TRIDIMENSIONAL DISSIPATIVE SEMI-NUMERICAL MODEL

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Abstract. Our research combines mean motion resonances and dissipative forces in the averaged elliptic spatial restricted three-body problem. The models presented can be applied in many contexts mixing resonances and dissipations, e.g., asteroid belt, transneptunian region, exoplanets, systems of planetary rings, etc. We propose a semi-numerical model that simulates the behaviour of test particles under the effect of generic forces, functions of powers of the position and/or of the velocity. This model is valid for any orbital eccentricities or inclinations, even at high values. Captures around symmetric and asymmetric equilibria are reproduced and the apparitions of a plateau of inclination for long periods of time are dectected.

1. Introduction

The Kuiper belt is a great laboratory to test theories about captures in resonance; indeed, the main mean motion external resonances with Neptune are characterized by concentrations of planetesimals. The long time stability of such regions seems quite strong and can be analyzed in different models.

If we use a very simple one degree of freedom model of resonance, with inclinations equal to zero, it has been shown how a dissipation (Stokes, Poynting-Robertson drags) could slowly push a small body into a resonance, and stabilize its motion "for ever" around a constant eccentricity, referred as "universal" in Beaugé and Ferraz-Mello (1993–1994).

Our purpose is to test how robust this behavior could be if we take the inclinations into account, and if we generalize this approach to any dissipation (presented as a generic function of powers of the position and velocity). It is quite clear that, with the introduction of a second degree of freedom, and for several types of forces, the capture will not be definitive anymore. However it could still be present for very long periods of time, during which the eccentricity and/or the inclination stay quasi constant, on a kind of plateau, to eventually leave the resonance and be trapped in the next one.

We propose a semi-numerical model to analyze this process of resonance trapping due to general dissipation forces in the frame of the spatial restricted three

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body problem and in the case of external mean motion resonances. We compute our simulations by using the 3-dimensional Extended Schubart Averaging (ESA) integrator developed by Moons (1994) for all mean motion resonances. We complete it by adding to the righthandside of the differential equations, the averaged contributions of general dissipative forces functions of powers of the position or/and velocity of the particle, following the same idea as Murray (1994) on the dynamical effects of any drag on the Lagrangian equilibria positions. We give our results for the 1:2 and 2:3 resonances with Neptune.

Details about the method and part of the results can be found in Jancart, Lemaitre and Letocart (2003).

2. The Resonant Integrator: ESA and Its Extension to the Dissipative Cases

ESA computes the averaged Hamiltonian $\mathcal{H}$ of the resonant spatial elliptic restricted three-body problem and its partial derivatives in closed form, for any value of the eccentricity and inclination of the perturbing body; it gives the averaged motion of a particle close to a mean motion resonance ($\frac{p+q}{p}$) with a perturbing planet. The resonant variables chosen in ESA (J, σ, J_z, σ_z, N and ν) are given by:

$$
\begin{aligned}
J &= L - G & \qquad q\,\sigma &= (p+q)\lambda' - p\lambda - q\,\varpi \\
J_z &= G - H & \qquad q\,\sigma_z &= (p+q)\lambda' - p\lambda - q\,\Omega \\
N &= H - L - \tfrac{q}{p}L & \qquad q\,\nu &= (p+q)\lambda' - p\lambda
\end{aligned}
$$

where $L = \sqrt{\mu a}$, $G = L\sqrt{1-e^2}$, $H = G\cos I$ are the Delaunay's momenta, expressed with respect to the semimajor axis a, the eccentricity e and the inclination I of the small body; $\mu = \mathcal{G}M_1$, with $\mathcal{G}$ the gravitation constant and M_1 the mass of the primary; λ and λ' represent the mean longitudes of the small body and of the secondary, ϖ is the longitude of the pericenter and Ω the longitude of the node of the small body. For an external mean motion resonance, the value of q is negative.

We calculate the closed forms of the generic drag terms written in powers of position and velocity variables (see formula (1)), with the same philosophy as ESA, which means without any series expansion in the eccentricity or the inclination, and add these averaged components to the averaged equations of motion given by ESA. The problem was to rewrite the forces in the averaged canonical variables suitable for the ESA integrator.

