0^2 x + \delta_0 t)} \end{pmatrix}, \tag{56}$$

we can derive by using Eq. (10),

$$M_j = \frac{e^{F_j} e^{dt + 2i(\gamma_0^2 x + \delta_0 t)}}{e^{ct}}, \quad 1 \le j \le 2N, \tag{57}$$

with $\nu_j^{(i)} = e^{F_j}(1 \le i \le 2, 1 \le j \le 2N)$.

(I) We take $N = 1$ with $\lambda = \lambda_j$ (j = 1, 2), and yield the 1-soliton solutions of the GNLS equation (4) as following:

$$\widetilde{u}(x,t) = \frac{\omega_0}{\beta\gamma_0}e^{-i(\gamma_0^2 x + \delta_0 t)} + \frac{\Delta B_1}{\Delta}, \quad \widetilde{r}(x,t) = -\widetilde{u}^*(x,t), \tag{58}$$

with

$$\Delta = \begin{vmatrix} \lambda_1^2 & M_1\lambda_1 \\ \lambda_2^2 & M_2\lambda_2 \end{vmatrix}, \ \Delta B_1 = \begin{vmatrix} \lambda_1^2 & -1 \\ \lambda_2^2 & -1 \end{vmatrix}, \ \Delta C_1 = \begin{vmatrix} -M_1 & \lambda_1^2 M_1 \\ -M_2 & \lambda_2^2 M_2 \end{vmatrix}. \tag{59}$$

(II) We take $N = 2$ in the N-times DT with $\lambda = \lambda_j (j = 1, 2, 3, 4)$. The linear algebraic system (9) leads to the 2-soliton solutions of GNLS Eq. (4) as following:

$$\widetilde{u}(x,t) = \frac{\omega_0}{\beta\gamma_0}e^{-i(\gamma_0^2 x + \delta_0 t)} + \frac{\Delta B_2}{\Delta}, \quad \widetilde{r}(x,t) = -\widetilde{u}^*(x,t), \tag{60}$$

with

$$\Delta = \begin{vmatrix} \lambda_1^2 & \lambda_1^4 & M_1\lambda_1 & M_1\lambda_1^3 \\ \lambda_2^2 & \lambda_2^4 & M_2\lambda_2 & M_2\lambda_2^3 \\ \lambda_3^2 & \lambda_3^4 & M_3\lambda_3 & M_3\lambda_3^3 \\ \lambda_4^2 & \lambda_4^4 & M_4\lambda_4 & M_4\lambda_4^3 \end{vmatrix}, \Delta B_2 = \begin{vmatrix} \lambda_1^2 & \lambda_1^4 & M_1\lambda_1 & -1 \\ \lambda_2^2 & \lambda_2^4 & M_2\lambda_2 & -1 \\ \lambda_3^2 & \lambda_3^4 & M_3\lambda_3 & -1 \\ \lambda_4^2 & \lambda_4^4 & M_4\lambda_4 & -1 \end{vmatrix}, \Delta C_2 = \begin{vmatrix} \lambda_1 & -M_1 & \lambda_1^2 M_1 & \lambda_1^4 M_1 \\ \lambda_2 & -M_2 & \lambda_2^2 M_2 & \lambda_2^4 M_2 \\ \lambda_3 & -M_3 & \lambda_3^2 M_3 & \lambda_3^4 M_3 \\ \lambda_4 & -M_4 & \lambda_4^2 M_4 & \lambda_4^4 M_4 \end{vmatrix}. \tag{61}$$

Some periodic solutions for GNLS equation (4) with seed $u = \frac{\omega_0}{\beta\gamma_0} e^{-i(\gamma_0^2 x + \delta_0 t)}$ are shown, we consider $N = 2$ and plot their structure figures in Fig. 3.

Case 3: We consider a seed solution $u = e^{i\theta}$ with $\theta = ax + bt$, $b = \frac{1+a}{a}\alpha\beta^2 + 2\alpha\beta + a\alpha$. We can yield the following systems from Eq. (5)

$$\begin{cases} -i\lambda^2\varphi_1 - i\lambda e^{i\theta}\varphi_2 = \varphi_{1x}, \\ -i\lambda e^{-i\theta}\varphi_1 + i\lambda^2\varphi_2 = \varphi_{2x}, \end{cases} \tag{62}$$

