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A nonlocal game for witnessing quantum networks

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Nonlocal game as a witness of the nonlocality of entanglement is of fundamental importance in various fields. The well-known nonlocal games or equivalent linear Bell inequalities are only useful for Bell networks consisting of single entanglement. Our goal in this paper is to propose a unified method for constructing cooperating games in network scenarios. We propose an efficient method to construct multipartite nonlocal games from any graphs. The main idea is the graph representation of entanglement-based quantum networks. We further specify these graphic games with quantum advantages by providing a simple sufficient and necessary condition. The graphic games imply a linear Bell testing of the nonlocality of general quantum networks consisting of EPR states. It also allows generating new instances going beyond CHSH game. These results have interesting applications in quantum networks, Bell theory, computational complexity, and theoretical computer science.

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INTRODUCTION

A nonlocal game is generally described as multiple space-separated players interacting with a referee. The communication between players are forbidden. Remarkably, it is related to Bell theory that is fundament in quantum mechanics[1] when players are allowed to share classical resources or quantum entangled resources. Clauser-Horne-Shimony-Holt (CHSH) game as a distributed evaluating of Boolean equation: $y_1 \oplus y_2 = x_1 \wedge x_2$, provides the first notable way for witnessing the nonlocality of Einstein-Podolsky-Rosen (EPR) state,[1–6] where x_i, y_i are respective binary input and output. The shared entanglement permits a quantum strategy with larger winning probability than all classical strategies.

Generally, a nonlocal game allows all the players to determine a joint strategy from the complete knowledge of inputs distribution and the predicate conditions for win. The corresponding procedure can be featured by a generalized hidden variable model with classical sources or quantum sources.[1] One fundamental problem is to determine whether quantum mechanics is superior to classical theory in terms of nonlocal tasks. These evaluations are equivalent to solving special optimization problems from Tsirelson's reductions.[7] Unfortunately, no efficient algorithm exists for general multipartite nonlocal games.[8–13] Besides verifying entanglement, nonlocal games have so far inspired numerous applications, such as multi-prover interactive proofs,[14–16] quantum proof verification,[17,18] hardness of approximation,[19–21] PCP conjecture,[22–24] separating correlations,[25–28] and communication complexity theory.[29,30]

There are various interesting nonlocal games except for CHSH games. XOR games are the most well studied.[31–34] The global task is to XOR of the outcomes of all players. Other games include Kochen-Specker game,[4,31] non-zero sum games,[35,36] magic square game,[37] graph isomorphism game,[38] Bayesian game,[39] and conflicting interest game.[40] The key to achieve a quantum advantage is the nonlocal correlations generated by local measurements on the shared entangled states. Similar correlations are not achievable by remote players using local strategies with any shared classical resources. These tasks show the important role of quantum entanglement in information processing.[41]

Comparison to single entanglement,[1] the network scenarios of multiple independent sources require the nonlinear Bell inequalities to test the nonlocality.[42–45] Hence, how to construct meaningful nonlocal games should be interesting in scaling applications of entangled resources. Two related problems are as follows:

Problem 1—How to efficiently construct nonlocal games for space-separated players who share a quantum network consisting of entangled states?

Problem 2—How to characterize nonlocal games with or without quantum advantages in network scenarios?

The goal of this paper is to address these problems in the context of cooperative nonlocal games, as shown in Fig. 1. We propose a unified method to construct multipartite games from graph representations of general multi-source networks. Each game is determined by some subgraphs that can be constructed efficiently. Note that for any graph with N nodes, the proposed method provides $O(4^{nN})$ different nonlocal games with n players. These graphic nonlocal games will further be classified in terms of the quantum advantage using a simple sufficient and necessary condition. The result holds for the same probability distribution of all inputs going beyond previous algorithmic results[46] or semi-definite programs.[7,10,11,47,48] Surprisingly, the present graphic games allow testing the nonlocality of multi-source quantum networks going beyond previous CHSH game for single entanglement.[1–3,36] Compared with the recent nonlinear witness,[42–45] the graphic game provides the first verification of general networks using linear Bell testing beyond semiquantum game.[49] The new result is also useful for nonlocal satisfiability problems going beyond CHSH game[3] and cubic game.[37] The graphic game finally extends the guessing your neighbor's input (GYNI) game.[50,51] The present model provides novel instances to feature nonlocal games with different performances.[52]