The final differential equations can be written as:

$$
\begin{aligned}
\frac{dJ}{dt} &= -\frac{\partial\mathcal{H}}{\partial\sigma} + \mathcal{F}_J & \qquad \frac{d\sigma}{dt} &= \frac{\partial\mathcal{H}}{\partial J} + \mathcal{F}_\sigma \\
\frac{dJ_z}{dt} &= -\frac{\partial\mathcal{H}}{\partial\sigma_z} + \mathcal{F}_{J_z} & \qquad \frac{d\sigma_z}{dt} &= \frac{\partial\mathcal{H}}{\partial J_z} + \mathcal{F}_{\sigma_z} \\
\frac{dN}{dt} &= -\frac{\partial\mathcal{H}}{\partial\nu} + \mathcal{F}_N & \qquad \frac{d\nu}{dt} &= \frac{\partial\mathcal{H}}{\partial N} + \mathcal{F}_\nu
\end{aligned}
$$

The calligraphic notations $\mathcal{F}_{x_i}$ or $\mathcal{F}_{y_i}$ indicate that the components of the force have been averaged over the mean anomaly M.

To obtain the dissipative contributions in ESA resonant variables, we use the transformation:

$$\begin{aligned}
\mathcal{F}_\sigma &= \mathcal{F}\,\nu - \mathcal{F}_\omega - \mathcal{F}_\Omega \,, & \mathcal{F}_J &= \mathcal{F}_L - \mathcal{F}_G \,, \\
\mathcal{F}_{\sigma_z} &= \mathcal{F}\,\nu - \mathcal{F}_\Omega \,, & \mathcal{F}_{J_z} &= \mathcal{F}_G - \mathcal{F}_H \,, \\
\mathcal{F}_\nu &= -\tfrac{p}{q}\,(\mathcal{F}_M + \mathcal{F}_\omega + \mathcal{F}_\Omega) \,, & \mathcal{F}_N &= \tfrac{q+p}{p}\mathcal{F}_L - \mathcal{F}_H \,.
\end{aligned}$$

3. The Averaged Dissipative Forces

Following Murray (1994), we selected two kinds of generic forces :

$$\begin{aligned}
\vec{R} &= -k^2 r^j \dot{r}^i \vec{v} \\
\vec{S} &= -k^2 r^j v^i \vec{v}
\end{aligned} \tag{1}$$

where r and v are the norms of the position $\vec{r}$ and of the velocity $\vec{v}$ of the third body, expressed in an inertial frame (not rotating) and k is a small coefficient. $\vec{R}$ and $\vec{S}$ have to be averaged over the mean anomaly M of the small body.

3.1. Forces of the first type: $\vec{R} = -k^2 r^j \dot{r}^i \vec{v}$

The components of the force $\vec{\mathcal{R}}$ in Delaunay's canonical variables M, ω, Ω, L, G and H, after averaging, are given by:

$$\begin{aligned}
\mathcal{R}_M &= \frac{B_{ij}}{e}\,\beta\,\mathcal{I}_2 \,, & \mathcal{R}_L &= C_{ij}\,(\mathcal{I}_3 + e\,\mathcal{I}_4) \,, \\
\mathcal{R}_\omega &= -B_{ij}\,e\,\mathcal{I}_1 - \beta\,\mathcal{R}_M \,, & \mathcal{R}_G &= C_{ij}\,\beta\,\mathcal{I}_5, \\
\mathcal{R}_\Omega &= 0 \,, & \mathcal{R}_H &= \mathcal{R}_G\,\cos I \,.
\end{aligned} \tag{2}$$

In the above expressions (2), $\beta = \sqrt{1-e^2}$, the coefficients B and C are given by

$$B_{ij} = 2\,C_{ij}\,\frac{1}{L} \quad \text{and} \quad C_{ij} = A_{ij}\,e^i \text{ with } A_{ij} = -k^2 a^{(i+j+2)} n^{(i+1)}$$

where n is the mean motion of the particle and $k^2 = 10^{-6}$.

The $\mathcal{I}_n$ are integrals from 0 to 2 π of power of the eccentricity with respect to the eccentric anomaly. These integrals are due to the averaged process (see Jancart et al. (2003)).