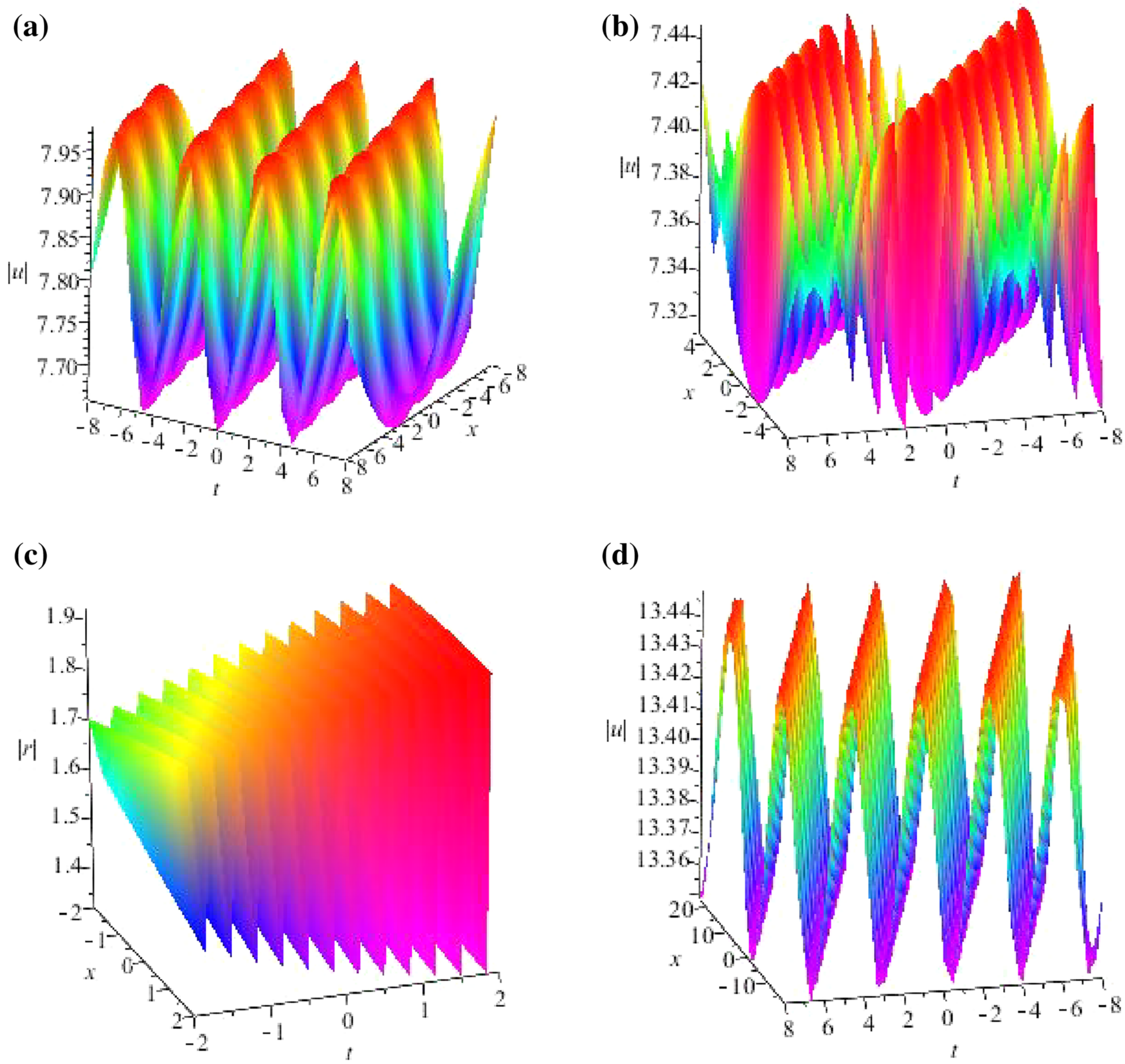

Figure 3. Profiles of intensity distribution (**a**) $|\widetilde{u}(x,t)|$ of Eq. (58) with parameters $\lambda_1 = 0.2, \lambda_2 = 0.3, \alpha = 0.3, \beta = 5, \gamma_0 = 0.5, \sigma = -1, \omega_0 = 0.4, F_1 = 0.3, F_2 = 0.4$; (**b**) $|\widetilde{u}(x,t)|$ of Eq. (58) with parameters $\lambda_1 = 0.3 + 0.2i, \lambda_2 = 0.3 - 0.2i, \alpha = 0.4, \beta = 5, \gamma_0 = 0.6, \sigma = -1, \omega_0 = 0.2, F_1 = 0.2 + 0.3i, F_2 = 0.2 - 0.3i$; (**c**) $|\widetilde{r}(x,t)|$ of Eq. (60) with parameters $\lambda_1 = 0.3i, \lambda_2 = -0.2i, \lambda_3 = 0.4i, \lambda_4 = -0.5i, \alpha = 0.4, \beta = 5, \gamma_0 = 0.2, \sigma = -1, \omega_0 = 0.3, F_1 = 0.3i, F_2 = -0.3i, F_1 = 0.5i, F_1 = -0.5i$; (**d**) $|\widetilde{u}(x,t)|$ of Eq. (60) with parameters $\lambda_1 = 0.3, \lambda_2 = -0.2, \lambda_3 = 0.4, \lambda_4 = 0.5, \alpha = 0.4, \beta = 8, \gamma_0 = 0.5, \sigma = -1, \omega_0 = 0.2, F_1 = 0.3i, F_2 = -0.3i, F_3 = 0.5i, F_4 = -0.5i$.

without loss of generality, we assume that $\sigma = -1, \varphi_1 = me^{c_1 x}, \varphi_2 = ne^{c_1 x - i\theta}$, then Eq. (62) is solved by

$$\begin{cases} n = \frac{1 \pm \sqrt{s} - 2\lambda^2}{2\lambda} m, \\ c_1 = \frac{1 \pm \sqrt{s}}{2i}. \end{cases} \tag{63}$$

We can obtain $s = 1 + 4\lambda^4$. We derive the system through Eq. (5),

$$\begin{cases} \left(\frac{i\alpha\beta^2}{2} - i\eta^2\right)\psi_1 + \left(\frac{i\alpha\beta^2}{2\lambda} - i\alpha\lambda\right)e^{i\theta}\psi_2 = \psi_{1t}, \\ \left(\frac{i\alpha\beta^2}{2} - i\alpha\lambda\right)e^{-i\theta}\psi_1 + \left(i\eta^2 - \frac{i\alpha\beta^2}{2}\right)\psi_2 = \psi_{2t}, \end{cases} \tag{64}$$