RESULT

Nonlocal multipartite cooperating games

An n-player nonlocal cooperating game consists of n players A_1, A_2, ..., A_n and a referee,[31] as shown in Fig. 1. All the players agree

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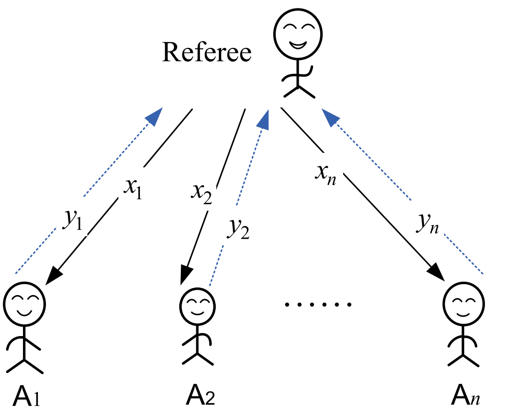

Fig. 1 Cooperating nonlocal game. A referee chooses questions x_ks from a finite set according to a prior distribution $p(\mathbf{x})$, and sends to players. Each player A_k should output an answer y_k based on the received question x_k and the shared resource without communicating with other players. The referee then evaluates a payoff function F with all outputs and inputs to determine win or lose for all the players

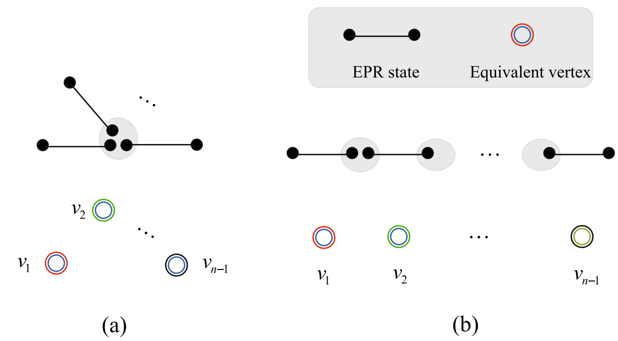

Fig. 2 Graphic representation of multi-source quantum networks consisting of EPR states. **a** Star-type network. Each of $n-1$ observers share one EPR state with the center observer. **b** Chain-type network of entanglement swapping in long distance. Any two adjacent observers share one EPR state. Each entanglement is schematically represented by one vertex in graph. The relationship that two observers share one entangled state is equivalent to two players sharing one vertex, where each player owns vertices with the same color

with a joint strategy beforehand but cannot communicate with each other during the game. The referee firstly chooses n questions: x_1, x_2, ..., x_n from a finite set $\mathcal{X} := X_1\times X_2\times ...\times X_n$ according to a known distribution $p(\mathbf{x})$. And then, sends x_i to A_i, $i=1, 2, ..., n$. All the players are now required to reply with answers y_1, y_2, ..., y_n chosen from a finite set $\mathcal{Y} := Y_1\times Y_2\times ...\times Y_n$ to the referee. Finally, the referee determines whether all the players win or lose the game according to a payoff function: F: $\mathcal{X}\times\mathcal{Y}\rightarrow\{0, 1\}$, i.e., win if $F(\mathbf{x}, \mathbf{y})=1$ and lose if $F(\mathbf{x}, \mathbf{y})=0$, where $\mathbf{x}=x_1x_2...x_n$, $\mathbf{y}=y_1y_2...y_n$. The optimal average winning probability of all the players, i.e., the nonlocal value of the game, is defined by:

$$\varpi_c=\sup_{\Omega}\sum_{\mathbf{x},\mathbf{y},\lambda}p(\mathbf{x})\mu(\lambda)F(\mathbf{x},\mathbf{y})P(\mathbf{y}|\mathbf{x},\lambda) \tag{1}$$

Here, $\mu(\lambda)$ is the probability measure of the hidden classical resource λ and satisfies $\sum_\lambda\mu(\lambda)=1$. $P(\mathbf{y}|\mathbf{x}, \lambda)$ denotes the joint conditional probability depending on the shared randomness λ. The supremum is over all the possible classical spaces Ω of hidden variables λ. Generally, it is hard to approximate the nonlocal value ϖ_c of multipartite games.[10,11,48] From the linearity of ϖ_c with respect to all probability distributions, it is sufficient use deterministic strategies for all the players, i.e., $p(\lambda)=1$ for some λ.