3.2. FORCES OF THE SECOND TYPE: $\vec{S} = -k^2 r^j v^i \vec{v}$

The components of the force $\vec{\mathcal{S}}$ in Delaunay's variables, after averaging, are given by:

$$\begin{array}{lll} \mathcal{S}_M = 0\,, & \qquad \mathcal{S}_L = A_{ij}\,\mathcal{J}_1\,, & \\ \mathcal{S}_\omega = 0\,, & \qquad \mathcal{S}_G = A_{ij}\,\beta\,\mathcal{J}_2\,, & \qquad (3) \\ \mathcal{S}_\Omega = 0\,, & \qquad \mathcal{S}_H = \mathcal{S}_G\,\cos I\,. & \end{array}$$

The $\mathcal{J}_n$ are integrals from 0 to 2 π of power of the eccentricity with respect to the eccentric anomaly. These integrals are due to the averaged process (see Jancart et al. (2003)). A_{ij} and β are the same as in Equations (2).

Interested in the role of the inclination in captures, we are looking, as a first step, for the presence of a plateau in inclination (period of time relatively long while the inclination is stabilized) and, in a second step, for values of initial inclinations to allow a capture. The two problems are treated separately.

4. The Inclinations after Capture: Case of the Resonance 1:2

Starting with an inclination of a few degrees, we find two main behaviors:

– for the first behavior, the particles are first trapped in the mean motion resonance (the angle σ is captured in one of the asymmetric equilibria* and librates with a decreasing amplitude); the eccentricity is pumped up, quite quickly, up to a constant value, the value of which depends on the force considered. Once the eccentricity is stabilized, we can notice that the equilibrium is not yet reached: the inclination is decreasing very slowly to zero, with the second angle σ_z always circulating. The only “equilibrium inclination” in that case is 0 and is reached asymptotically, after several millions of years (see Figures 1 and 2).

– for the second behavior, we have the capture of σ in asymmetric equilibrium, with a decreasing amplitude, together with the increasing of the eccentricity, up to its plateau value. The inclination is oscillating with small (slightly increasing) amplitude about its initial value of a few degrees, with a quasi constant mean value while the angle σ_z is circulating. Once the eccentricity has reached its maximal value and is stabilized (universal eccentricity due to the combination of the resonance and the drag effects), the second stage starts up, with a rapid increase of the inclination, connected to a capture in resonance of the second angle σ_z. This case coincides with a secular resonance case called “Kozai resonance”. The behavior of this second degree of freedom (I and σ_z) is very similar to the first one (e and σ) but is less stable in the sense

* These equilibria corresponding to values of $\sigma \neq 0$ or 180 degrees may appear in captures in the 1:2 resonance (Beaugé, 1994)