without loss of generality, we assume that $\psi_1 = pe^{s_1 t}, \psi_2 = qe^{s_1 t - i\theta}, \alpha = 1, \beta = -1, \eta = \sqrt{\alpha}(\lambda - \frac{\beta}{2\lambda})$, then Eq. (64) is solved by

$$\begin{cases} p = \frac{i - 2i\lambda^2}{4i\lambda^4 + 2i\lambda^2 + 4\lambda^2 s_1 + i} q, \\ s_1 = \frac{-(40i\lambda^4 + 8i\lambda^2) \pm \sqrt{z^2 - 64\lambda^4 y}}{32\lambda^4}, \end{cases} \tag{65}$$

we can obtain z and y as following : $z = 40i\lambda^4 + 8i\lambda^2, y = 16\lambda^8 - 24\lambda^6 - 8\lambda^5 - 8\lambda^4 - 4\lambda^3 - 8\lambda^2 + 1$.
It is easy to find two basic solutions for Eqs. (63) and (65):

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} C_5 e^{\frac{(\lambda + \lambda\sqrt{Q})x + (3 - 4\sqrt{9\lambda^2 - \Delta_2})t}{2i\lambda}} \\ C_6 e^{\frac{2\lambda(1 + \sqrt{Q})x + (3 - 4\sqrt{9\lambda^2 - \Delta_2})t}{2i\lambda} - 2i\theta} \end{pmatrix}, \tag{66}$$

we can obtain that : $\Delta_2 = \lambda^2(\alpha^2\beta^4 - 4\alpha\eta^2\beta^2 + 6\alpha\beta^2 + 4\eta^4 - 12\eta^2 - 4\alpha^2\beta^2 + 4\alpha^2\lambda^2) + \alpha\beta^4, C_5 = \frac{-2\alpha\lambda^2 + \alpha\beta^2}{\alpha\lambda\beta^2}$, $C_6 = \frac{2\lambda^2 - \sqrt{Q} - 1}{2\lambda}, Q = 3 - 4\sqrt{9\lambda^2 - \Delta_2}$.
According to Eq. (10), we obtain

$$M_j = e^{\frac{(\lambda + \lambda\sqrt{Q})x}{2i\lambda} + F_j - 2i\theta}, \quad 1 \le j \le 2N, \tag{67}$$

with $v_j^{(i)} = e^{F_j}(1 \le i \le 2, 1 \le j \le 2N)$.