In quantum scenarios, players are allowed to share various entangled states and perform quantum measurements (associated with noncommuting operators which are different from commuting operators in classical physics) to obtain answers. Let ρ be the shared entangled state. Denote positive-operator valued measurements (POVMs) of all the observers as $\{M_{y_1}^{x_1}\}$, $\{M_{y_2}^{x_2}\}$,..., $\{M_{y_n}^{x_n}\}$, which depend on the received questions x_1, x_2, ..., x_n, respectively. These operators satisfy the normalization conditions: $\sum_{y_i}M_{y_i}^{x_i}=\mathbb{I}$ for $i=1, 2, ..., n$, where $\mathbb{I}$ is the identity operator. The optimal winning probability for all the quantum players to player a cooperating nonlocal game defined above is defined by:

$$\varpi_q=\sup\sum_{\mathbf{x},\mathbf{y}}p(\mathbf{x})F(\mathbf{x},\mathbf{y})\mathrm{Tr}[\otimes_{j=1}^{n}M_{y_j}^{x_j}\rho], \tag{2}$$

where the supremum is over all the possible quantum states and local POVMs. Note that each linear Bell-type inequality is equivalent to a cooperating nonlocal game.[53] The nonlocal value ϖ_q is related to the Tsirelson bound of linear Bell inequality.[7] A central problem in the nonlocality theory is to find computable good bounds of nonlocal value ϖ_q.[7,31] Unfortunately, it is QMA-hard to approximate the nonlocal value of ϖ_q.[17,54]

Graphic games

CHSH game provides a novel idea to witness a bipartite entanglement.[2,3] As a natural extension, it would be of interesting to characterize multiple entangled states shared by space-separated observers in a network scenarios. Although the linear testing of single entanglement[53,55–59] allows designing equivalent nonlocal game, it is unknown how to construct meaningful nonlocal games for multi-source quantum networks that requires nonlinear testing[42–45] or semiquantum testing.[49] To address this problem, we propose a general method to construct nonlocal games from any graphs with only vertices. Surprisingly, the graphic games can be completely classified with respect to the nonlocal values. To explain the main result, we firstly present the following definitions.

Definition 1—(The graphic representation of quantum networks) Consider a multi-source network $\mathcal{N}_q$ consisting of multiple independent EPR states shared by different observers. Its graphic representation is defined as a vertex graph $\mathcal{G}_q$, where each node denotes one entanglement.

Some typical examples of quantum networks are shown in Fig. 2. Generally, for any graph, we can assign a subgraph to each player according to the input. In this case, the relation that two observers share one entangled state is equivalent to two players sharing one vertex, see Fig. 2, where each player owns some vertices with the same color. These nonlocal games can be then characterized by using the consistency conditions on the common vertices shared by different players. Formally, the graphic game is defined as follows:

Definition 2—An n-partite graphic game is a six-tuple: $(\mathcal{G},\mathcal{A},\mathcal{V},\mathcal{X},\mathcal{Y},\mathcal{F})$. $\mathcal{G}$ is a general graph with vertex set V (the edges are not required in this application). $\mathcal{A}$ denotes the set of all the players A_is and referee. $\mathcal{V}$ denotes the set of all the vertex sets $V^{x_i}\subseteq V$ owned by players, where V^{x_i} denotes the set of vertices assigned to the player A_i according to the input question x_i. Assume that V^{x_i}s satisfy the following conditions: $V^{x_i}\cap V^{x_j}=\emptyset$ for $x_i=x_j=1$ with $1\le i<j\le m$, where m is a parameter satisfying $1\le m<n$. $\mathcal{X}$ denotes the set of binary problems $x_1, x_2, ..., x_n\in\{0, 1\}$. $\mathcal{Y}$ denotes the set of all the assignments on vertices (as outputs) by players. F: $\mathcal{X}\times\mathcal{Y}\rightarrow\{0, 1\}$ is a payoff function depending on all the inputs and outputs of players. A graphic game is implemented as follows:

Input—The referee randomly chooses binary question x_i according to the distribution $\{p, 1-p\}$, and sends to the player A_i, $i=1, 2, ..., n$.

Output—The player A_i assigns an integer $y_i \in \{\pm 1\}$ on each vertex in the set V^{x_i}, and sends all the assignments to the referee secretly.

Winning conditions—All the players win the graphic game, i.e., $F(\mathbf{x}, \mathbf{y}) = 1$, if and only if their assignments satisfy the following consistency conditions:

(a) $S^{x_i} = 1$ for $i = m+1, m+2, ..., n$;
(b) $S^{x_i;x_j} = -1$ when $x_i = x_j = 1$, or 1 otherwise, for $1 \le i \le m < j \le n$;
(c) $S^{x_i;x_j} = S^{x_j;x_i} = 1$ for $m < i < j \le n$;

where S^{x_i} denotes the product of all the assignments of the player A_i, and $S^{x_i;x_j}$ denotes the product of all the assignments by the player A_i on the vertices shared with the player A_j.

Definition 2 contains previous nonlocal games[3,31,37] as special cases. The consistency conditions (the conditions (b) and (c) in Definition 2) are important for the most of nonlocal games.[3] Quantum correlations derived from entangled resources can satisfy these kinds of requirements with higher winning probability over all classical achievable correlations. Here, m is a free parameter satisfying $1 \le m < n$, which is used to feature the players whose assignments satisfy the consistency conditions. Specifically, the consistency conditions should be satisfied by the outputs of all the players to win a nonlocal game. Our graphic game is then similar to a nonlocal computation such as CHSH game with nonlinear restrictions shown in conditions (a)–(c). Generally, for a graph $\mathcal{G}$ with N vertices, there are $2^N - 1$ nontrivial subgraphs, which imply $O(4^{nN})$ different graphic games. It is hard to evaluate or approximate the nonlocal values for all these games. Hence, it should be interesting if some restricted games can be distinguished.

Toy example

Take the graph $\mathcal{G}$ shown in Fig. 3a as an example. There are two players in Fig. 3b. Each player holds one vertex independent of inputs, where their outputs are $1 - 2x_1$, $1 - 2x_2$. There is no consistency requirement for two players with $m = 1$. It is easy to obtain $\varpi_c = 1$ for classical players. Hence, there is no quantum advantage from $\varpi_q = \varpi_c = 1$. For the game in Fig. 2c, the player A_1 holds different vertices v_{x_1+1} according to the input x_1 while A_2 owns the whole graph independent of the input. In this case, two players have to satisfy the consistency conditions (b) and (c), i.e., $y_{1;A1} = y_{1;A2} = 1$ and $y_{2;A1} = y_{2;A2} = -1$, where $y_{i;Aj}$ denotes the assignment on the vertex v_i by the player A_j. Now, assume that the player A_1 assigns $1 - 2x_1$ on the shared one vertex according to the input x_1. The player A_2 assigns the same value of $1 - 2x_2$ on two vertices. With these assignments, the graphic game is equivalent to CHSH game.[3] This implies a strict quantum advantage for two players who share an EPR state.[2] Similar result holds for the graphic game shown in Fig. 3d. To explain the main result, some definitions will be introduced beforehand.