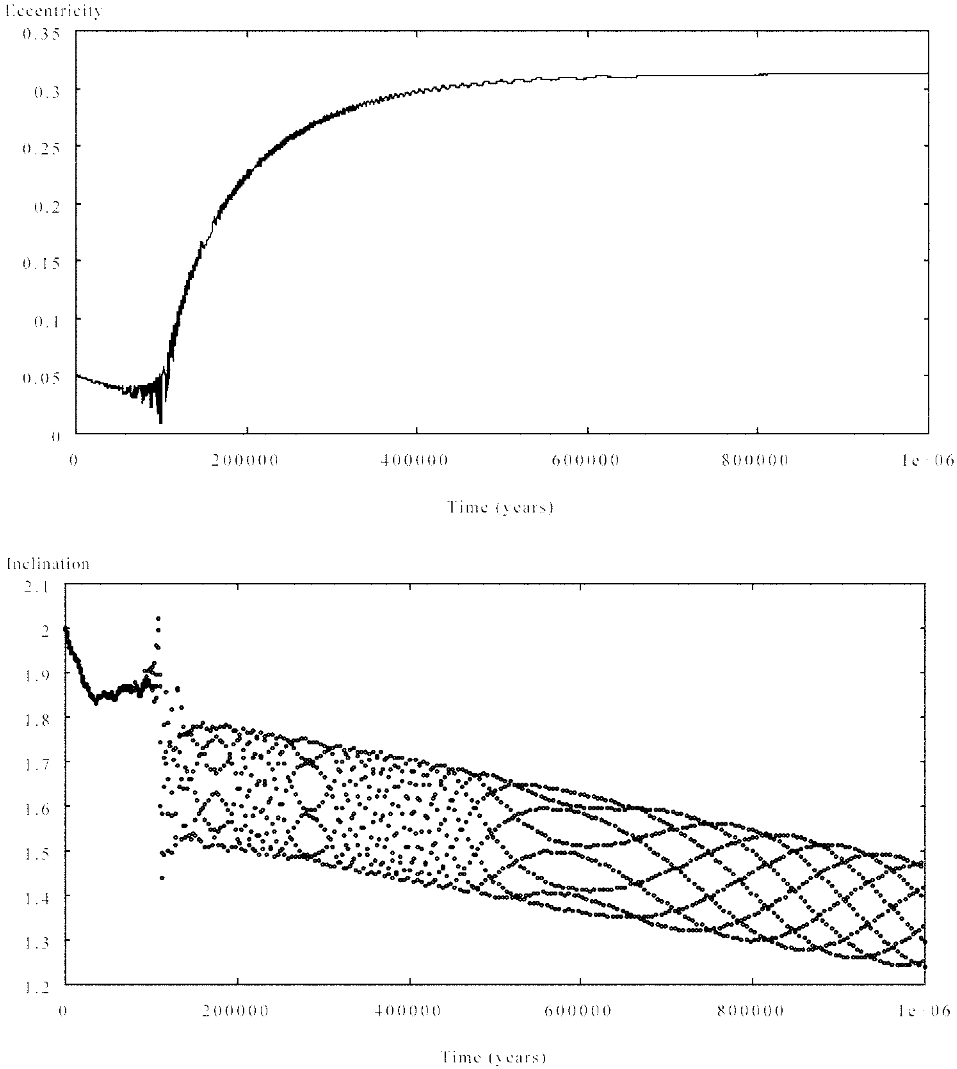

Figure 1. S-force, with $i = 2$, $j = 2$ and $k^2 = 10^{-6}$: representation of e and I over 1,000,000 years. The inclination falls down to 0.

that the increase of the inclination is very sharp, up to values of I of 30°, and very often, after a few hundreds of thousands of years of this regime, the small body is ejected out of the resonance. This escape is due to the presence of the drag which is still acting on the particle. This capture around high values of I is quite short and as soon as the angle σ_z starts to circulate once again, the inclination goes down to 0.

A typical case of this generic behavior is given in Figures 3 and 4, for a S force, characterized by $i = 0$ and $j = -2$, and $k^2 = 10^{-6}$.

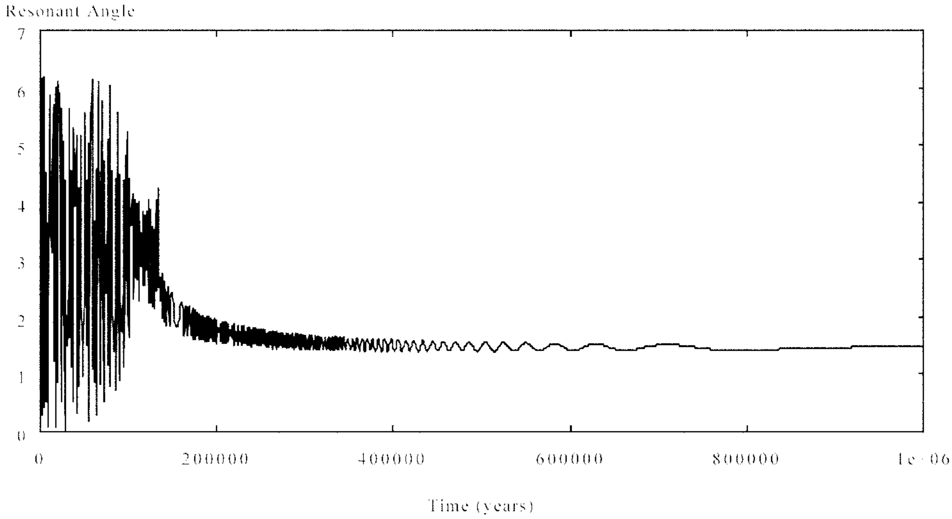

Figure 2. S-force, with $i = 2$, $j = 2$ and $k^2 = 10^{-6}$: representation of σ over 1,000,000 years. The angle σ_z always circulates.