(I) We take $N = 1$ with $\lambda = \lambda_j$ (j = 1, 2) and derive the 1-breather solutions of the GNLS equation (4) as following:

$$\widetilde{u}(x,t) = e^{i\theta} + \frac{\Delta B_1}{\Delta}, \quad \widetilde{r}(x,t) = -\widetilde{u}^*(x,t), \tag{68}$$

with

$$\Delta = \begin{vmatrix} \lambda_1^2 & M_1\lambda_1 \\ \lambda_2^2 & M_2\lambda_2 \end{vmatrix}, \Delta B_1 = \begin{vmatrix} \lambda_1^2 & -1 \\ \lambda_2^2 & -1 \end{vmatrix}, \Delta C_1 = \begin{vmatrix} -M_1 & \lambda_1^2 M_1 \\ -M_2 & \lambda_2^2 M_2 \end{vmatrix}. \tag{69}$$

(II) We take $N = 2$ in the N-times DT with $\lambda = \lambda_j (j = 1, 2, 3, 4)$. The linear algebraic system (9) leads to the 2-breather solutions of GNLS Eq. (4) as following:

$$\widetilde{u}(x,t) = e^{i\theta} + \frac{\Delta B_2}{\Delta}, \quad \widetilde{r}(x,t) = -\widetilde{u}^*(x,t), \tag{70}$$

with

$$\Delta = \begin{vmatrix} \lambda_1^2 & \lambda_1^4 & M_1\lambda_1 & M_1\lambda_1^3 \\ \lambda_2^2 & \lambda_2^4 & M_2\lambda_2 & M_2\lambda_2^3 \\ \lambda_3^2 & \lambda_3^4 & M_3\lambda_3 & M_3\lambda_3^3 \\ \lambda_4^2 & \lambda_4^4 & M_4\lambda_4 & M_4\lambda_4^3 \end{vmatrix}, \Delta B_2 = \begin{vmatrix} \lambda_1^2 & \lambda_1^4 & M_1\lambda_1 & -1 \\ \lambda_2^2 & \lambda_2^4 & M_2\lambda_2 & -1 \\ \lambda_3^2 & \lambda_3^4 & M_3\lambda_3 & -1 \\ \lambda_4^2 & \lambda_4^4 & M_4\lambda_4 & -1 \end{vmatrix}, \Delta C_2 = \begin{vmatrix} \lambda_1 & -M_1 & \lambda_1^2 M_1 & \lambda_1^4 M_1 \\ \lambda_2 & -M_2 & \lambda_2^2 M_2 & \lambda_2^4 M_2 \\ \lambda_3 & -M_3 & \lambda_3^2 M_3 & \lambda_3^4 M_3 \\ \lambda_4 & -M_4 & \lambda_4^2 M_4 & \lambda_4^4 M_4 \end{vmatrix}. \tag{71}$$

Some breather solutions for GNLS equation (4) with seed $u = \frac{\omega_0}{\beta\gamma_0} e^{-i(\gamma_0^2 x + \delta_0 t)}$ are shown, we consider $N = 2$ and plot their structure figures in Fig. 4.

Rational wave solutions for GNLS Eq. (4)

In this section, we construct the rational wave solutions of the GNLS Eq. (4). In fact, the rational wave solutions can be obtained by the limits of the eigenfunctions or the limits of the breather solutions.
Based on Eq. (66), we can get a new eigenfunction of the Lax pair (5)