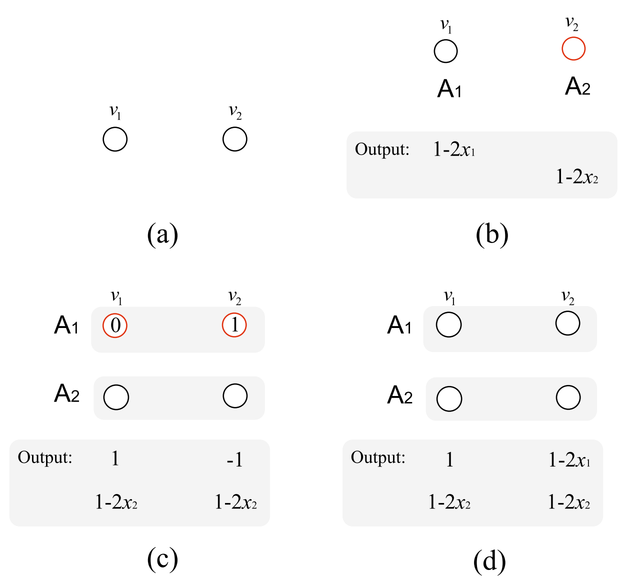

Fig. 3 A graphic nonlocal game. **a** A graph with two vertices. **b** A graphic game with two players. The player A_i holds the vertex v_i independent of the input, $i = 1$, 2. **c** A graphic game with two players. The player A_1 holds the vertex v_{x1} depending on its input x_1 while the player A_2 owns the vertices v_1, v_2 independent of the input. **d** A graphic game with two players. The player A_i holds the vertexes v_1, v_2 independent of the input, $i = 1$, 2. The players can share classical or quantum resources

Definition 3—Consider a graphic game with n players $A_1, A_2, ..., A_n$. For each i with $1 \le i \le m$, A_i denotes the set of players A_js ($m + 1 \le j \le n$) who share vertices with A_i for each input $x_i x_j$, i.e.,

$$\mathcal{A}_i = \{A_j | V^{x_i} \cap V^{x_j} \ne \emptyset, \forall x_i, x_j = 0, 1\}. \tag{3}$$

Definition 4—Denote $\mathcal{A}_i^s$ as the set of players as follows:

$$\mathcal{A}_i^s = \Big\{(A_i, A_{i_1}, \ldots, A_{i_{s-1}}) | A_{i_j} \in \mathcal{A}_i, V^{x_i} \cap (\cap_{j=1}^{s-1} V^{x_{i_j}}) \ne \emptyset, \forall x_i, x_{i_1}, \ldots, x_{i_{s-1}} = 0, 1\Big\}, \tag{4}$$

where each element consists of s players $A_i, A_{i_1}, \ldots, A_{i_{s-1}}$ who share common vertices for each input $x_i x_{i_1} \ldots x_{i_{s-1}}$, and $2 \le s \le n - m$.

$\mathcal{A}_i^s$ characterizes the consistency requirements among s players of A_js and A_i. In what follows, we omit the case that all the players have no common vertex because of $\varpi_c = \varpi_q$.

Definition 5—Let l_i be an integer associated with $\mathcal{A}_i^s$ as:

$$l_i = \min\{s | \mathcal{A}_i^s \ne \emptyset\}, \forall i = 1, 2, \ldots, m. \tag{5}$$

With these definitions, we find all the graphic games with quantum advantages when proper consistency conditions are satisfied.

Theorem 1—For any graphic game with $\varpi_c \ne 1$, there is a quantum advantage, i.e., $\varpi_q > \varpi_c$, if and only if $\min\{l_1, l_2, ..., l_m\} = 2$.

