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Locally Greater Vulnerability to Background Risk

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Abstract

Willingness to take on risk is influenced by the presence of fair and unfair background risks for decision makers who are risk vulnerable as defined by Gollier and Pratt [1996], for these decision makers are more risk averse when they possess such an uninsurable background risk. We present an alternative derivation of the index of local vulnerability based on Diamond and Stiglitz [1974] compensated increases in risk, such that risk aversion increases with the introduction of any small fair background risk if and only if the index of local vulnerability is positive. We establish that the increase in risk aversion is greater for those who are more vulnerable as measured by the index of local vulnerability.

Key words: local vulnerability, risk aversion, portfolio choice, insurance

JEL Classification No.: D80, D81, G11, G22

1. Introduction

It is now widely recognized that background risks, even if they are independent of the foreground risk under analysis, can have a substantial effect on a decision maker's willingness to bear the foreground risk. Examples include portfolio choice and demand for insurance in the presence of background uncertainty about human capital and its rate of return. In this paper we derive the index of local vulnerability by applying the Diamond and Stiglitz [1974] characterization of greater risk aversion. Their characterization exploits compensated increases in risk to identify the index of risk aversion. Taking the compensated approach, we establish not only that exposure to a small fair background risk increases the degree of risk aversion for a locally vulnerable decision maker, but also that such exposure results in a greater increase in risk aversion for decision makers who are more vulnerable as indicated by a higher value for the index of local vulnerability.

Gollier and Pratt [1996] introduced the concept of *risk vulnerability* to describe decision makers $U(\theta)$ who become more averse to risk about θ upon the introduction of an additive, independent background risk, be it fair (zero-mean) or unfair (negative-mean), in the sense that the index of risk aversion, $R(\theta) \equiv -U_{\theta\theta}/U_\theta$, is higher in the presence of background risk. We define as *vulnerable* those decision makers who become more risk averse with the introduction of any fair background risk, and we develop a corresponding index of *local vulnerability* by applying the insights of Diamond and Stiglitz. They observe that any compensated increase in risk for one decision maker would reduce the expected utility of a more risk-averse decision maker. From this observation, we conclude that a vulnerable decision maker must be made worse off by any compensated increase in risk that is accompanied by the introduction of a fair background risk. By exploiting this characterization of vulnerability, we identify an index $V(\theta)$ such that (i) V is positive if and only if U is locally vulnerable, and (ii) the increase in risk aversion is greater with higher values for the index V.

Gollier and Pratt conclude that a positive value for $R_{\theta\theta} - 2RR_\theta$ is necessary and sufficient for the introduction of a small fair background risk to increase risk aversion. Therefore, the index V is necessarily equivalent to $R_{\theta\theta} - 2RR_\theta$. However, our approach, using the concept of compensated increases in risk, has the advantage of showing that the measure V is indeed a true index of local vulnerability. In particular, the magnitude of V indicates the strength of the response to a small fair background risk. Thus, by linking the identification of vulnerable decision makers to compensated increases in risk, our analysis sheds new light on the results obtained by Gollier and Pratt, and adds to their work by showing that the introduction of a small fair background risk causes a greater increase in risk aversion for decision makers with a higher index of local vulnerability.

The derivation of the index of local vulnerability is presented in the next section. In Section 3, we establish that higher values for the index indicate greater local vulnerability. A summary and concluding remarks on the implications of greater vulnerability are presented in Section 4.

2. The index of local vulnerability

Consider a decision maker whose utility $U(\theta + \varepsilon)$ depends on a random variable θ and an additive, independent random variable ε representing background risk. The utility function $U(\theta)$ is assumed to be strictly increasing and strictly concave, reflecting risk aversion. The foreground and background risks have cumulative distribution functions denoted, respectively, by $F(\theta)$ and $H(\varepsilon)$. We shall refer to *admissible* foreground and background risks as those for which each realization $(\theta + \varepsilon)$ belongs to a compact interval $[a, b]$ on which U is defined, and we shall denote by b_θ and b_ε, respectively, the upper limits of the supports for an admissible pair $F(\theta)$ and $H(\varepsilon)$.