$$R_1(\varepsilon) = (f_1, g_1)^T, \tag{72}$$

with

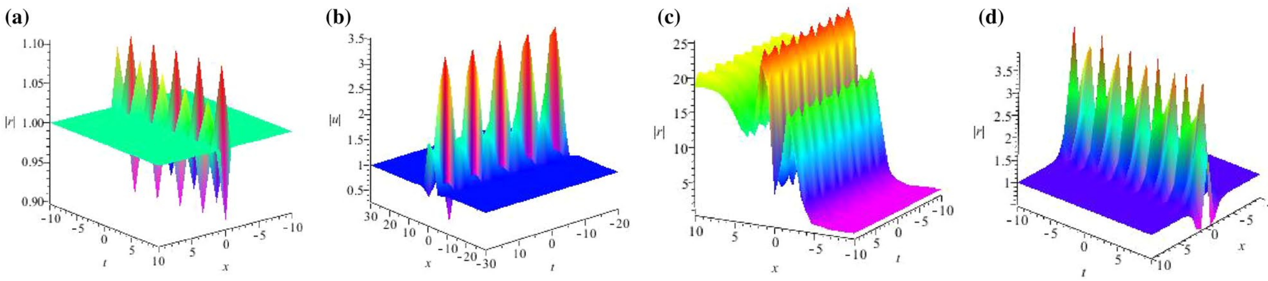

Figure 4. Profiles of intensity distribution (**a**) $|\widetilde{r}(x,t)|$ of Eq. (68) with parameters $\lambda_1 = -0.3+5i, \lambda_2 = 0.3+4i, \alpha = 1, \beta = -1, a = -1, b = 3, \sigma = -1, F_1 = i, F_2 = 2i$; (**b**) $|\widetilde{u}(x,t)|$ of Eq. (68) with parameters $\lambda_1 = 0.5i, \lambda_2 = 0.3i, \alpha = 1, \beta = -1, a = -1, b = 3, \sigma = -1, F_1 = i, F_2 = 2i$; (**c**) $|\widetilde{r}(x,t)|$ of Eq. (70) with parameters $\lambda_1 = 0.5i, \lambda_2 = -0.3i, \lambda_3 = 0.2i, \lambda_4 = -0.4i, \alpha = 1, \beta = -1, a = -1, b = 3, \sigma = -1, F_1 = i, F_2 = 2i, F_3 = 3i, F_4 = 2i$; (**d**) $|\widetilde{r}(x,t)|$ of Eq. (70) with parameters $\lambda_1 = 0.03+0.5i, \lambda_2 = 0.03-0.5i, \lambda_3 = 0.02+0.3i, \lambda_4 = 0.02-0.3i, \alpha = 1, \beta = -1, a = -1, b = 3, \sigma = -1, F_1 = i, F_2 = 2i, F_3 = 3i, F_4 = 2i$.

$$\begin{aligned} f_1 =& C_5 e^{\frac{(\lambda+\lambda\sqrt{Q})x+(3-4\sqrt{9\lambda^2-\Delta_2})t}{2i\lambda}},\ g_1 = C_6 e^{\frac{2\lambda(1+\sqrt{Q})x+(3-4\sqrt{9\lambda^2-\Delta_2})t}{2i\lambda}-2i\theta},\\ C_5 =& \frac{-2\alpha\lambda^2+\alpha\beta^2}{\alpha\lambda\beta^2},\ C_6 = \frac{2\lambda^2-\sqrt{Q}-1}{2\lambda},\ Q = 3-4\sqrt{9\lambda^2-\Delta_2},\\ \Delta_2 =& \lambda^2(\alpha^2\beta^4 - 4\alpha\eta^2\beta^2 + 6\alpha\beta^2 + 4\eta^4 - 12\eta^2 - 4\alpha^2\beta^2 + 4\alpha^2\lambda^2) + \alpha\beta^4, \end{aligned}$$

where ε is a small parameter, if we fix $\lambda_1 = \frac{1}{2}+\frac{1}{2}i$, and let $\lambda = \frac{1}{2}+\frac{1}{2}i+\varepsilon^2$, then $R_1(\varepsilon)$ can be expanded at $\varepsilon = 1$, so we have

$$R_1(\varepsilon) = R_1^{[0]} + R_1^{[1]}\varepsilon^2 + R_1^{[2]}\varepsilon^4 + R_1^{[3]}\varepsilon^6 + \cdots \tag{73}$$

where

$$R_1^{[0]} = \begin{pmatrix} C_5 e^{\frac{Fx+Qt}{i-1}} \\ C_6 e^{\frac{2Fx+Qt+2\theta(i-1)}{i-1}} \end{pmatrix}, \tag{74}$$

and

$$R_1^{[1]} = \begin{pmatrix} \frac{-2i\varepsilon^2(Fx+Qt)}{(i-1)^2} C_5 e^{\frac{Fx+Qt}{i-1}} \\ \frac{4\theta(i-1)-2i[2Fx+Qt+2\theta(i-1)]}{i-1} C_6 e^{\frac{2Fx+Qt+2\theta(i-1)}{i-1}} \end{pmatrix}, \tag{75}$$