The two behaviors are already mentioned by Gomes (1997), where he presented results connected to a universal inclination considering restricted hypothesis with respect to the dissipative forces or the type of resonances.

To explain the different behaviors and the stability of the equilibrium we develop a toy model of dissipation in the case 1 : 2, using an Hamiltonian (three dimensional and truncated). The results obtained are presented in Jancart et al. (2003).

The specific results for the S forces and the R forces are specified in Jancart et al. (2003) for many values of i and j and for the 1 : 2 and the 2 : 3 mean motion resonances for a body like Jupiter.

Below, we concentrate on results in the 2 : 3 mean motion resonance with a body like Neptune (orbital elements and mass).

5. The Initial Inclinations: Case of the Resonance 2 : 3

Since the capture is dependent on the initial conditions, we tested a series of forces R and S for (i, j) equal $(0, 0)$ or $(0, -2)$ or $(2, 2)$, always with the same initial conditions in semi major axis and eccentricity ($a = 1.35$, $e = 0.05$) but with increasing values of the initial inclination from 0 to 50 degrees (we consider $I = 0, 4, 10, 20, 30, 40, 50°$).

In any case of dissipation except for the R-force, $(2, 2)$, for inclinations under 20°, the capture is always present; on the other hand, if the initial inclination is higher than 50° there is no capture.