The initial absence of background risk is represented by the degenerate distribution

$$\tilde{H}(\varepsilon) = 0 \quad \text{if } \varepsilon < 0 \quad \text{and} \quad \tilde{H}(\varepsilon) = 1 \quad \text{if } \varepsilon \geq 0, \tag{1}$$

which has a mean value of zero. The introduction of a fair background risk $H(\varepsilon)$ can then be represented as a mean preserving spread of the improper distribution $\tilde{H}(\varepsilon)$. Hence, we assume that $H(\varepsilon) - \tilde{H}(\varepsilon)$ satisfies the integral conditions characteristic of mean preserving spreads,

$$\int^{y} [H(\varepsilon) - \tilde{H}(\varepsilon)]\, d\varepsilon \geq 0 \quad \forall\, y < b_\varepsilon \text{ with equality at } y = b_\varepsilon, \tag{2}$$

as established by Rothschild and Stiglitz (1970).[1]

We define a decision maker to be (*locally*) *vulnerable* if risk aversion increases with the introduction of a (small) fair background risk. By a *small* background risk, we mean that the interval of its support around zero is arbitrarily small.[2] For a locally vulnerable decision maker $U(\theta)$, and a small fair background risk $H(\varepsilon)$, the derived utility function $\psi(\theta) \equiv \int U(\theta+\varepsilon)\, dH(\varepsilon)$ is more risk averse than $U(\theta)$, so that $R^{\psi}(\theta) \equiv -\psi_{\theta\theta}(\theta)/\psi_{\theta}(\theta)$ is greater than $R(\theta)$.

Diamond and Stiglitz [1974] introduced compensated increases in risk to identify decision makers with greater risk aversion. By their definition, a mean utility preserving spread is a shift $F_{\bar{r}}(\theta, \bar{r})$ in the distribution for θ that induces a mean preserving spread in the distribution of ex post utility $U(\theta)$. They showed that $F_{\bar{r}}(\theta, \bar{r})$ must satisfy the integral conditions

$$\int^{y} U_{\theta}(\theta) F_{\bar{r}}(\theta, \bar{r})\, d\theta \geq 0 \quad \forall\, y < b_{\theta} \text{ with equality at } y = b_{\theta}, \tag{3}$$

which ensure that expected utility remains constant while the distribution of utility has greater spread.[3] To identify the index of risk aversion, Diamond and Stiglitz exploited the fact that a compensated increase in risk for one decision maker would reduce the expected utility of anyone more risk averse. We apply this idea in the following Theorem to identify the restriction on risk preferences necessary and sufficient to ensure that a decision maker is locally vulnerable. (Proofs are presented in Appendix A.)

Theorem 1: *The following are equivalent definitions for utility functions $U(\theta)$ showing local vulnerability*:

(a) *Given any admissible risk $F(\theta, \bar{r})$, any compensated increase in risk $F_{\bar{r}}(\theta, \bar{r})$ satisfying* (3) *that is accompanied by the introduction of a small fair background risk $H(\varepsilon)$ satisfying* (2) *reduces expected utility*, *that is*,

$$\iint U(\theta+\varepsilon)\, dF_{\bar{r}}(\theta, \bar{r})\, d[H(\varepsilon) - \tilde{H}(\varepsilon)]$$

is negative.

(b) *The introduction of any small fair background risk $H(\varepsilon)$ satisfying* (2) *makes the decision maker more risk averse*, *that is*,

$$R^{\psi}(\theta) - R(\theta)$$

is positive.

(c) *For all θ and any small fair background risk $H(\varepsilon)$*,

$$-\partial[U_{\theta\theta\theta}(\theta+\varepsilon)/U_{\theta}(\theta)]/\partial\theta$$

is positive.

The equivalence between parts (a) and (b) uses the relationship between compensated increases in risk and greater risk aversion in an intuitive manner. Part (a) requires that expected utility decline when any compensated increase in risk is accompanied by the introduction of fair background risk. Thus, a compensated increase in risk, which has no effect on the decision maker's expected utility in the absence of background risk, would reduce the same decision maker's expected utility in the presence of a fair background risk. Applying Diamond and Stiglitz's result, the decision maker must be more risk averse in the presence of the background risk, as stated in part (b), indicating that the decision maker is vulnerable.[4]