Theorem 1 presents a sufficient and necessary condition for graphic games with quantum advantage. One example is shown in Fig. 3b. This provides the first instance for Problem 2 with a deterministic separation of classical correlations and quantum correlations in correlation space. It is interesting to intuitively show the relationship between l_i and quantum advantage. Note that $\mathcal{A}_i^s$ shown in Eq. (4) features the number of players who have the consistency requirements with A_i independent of inputs. If one divides a network into m subnetworks according to players A_is ($i = 1, 2, ..., m$) who have no shared vertices, l_i is then used to describe the topology characters of a given network. Specially, it means that the network consists of star-type in Fig. 2a networks or chain-type subnetworks in Fig. 2b for $l_i = 2$. For $l_i = k > 2$, the network consists of various cyclic subnetworks. In this case, similar to previous nonlinear method[42–45] Theorem 1 implies that there is no useful linear testing scheme for cyclic networks. One example is triangle network.[42,45] The proof is inspired by the unbalanced CHSH game shown in Supplementary Notes 1 and 2.[6,37] To explain the main idea, consider a graphic game $\mathcal{G}$ with $m = 1$ consisting of three players A_1, A_2, A_3. Two different games $\mathcal{G}_i$ are defined as follows. $\mathcal{G}_1$ requires that the outputs of two pairs of players {A_1, A_2}, and {A_1, A_3} are consistency simultaneously. $\mathcal{G}_2$ requires that the outputs of three pairs of players {A_1, A_2}, and {A_1, A_3}, and {A_2, A_3} are consistency simultaneously. Both games can be regarded as two and three simultaneous CHSH games, respectively. Theorem 1 means that $\mathcal{G}_1$ has a quantum advantage with $l_1 = 2$ while $\mathcal{G}_2$ has no quantum advantage with $l_1 = 3$. The main reason is that there are too many restrictions in $\mathcal{G}_2$ to achieve a quantum advantage. This provides a new witness for entanglement-based quantum networks[45] and nonlocal games.[33]

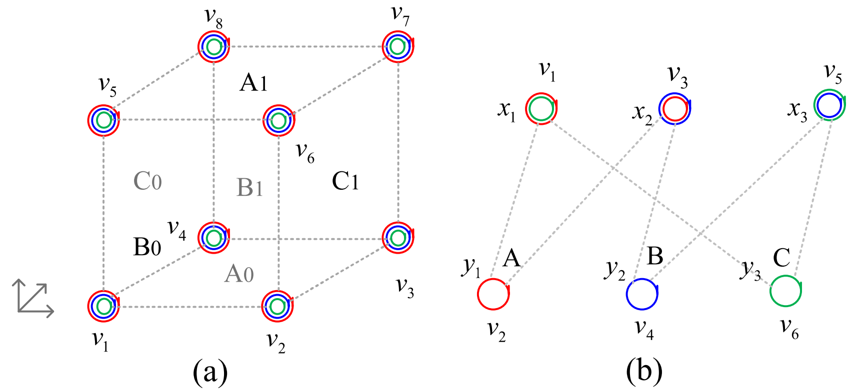

Fig. 4 Some graphic games. **a** Cubic game. Each player owns all the vertices on a 2-dimensional plane (denoted by A_i, B_j, C_k) with different colors depending on the input. **b** The equivalent graphic game of GYNI game.[50] Each player owns three vertices of one subgraph (denoted by *A*, *B*, *C*) with different colors. Here, the edges with dot lines are used for visualization

Witnessing multi-source quantum networks

Multi-source quantum networks can extend applications of single-source Bell network in large scale.[60,61] However, it is hard to verify these distributive entangled states in global pattern due to the non-convexity of quantum correlations. One useful way is to make use of some nonlinear Bell-type inequalities.[42–45] Interestingly, the present model provides another witness for general quantum networks. Specifically, let one vertex schematically represent one EPR state, as shown in Definition 1.[2] Any multi-source quantum network consisting of EPR states is equivalent to a graph with lots of vertices. Some examples are shown in Fig. 2. Figure 2a presents a star-type quantum network, where each pair of B_i and A share one EPR state $|\Phi\rangle_i$, $i = 1, 2, ..., n-1$. Its graphic representation consists of $n-1$ vertices $v_1, v_2, ..., v_{n-1}$. The assumption that each pair of players share one EPR state can be schematically represented by sharing one vertex in terms of the graphic game in Definition 2. Similar representation holds for chain-type network shown in Fig. 2b. From the equivalent reformations, there are lots of testing to achieve a quantum advantage when the graphic games are defined. Generally, from Theorem 1 we obtain a corollary as:

Corollary 1—For any quantum network $\mathcal{N}_q$ consisting of EPR states shared by *n* observers, there are nonlocality witnesses in terms of graphic game if $\mathcal{N}_q$ is *k*-independent with $k \geq 2$.