with

$$Q = 3-4\sqrt{9\lambda^2-\Delta_2},\ \ F = \lambda+\lambda\sqrt{Q}.$$

We present the rational wave solution of the GNLS Eq. (4) as following:

$$u_R = u + \frac{f_1^{[1]} g_1^{[1]^*}(\lambda^2 - {\lambda^*}^2)}{|\lambda|^2(|f_1^{[1]}|^2\lambda + |g_1^{[1]}|^2\lambda^*)}, \tag{76}$$

with

$$f_1^{[1]} = C_5 \frac{-2i\varepsilon^2(Fx+Qt)}{(i-1)^2},\ \ g_1^{[1]} = C_6 \frac{4\theta(i-1)-2i[2Fx+Qt+2\theta(i-1)]}{i-1}.$$

Some rational wave solutions for GNLS equation (4) are shown with the limits of the breather solutions, we plot their structure figures in Fig. 5.

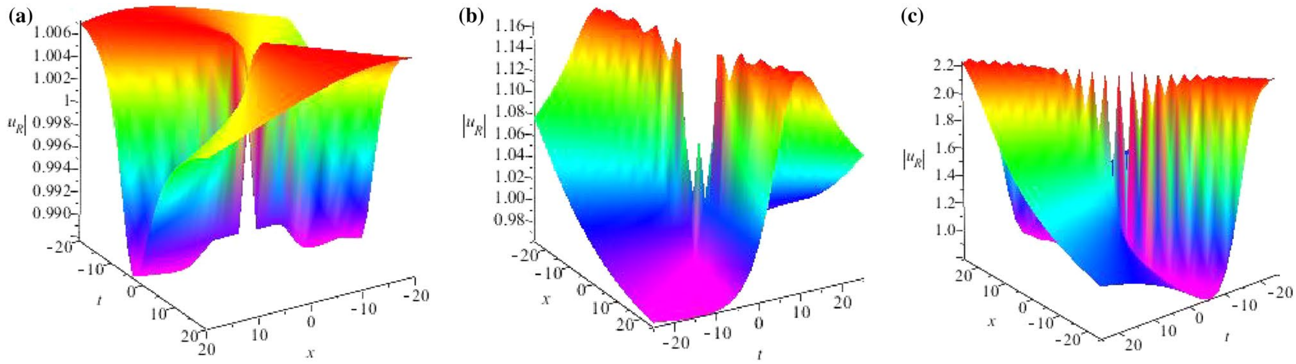

Figure 5. Profiles of intensity distribution (**a**) $|\widetilde{u}_R(x,t)|$ of Eq. (76) with parameters $\lambda=\frac{5}{2}+\frac{1}{2}i, \alpha=-0.3, \beta=0.5, a=-1, b=3$; (**b**) $|\widetilde{u}_R(x,t)|$ of Eq. (76) with parameters $\lambda=\frac{3}{2}+\frac{1}{2}i, \alpha=0.9, \beta=-0.8, a=-1, b=3$; (**c**) $|\widetilde{u}_R(x,t)|$ of Eq. (76) with parameters $\lambda=\frac{1}{2}-i, \alpha=0.6, \beta=-0.6, a=-1, b=3$.

Data availability

All data generated or analysed during this study are included in this published article.