The Theorem is restricted to small background risks, since otherwise our appeal to compensated increases in risk would not lead to an unambiguous characterization of vulnerability. In particular, a compensated increase in risk about θ must be tailored to the decision maker's risk preferences regarding θ, and these preferences are influenced by the presence of background risk when the decision maker is vulnerable. As a result, a compensating change in $F(\theta)$ designed before the introduction of background risk generally differs from a compensating change designed after its introduction. These alternative compensation schemes may not yield mutually consistent characterizations of vulnerability for large background risks, but they do yield consistent characterizations in cases of small background risks.[5]

The condition in part (c) of the Theorem requires that the derivative have a uniformly positive sign for all ε for any small fair background risk. When ε equals zero, the derivative in part (c) equals

$$V(\theta) \equiv -\partial[U_{\theta\theta\theta}(\theta)/U_{\theta}(\theta)]/\partial\theta. \tag{4}$$

We conclude that a positive value for $V(\theta)$ is necessary and sufficient for local vulnerability.

Corollary 1: *Risk aversion always increases with the introduction of any admissible small fair background risk if and only if the index of local vulnerability is positive, that is, we must have* $V(\theta) > 0$.

By differentiating R_{θ}, it is straightforward to show that $V = R_{\theta\theta} - 2RR_{\theta}$. Thus, the index of local vulnerability coincides with the expression derived by Gollier and Pratt to characterize local vulnerability, confirming the validity of Theorem 1.

3. Greater local vulnerability

In this section we show that $V(\theta)$ indeed provides an index of the degree of local vulnerability in the intuitive sense that the introduction of any small fair background risk causes a greater increase in risk aversion for the more vulnerable decision maker, for whom $V(\theta)$ is uniformly greater. To establish this result, we assume that the decision maker's utility function $U(\theta, \nu)$ belongs to a family Φ whose members are ranked by the degree of local vulnerability ν, and we introduce the following definition of equivalent compensated increases in risk for members of such a family.

Definition: Given a family Φ of decision makers $U(\theta, \nu)$ ranked by degree of local vulnerability ν, and an admissible risk $F(\theta, \bar{r}, \nu)$, the shift $F_{\bar{r}}(\theta, \bar{r}, \nu)$ belongs to a class of equivalent compensated increases in risk for the members of Φ if, for all ν,

$$\int^{y} U_{\theta}(\theta, \nu) F_{\bar{r}}(\theta, \bar{r}, \nu)\, d\theta \equiv G(y, \bar{r}) \geq 0 \tag{3'}$$

for all y in the support of $F(\theta, \bar{r}, \nu)$, with equality at $y = b_{\theta}$.

The initial risk $F(\theta)$ is the same for all members of the family Φ, but the compensated increases in risk are unique to each decision maker in the family, and for this reason each shift $F_{\bar{r}}(\theta, \bar{r}, \nu)$ depends on the preference parameter ν. We may select any given decision maker $U(\theta, \nu)$, along with a compensated increase in risk $F_{\bar{r}}(\theta, \bar{r}, \nu)$ for that decision maker, to determine the magnitude of $G(y, \bar{r})$ for each value of y. Condition (3′) then identifies a class of compensated increases in risk such that (i) each member of the class is associated to a particular decision maker who can be ranked by local vulnerability in comparison to the given decision maker, and (ii) the utility distribution for each decision maker undergoes the same mean preserving spread.[6]

We can now establish that $V(\theta)$ is an index of the degree of local vulnerability by extending Theorem 1 to a comparison of decision makers with respect to greater local vulnerability.[7]

Theorem 2: *The following are equivalent definitions for a family Φ of utility functions $U(\theta, \nu)$ showing greater local vulnerability with higher values of the index ν:*

(a) *Given any admissible risk $F(\theta, \bar{r}, \nu)$ and any member $U(\theta, \nu)$ of Φ, any equivalent compensated increase in risk $F_{\bar{r}}(\theta, \bar{r}, \nu)$ satisfying* (3′) *that is accompanied by the introduction of a small fair background risk $H(\varepsilon)$ satisfying* (2), *which necessarily reduces the expected utility of a vulnerable decision maker, causes a greater reduction in expected utility for the members of family Φ with higher values of the index ν, that is,*

$$\frac{\partial}{\partial \nu} \iint U(\theta + \varepsilon, \nu)\, dF_{\bar{r}}(\theta, \bar{r}, \nu)\, d[H(\varepsilon) - \tilde{H}(\varepsilon)]$$

is negative.