In Corollary 1, *k*-independence means that there are a set of *k* observers who do not share any entanglement among themselves. Note that for verifying quantum networks, all the shared vertices V^{x_i}s are independent of inputs. From the equivalent reduction,[45] any *k*-independent network is equivalent to a generalized star-type network, as shown in Fig. 2a. The proof of Corollary 1 follows easily from the corresponding graphic games, where $\min_i l_i = 2$ because no more than two players share vertices. Take the star-type network shown in Fig. 2a as an example. Each pair of players $\{B_i, A\}$ shares one vertex v_i independent of their inputs, $i = 1, 2, ..., n-1$. There exists quantum advantage because of $l_1 = 2$ by assuming $m = 1$ (or $m = n-1$). For the chain-type network shown in Fig. 2b, each vertex v_i is owned by two adjacent players A_i and A_{i+1}. One can define different *m* by choosing the players without sharing vertices (independent in quantum networks).[45] Quantum advantages hold for these graphic games. Generally, the present graphic games allow the first linear testing for multi-source quantum networks consisting of EPR states.

CHSH game

Two equivalent forms of CHSH game in ref. [3] are shown in Fig. 3c, d. Another extension is cube game, which is defined over an *n*-dimensional cube graph $\mathcal{G}$ with 2^n vertices represented by $\{(q_1, q_2, ..., q_n), \forall q_i = 0, 1\}$, as shown in Fig. 4a, where $m = 1$. Specially, the referee randomly chooses binary question $x_i \in \{0, 1\}$ according to a given distribution $\{p, 1-p\}$, and sends it to the player A_i. Each player assigns 1 or -1 to their own vertices on $n-1$-dimensional hyperplane defined by $\{(q_1, q_2, ..., q_n) | q_i = x_i\}$ and sends to the referee. All the players win the game if and only if their assignments satisfy the requirements in Definition 2. From Theorem 1, it follows that the cubic game has no quantum advantage over all the classical strategies when $n \geq 3$, where $l_1 = n$. Our result is different from a recent result with relaxed consistency conditions.[35] Another simple extension with $m \geq 1$ is defined as: the player A_i owns the local subgraph defined by $\{(q_1, q_2, ..., q_n) | q_1 = q_2 = ... = q_m = x_i\}$ for $i = 1, 2, ..., m$, and $\{(q_1, q_2, ..., q_n) | q_j = x_j\}$ for $j = m+1, m+2, ..., n$. Note that any two players of $A_1, A_2, ..., A_m$ have no common subgraph for all the inputs. It is straightforward to check that this generalization has quantum advantage when $m = n-1$.

Distributive SAT problems

A Boolean satisfiability problem (SAT) aims to check some given equations.[62] The decision problem is of central importance in theoretical computer science, complexity theory and algorithmic theory.[63] Interestingly, each graphic game is equivalent to a distributive SAT problem. Take the game shown in Fig. 4a as an example. Let y_{i,x_j} be assignment on the vertex v_i by A_j according to the input x_j, $i = 1, 2, ..., 8$; $j = 1, 2, 3$. From Definition 2, the winning conditions are equivalent to Boolean equations with 48 variables. Theorem 1 shows a qualitative decision without quantum advantage from shared entangled resources. Generally, for an *n*-player graphic game involving *N* vertices, there are $O(nN)$ conditions, which should be checked.[64] Theorem 1 provides an intuitive completion for distributive SAT problems with different resources using graphic game model. Hence, in applications, an equivalent graphic game should be found for special decision problems. One possible way is to reduce the required clauses firstly even if it is hard. And then, define a graphic game clause-by-clause. Special output strategies may be applied as guess your neighbour's input (GYNI) game.[50]