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References

1. Latchio Tiofack, C. G. *et al.* Exact quasi-soliton solutions and soliton interaction for the inhomogeneous coupled nonlinear Schrödinger equations. *J. Mod. Optic.* **57**(4), 261–272 (2010).
2. Vijayalekshmi, S., Mahalingam, A. & Mani-Rajan, M. S. Symbolic computation on tunable nonautonomous solitons in inhomogeneous NLS system with generalized external potential. *Optik* **145**, 240–249 (2017).
3. Ablowitz, M. J. & Musslimani, Z. H. Integrable nonlocal nonlinear Schrödinger equation. *Phys. Rev. Lett.* **110**(6), 064105 (2013).
4. Yan, Z. Y. Integrable PT-symmetric local and nonlocal vector nonlinear Schrödinger equations: A unified two parameter model. *Appl. Math. Lett.* **47**, 61–68 (2015).
5. Zhou, Z. X. Darboux transformations and global solutions for a nonlocal derivative nonlinear Schrödinger equation. *Commun. Nonlinear. Sci. Numer. Simul.* **62**, 480–488 (2018).
6. Song, C. Q., Xiao, D. M. & Zhu, Z. N. Solitons and dynamics for a general integrable nonlocal coupled nonlinear Schrödinger equation. *Commun. Nonlinear. Sci. Numer. Simul.* **45**, 13–28 (2017).
7. Morgan, S. A., Ballagh, R. J. & Burnett, K. Solitary-wave solutions to nonlinear Schrödinger equations. *Phys. Rev. A.* **55**(6), 4338–4345 (1997).
8. Guo, B. L., Ling, L. M. & Liu, Q. P. Nonlinear Schrödinger equation: Generalized Darboux transformation and rogue wave solutions. *Phys. Rev. E.* **85**, 026607 (2012).
9. Biswas, A. & Milovic, D. Bright and dark solitons of the generalized nonlinear Schrödinger equation. *Commun. Nonlinear. Sci. Numer. Simul.* **15**(6), 1473–1484 (2010).
10. Hirota, R. *The Direct Method in Solition Theory* (Cambrige University Press, 2004).
11. Yu, F. J. Inverse scattering solutions and dynamics for a nonlocal nonlinear Gross-Pitaevskii equation with PT-symmetric external potentials. *Appl. Math. Lett.* **92**, 108–114 (2019).
12. Ablowitz, M. J. & Clarkson, P. A. *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, 1991).
13. Gerdjikov, V. S. Bose–Einstein condensates and spectral properties of multicomponent nonlinear Schrödinger equations. *Discret. Cont. Dyn.* **4**(5), 1181–1197 (2011).
14. Wang, M. L. Solitary wave solutions for variant Boussinesq equations. *Phys. Lett. A.* **199**, 169–172 (1995).
15. Fan, E. G. & Zhang, H. Q. Some new applications of homogeneous balance method. *Acta. Math.* **19**(3), 286–292 (1999).
16. Li, Y. S. & Zhang, J. E. Darboux transformations of classical Boussinesq system and its multi-soliton solutions. *Phys. Lett. A.* **284**(6), 253–258 (2001).
17. Zhao, Y. N-fold Darboux transformation for a nonlinear evolution equation. *Appl. Math.* **3**(8), 943–948 (2012).
18. Yu, F. J. Dynamics of nonautonomous discrete rogue wave solutions for an Ablowitz–Musslimani equation with PT-symmetric potential. *Chaos* **27**(2), 023108 (2017).
19. Fiacco, A. V. Second-order sufficient conditions for weak and strict constrained minima. *SIAM. J. Appl. Math.* **16**, 105 (1968).
20. Ablowitz, M. J. & Musslimani, Z. H. Integrable nonlocal nonlinear equation. *Stud. Appl. Math.* **139**(1), 7–59 (2017).
21. Ji, J. L. & Zhu, Z. N. On a nonlocal modified Korteweg–de Vries equation: Integrability, Darboux transformation and soliton solutions. *Commun. Nonlinear. Sci. Numer. Simul.* **42**, 699–708 (2017).
22. Li, L., Wang, L. & Yu, F. J. Some general bright soliton solutions and interactions for a (2+1)-dimensional nonlocal nonlinear Schrödinger equation. *Appl. Math. Lett.* **141**, 108600 (2023).
23. Li, L., Yu, F. J. & Duan, C. N. A generalized nonlocal Gross–Pitaevskii (NGP) equation with an arbitrary time-dependent linear potential. *Appl. Math. Lett.* **110**, 106584 (2020).
24. Yu, F. J., Liu, C. P. & Li, L. Broken and unbroken solutions and dynamic behaviors for the mixed local-nonlocal Schrödinger equation. *Appl. Math. Lett.* **117**, 107075 (2021).
25. Yu, F. J. & Li, L. Vector dark and bright soliton wave solutions and collisions for spin-1 Boses Einstein condensate. *Nonlinear. Dyn.* **87**(4), 2697–2713 (2017).
26. Shi, W. & Zhaqilao, S. Higher-order mixed solution and breather solution on a periodic background for the Kundu equation. *Commun. Nonlinear. Sci. Numer. Simul.* **119**, 107134 (2023).
27. Tang, Y. N., He, C. H. & Zhou, M. L. Darboux transformation of a new generalized nonlinear Schrödinger equation: Soliton solutions, breather solutions, and rogue wave solutions. *Nonlinear. Dyn.* **92**, 2023–2036 (2018).

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L.L. and F.J.Y.: supervision; writing—original draft; funding acquisition. C.F.: validation; editing.

Competing interests
The authors declare no competing interests.

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