(b) *The increase in risk aversion caused by the introduction of a small fair background risk $H(\varepsilon)$ satisfying* (2) *increases with ν, that is,*

$$R_{\nu}^{\psi}(\theta, \nu) - R_{\nu}(\theta, \nu)$$

is positive.

(c) *The index of local vulnerability $V(\theta, \nu)$ increases with ν, that is,*

$$V_{\nu}(\theta, \nu) \equiv -\partial^2 [U_{\theta\theta\theta}(\theta, \nu)/U_{\theta}(\theta, \nu)]/\partial\theta\, \partial\nu$$

is positive.

This theorem recognizes that compensated increases in risk depend on the decision maker's utility function $U(\theta)$. Therefore, in order to identify one decision maker as more vulnerable than another in the manner suggested by Theorem 1, the introduction of background risk must be accompanied by compensated increases in risk that are comparable for the two decision makers. The requisite comparability is achieved by the equivalence condition (3′). Using this condition, part (a) of Theorem 2 states that the introduction of a small fair background risk, accompanied by compensated increases in risk, results in a greater reduction in expected utility for the more vulnerable decision maker. Part (b) equates greater local vulnerability to a greater increase in the degree of risk aversion resulting from a small fair background risk, and part (c) identifies V as the index of local vulnerability. The natural characterizations of greater local vulnerability stated in parts (b) and (c) are linked by their equivalence to part (a), which exploits the compensated approach to identifying the degree of local vulnerability.

The only restriction placed on the members of a family of decision makers Φ is that they be ranked by degree of local vulnerability. However, the comparative statics significance of a greater increase in risk aversion is clearest when the decision makers have the same degree of risk aversion in the absence of background risk. In Appendix B, we present an illustration of Theorem 2 for decision makers who initially have the same degree of constant relative risk aversion. In the simple two-asset portfolio problem, the proportion of wealth invested in the risky asset is independent of the amount of wealth, and decreases with greater relative risk aversion. Our second Theorem shows that the introduction of a small fair background risk causes a greater reduction in the proportion of wealth invested in the risky asset for those decision makers who are more vulnerable. As we show in Appendix B, decision makers with equal and constant relative risk aversion are more vulnerable as the amount of endowed wealth is smaller.

4. Conclusions

Decision makers who are more risk averse in the presence of fair background risk are defined to be vulnerable. Although no index of vulnerability exists for arbitrary background risks, the index of local vulnerability, $V(\theta)$, not only indicates whether risk aversion increases or decreases upon the introduction of any small fair background risk, but a higher value for the index $V(\theta)$ indicates greater local vulnerability in the intuitive sense that introducing a small fair background risk results in a greater increase in risk aversion.

Positive vulnerability is relevant to asset and insurance markets when labor-income risk is uninsurable. Heaton and Lucas [2000] survey and extend the literature analyzing portfolio choice in the presence of background risk. Elmendorf and Kimball [2000] examine the saving-portfolio choice problem and find that labor-income risk tends to reduce investment in a risky asset. Eeckhoudt and Kimball [1992] show that the presence of an uninsurable, small fair background risk leads to a greater demand for insurance against another, insurable risk, under conditions that imply local vulnerability.[8] Meyer and Meyer [1998] show that, under the same conditions, "strong" and "simple" increases in fair background risk lead to an increase in the demand for insurance. Our analysis suggests that the influence of background risk is magnified for decision makers with greater vulnerability, other things equal. Thus, for example, in the simple portfolio problem where the introduction of a small fair background risk reduces a vulnerable investor's demand for the risky asset, the decline in demand is greater for equally risk-averse, but more vulnerable decision makers.

The degree of vulnerability may also be important for other reasons. Franke, Stapleton, and Subrahmanyam [1998] show that, among investors with hyperbolic absolute risk aversion, those with low background risk sell options on the market portfolio that are bought by those with high background risk. For the parametrized class of utility functions they consider, the index of local vulnerability is positive. Our analysis suggests that investors facing similar background risks may buy or sell options as they are more or less vulnerable.