GYNI game

Guess your neighbour's input (GYNI) game has recently been proposed to separate the multipartite quantum correlations from classical correlations.[50] This game is equivalently represented by a graphic game, as shown in Fig. 4b. The original consistency requirements are equivalently reduced to special output strategies for all the players, where the input is assigned the vertex v_{2i-1} for the players A_i, $i = 1, 2, 3$. This kind of graphic game has no quantum advantage for any quantum resources. It can be viewed as a relaxed graphic game satisfying partial consistency conditions in Definition 2. Here, we prove a more generalized result. Specifically, consider the following winning condition, i.e., $F(\mathbf{x}, \mathbf{y}) = 1$ if $y_i = f_i(\mathbf{x})$ for all is; Otherwise, $F(\mathbf{x}, \mathbf{y}) = 0$. Here, f_i denotes some function of the input bit series $\mathbf{x}$. As an example, $f_i(\mathbf{x})$ can be regarded as the restriction of the product of all the assigned values in the graphic games.

Theorem 2—There is no quantum advantage for a multipartite nonlocal game if $\mathcal{F}:\mathbf{x}\mapsto(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ is injective.

The proof is similar to its in ref., [50] which is shown in Supplementary Note 3. This provides another feature to address Problem 2. Some examples are presented as follows.

Example 1—Consider f_1, f_2, ..., f_n be any permutation in the permutation group S_n. Take $n = 3$ as an example. All the permutations are given by $S_3 = \{(1), (1, 2), (1, 3), (2, 3), (1, 2, 3), (1, 3, 2)\}$, where (1) denotes the identity operator, (i, j) denotes the permutation of $i \to j$ and $j \to i$, and (i, j, k) denotes the permutation of $i \to j$, $j \to k$ and $k \to i$. Now, define the expected output of three parties as $g(i, j, k)$ for the corresponding inputs i, j, k where $g \in S_3$. For the games with $(f_1, f_2, f_3) = (i, j, k)$ are previous games of guessing the neighbor' input.[50] From Theorem 2, there is no quantum advantage for this kind of games over classical resources.

Example 2—Consider f_1, f_2, ..., f_n be any injective mapping going beyond the permutation group S_n, where $\{0, 1\}^n$ is input space while $\mathbb{R}^n$ is output space. Take $n = 3$ as an example. Consider the following mappings: $\mathcal{F}_1:(x_1,x_2,x_3)\mapsto(x_2+x_3, x_1+x_3, x_1+x_2)$ and $\mathcal{F}_2:(x_1,x_2,x_3)\mapsto(2^{x_1}-2^{x_2+x_3}, 2^{x_2}-2^{x_1+x_3}, 2^{x_3}-2^{x_1+x_2})$. Both mappings are injective. If the mappings are used as the winning conditions of players, Theorem 2 implies no quantum advantages for these games.

Example 3—Define $f_i = \prod_{j=1}^{k_i} s_{i_j}$, for $i = 1, 2, ..., n$. Assume that $s_{i_1}, s_{i_2}, \ldots, s_{i_{k_i}}$ denote special assigns on the subgarph shared by the player A_j. If $s_i \in \{\pm 1\}$, the new game falls into graphic game proved in Theorem 1 when f_is are injective mappings. There is no quantum advantage from Theorem 1 or Theorem 2. Interestingly, similar result holds for $s_i \in \mathbb{R}$ going beyond the games stated in Theorem 1.

DISCUSSION

Theorem 1 provides a result for separating quantum correlations from classical correlations in terms of multi-source networks. The consistency conditions in Definition 2 characterize the "incompatible measurements" derived from quantum mechanics. These conditions should be carefully designed for special goals in applications. Unfortunately, the graphic game cannot solve generalized XOR game, which is an extension of CHSH game.[1,3,31] Another interesting problem is how to define meaningful consistency conditions such as assignments on edges going beyond Definition 2 for different goals including verifying general cyclic networks.[42,45]

Generally, we presented a unified way to construct multipartite nonlocal games using graph representations of entanglement-based quantum networks. The graphic games with quantum advantage are distinguished from others for general graphs. The new model is useful for witnessing general quantum networks consisting of EPR states. This can be regarded as a new feature of multi-source networks going beyond nonlinear Bell-type inequalities and semiquantum game. Our results are interesting in Bell theory, quantum Internet, theoretical computer science, and distributive computations.

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COMPETING INTERESTS

The author declares no competing interests.

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