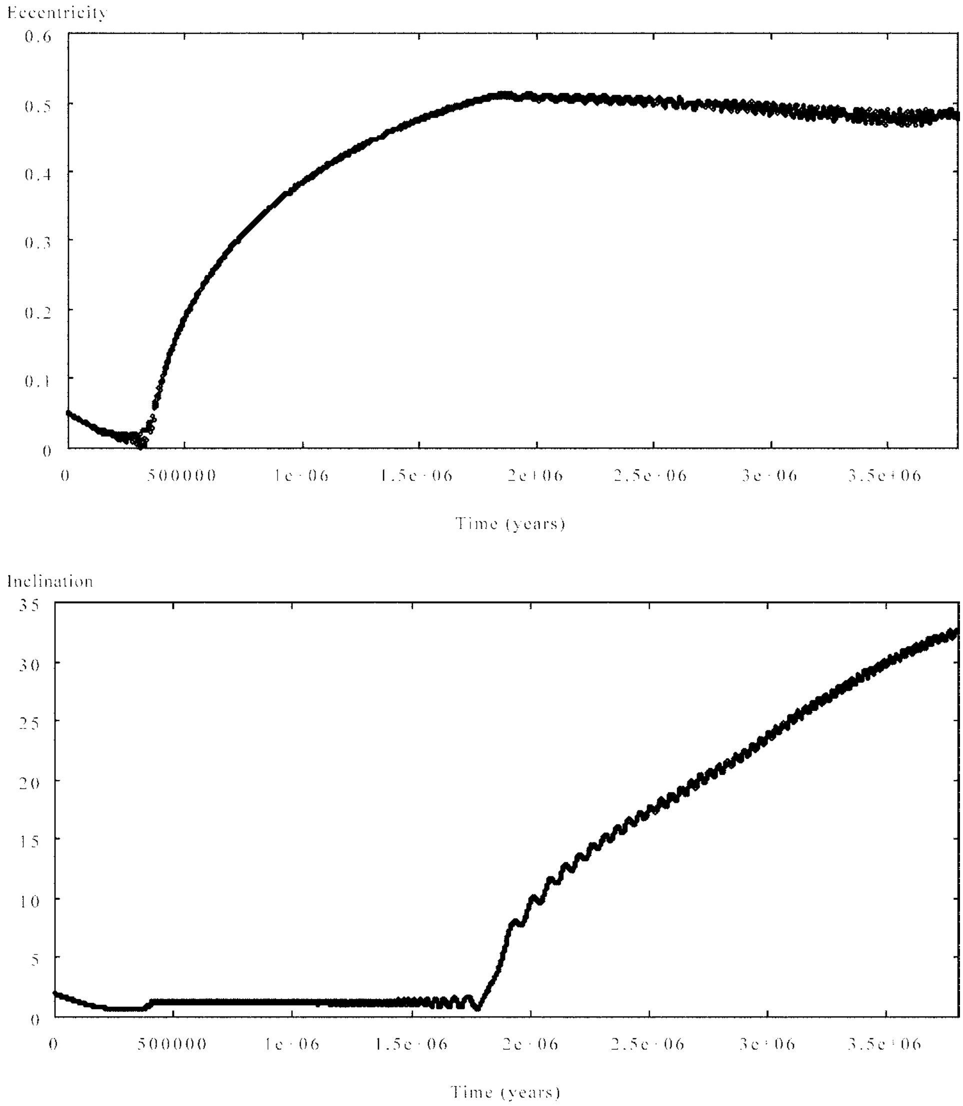

Figure 3. S-force, with $i = 0$, $j = -2$ and $k^2 = 10^{-6}$: representation of e and I over 3,500,000 years, limit of the capture.

Between 20 and 50 degrees, the capture depends on the force. For some forces, the succession of captures and escapes in successive different resonances of the eccentricity is very irregular and the motions show a chaotic behavior. In some cases either the semi major axis stays around its inital value or falls down to values smaller than 1 and we have no capture at all.

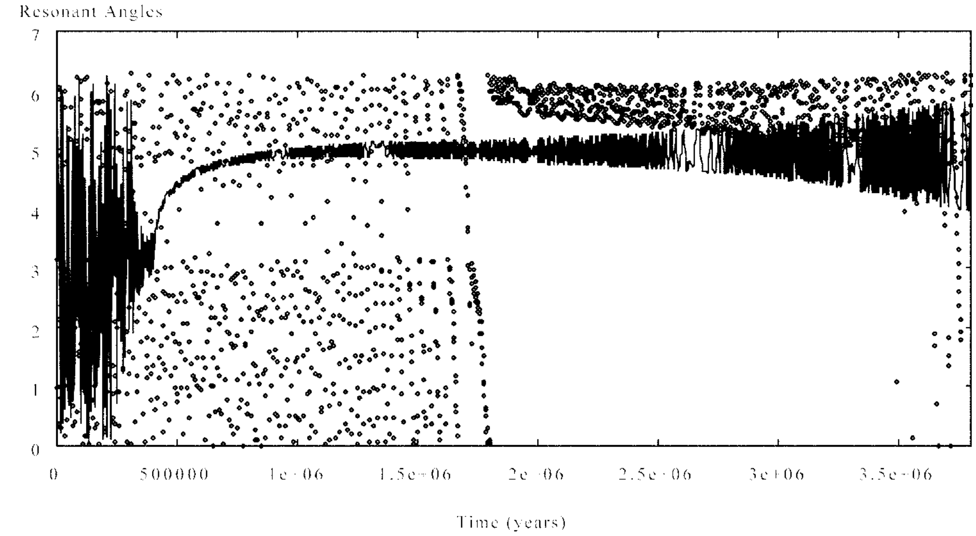

Figure 4. S-force, with $i = 0$, $j = -2$ and $k^2 = 10^{-6}$: representation of σ_z (dots) and σ (continuous line) over 3,500,000 years, limit of the capture.

6. Conclusion

We present here the results for generic classes of drag forces where the "universal eccentricity" appears systematically, in each case, as an equilibrium in the planar case. The idea of testing several dissipations was already discussed by Gomes (1995–1997) but in the planar case or for a specific set of forces.

We do not find a universal inclination, as Gomes (1997) did, for the forces that we have analyzed; the explanation is that in our generic forces, the term $S_{\cos i}$ obtained by Gomes, does not vanish. However we detect the apparition of a very long plateau of quasi constant inclination, very slowly decreasing to 0 for long periods of time.

We show that for a generic class of dissipations, acting on a resonant three dimensional averaged model, two different behaviors can be detected; a case where the planar case is always the attractive one, for long periods of time the inclination tends to 0, and a second case of double capture, corresponding to higher values of I but starting from any value of the initial inclination below 50°.

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