Finally, Guiso, Jappelli, and Terlizzese [1996] present empirical evidence showing that riskier background wealth in the form of human capital leads to more risk-averse investment behavior, and the empirical evidence presented by Guiso and Jappelli [1998] indicates that background risk has a positive effect on demand for insurance. Our analysis suggests that differences in risk-taking behavior may also be attributed to differences in vulnerability. Although the index of vulnerability is only valid for small background risks, our results point to a complementary avenue for understanding observed differences in risk-taking behavior.

Appendix A

In this appendix we present proofs for the results stated in the text.

Proof of Theorem 1. To equate part (a) with parts (b) and (c), we first rewrite the expression for the change in expected utility given in part (a) as

$$\begin{aligned}
\bar{U}_{\bar{r}} &\equiv \iint U(\theta+\varepsilon)\,dF_{\bar{r}}(\theta,\bar{r})\,d[H(\varepsilon)-\tilde{H}(\varepsilon)] \\
&= \int \left\{ U(\theta) + \int U(\theta+\varepsilon)\,d[H(\varepsilon)-\tilde{H}(\varepsilon)] \right\} dF_{\bar{r}}(\theta,\bar{r}) \\
&= \iint U(\theta+\varepsilon)\,dH(\varepsilon)\,dF_{\bar{r}}(\theta,\bar{r}) \\
&= \int \psi(\theta)\,dF_{\bar{r}}(\theta,\bar{r}). \qquad\qquad (A.1)
\end{aligned}$$

The second line follows from the first after reversing the order of integration and adding $\int U(\theta)\,dF_{\bar{r}}(\theta,\bar{r})$, which equals zero since $F_{\bar{r}}(\theta,\bar{r})$ is a compensated increase in risk for $U(\theta)$. The third line then follows from the assumption that $\tilde{H}(\varepsilon)$ is improper with $\tilde{H}(0)=1$, so that $U(\theta)=\int U(\theta+\varepsilon)\,d\tilde{H}(\varepsilon)$. Finally, the last line follows from the definition $\psi(\theta)\equiv \int U(\theta+\varepsilon)\,dH(\varepsilon)$.

Applying integration by parts twice with respect to θ to evaluate the last integral in (A.1), we obtain

$$\begin{aligned}
\bar{U}_{\bar{r}} &= -\int \psi_\theta F_{\bar{r}}\,d\theta \\
&= -\int (\psi_\theta/U_\theta)U_\theta F_{\bar{r}}\,d\theta
\end{aligned}$$

$$= \int \partial(\psi_\theta/U_\theta)/\partial\theta \int^\theta U_\theta F_{\bar{r}}\,dy\,d\theta$$
$$= \int (R - R^\psi)(\psi_\theta/U_\theta) \int^\theta U_\theta F_{\bar{r}}\,dy\,d\theta. \tag{A.2}$$

Note that, since we restrict attention to compensated increases in risk for which $F_{\bar{r}}(\theta, \bar{r})$ equals zero at both limits of the support of $F(\theta, \bar{r})$, no terms involving these limits appear in this expression. From the integral conditions (3), we conclude that this expression is negative if and only if R^ψ exceeds R, thus establishing equivalence between parts (a) and (b).

Applying integration by parts to the expression in the first line of (A.1), twice with respect to ε and then twice with respect to θ, yields

$$\bar{U}_{\bar{r}} = \iint \partial(U_{\theta\varepsilon\varepsilon}/U_\theta)/\partial\theta \int^\varepsilon (H - \tilde{H})\,dx \int^\theta U_\theta F_{\bar{r}}\,dy\,d\theta\,d\varepsilon, \tag{A.3}$$

where $U_{\theta\varepsilon\varepsilon} \equiv \partial^2 U_\theta(\theta + \varepsilon)/\partial\varepsilon^2$ and $U_\theta \equiv U_\theta(\theta)$. Since $F_{\bar{r}}$ satisfies the integral conditions (3) for a compensated increase in risk for $U(\theta)$, the term $\int^\theta U_\theta F_{\bar{r}}\,dy$ is positive. Similarly, $H - \tilde{H}$ satisfies the inequality conditions (2) for the introduction of risk about ε, implying that $\int^\varepsilon (H - \tilde{H})\,dx$ is positive. Therefore, $\bar{U}_{\bar{r}}$ is negative if and only if

$$-\partial(U_{\theta\varepsilon\varepsilon}/U_\theta)/\partial\theta = -\partial[U_{\theta\theta\theta}(\theta + \varepsilon)/U_\theta(\theta)]/\partial\theta$$

is positive for all θ and ε, thereby establishing equivalence between parts (a) and (c). □

Proof of Theorem 2. By adapting in an obvious manner the notation introduced in proving Theorem 1, we can write the statement in part (a) of Theorem 2 as requiring $\partial\bar{U}_{\bar{r}}(\nu)/\partial\nu < 0$ for an equivalent compensated increase in risk for $U(\theta, \nu)$, $F_{\bar{r}}(\theta, \bar{r}, \nu)$ satisfying (3′). Thus, from the last line of Eq. (A.2), we must have

$$\left(R_\nu - R_\nu^\psi\right)(\psi_\theta/U_\theta) + (R - R^\psi)\frac{\partial}{\partial\nu}(\psi_\theta/U_\theta) < 0,$$

since $\frac{\partial}{\partial\nu}\int^\theta U_\theta(y, \nu)F_{\bar{r}}(y, \bar{r}, \nu)\,dy = \frac{\partial}{\partial\nu}G(\theta, \bar{r}) = 0$. With a sufficiently small background risk, we have $\psi_\theta(\theta, \nu) \cong U_\theta(\theta, \nu)$, implying

$$R_\nu^\psi(\theta, \nu) - R_\nu(\theta, \nu) > 0$$

under part (a), thus establishing equivalence between parts (a) and (b).

Similarly, from Eq. (A.3) and the definition of an equivalent compensated increase in risk for $U(\theta, \nu)$, part (a) requires that we have

$$\partial^2[U_{\theta\varepsilon\varepsilon}(\theta + \varepsilon, \nu)/U_\theta(\theta, \nu)]/\partial\theta\,\partial\nu < 0.$$

For sufficiently small background risks, this inequality holds if and only if $V_\nu(\theta, \nu)$ is positive for all θ, establishing equivalence between parts (a) and (c). □

Appendix B

In this appendix we present an example that illustrates Theorem 2. Consider the isoelastic utility function $U(\theta) = [1/(1-\gamma)]\theta^{1-\gamma}$ with $0 < \gamma \neq 1$, for which the index of relative risk aversion is $\theta R(\theta) = \gamma$, and the index of vulnerability is

$$V(\theta) = 2\gamma(1+\gamma)\theta^{-3}. \tag{B.1}$$

In the simple, two-asset portfolio problem, terminal wealth is given by

$$\theta = W(1 + xr), \tag{B.2}$$

where W is the endowed wealth, x is the proportion of this wealth invested in the risky asset whose random rate of return is r, and the rate of return on the safe asset is normalized to zero. It follows from (B.1) and (B.2) that decision makers with the same degree of risk aversion but smaller wealth endowments are more vulnerable.

After the introduction of background risk, the decision maker's utility function is

$$\psi(\theta) = [1/(1-\gamma)] \int (\theta + \varepsilon)^{1-\gamma}\, dH(\varepsilon) = [1/(1-\gamma)][\theta - \pi(\theta)]^{1-\gamma}, \tag{B.3}$$

where $\pi(\theta)$ is the absolute risk premium for the background risk. Since $V(\theta)$ is positive, $R^{\psi}(\theta) > R(\theta)$. For a richer decision maker with endowed wealth $\hat{W} > W$, let $\hat{\psi}(\theta)$ denote the utility function in the presence of the background risk $H(\varepsilon)$. This decision maker is less vulnerable, and Theorem 2 implies that we have $R^{\psi}(\theta) > R^{\hat{\psi}}(\theta)$.

To verify this inequality, observe first that both decision makers invest the same proportion of their wealth in the risky asset. Hence, from (B.2) and (B.3) we have

$$\hat{\psi}(\theta) = \psi(k\theta), \tag{B.4}$$

where $k = \hat{W}/W > 1$. Next observe that

$$\psi_{\theta}(\theta) = \int (\theta + \varepsilon)^{-\gamma}\, dH(\varepsilon) \tag{B.5}$$

and

$$\psi_{\theta\theta}(\theta) = -\gamma \int (\theta + \varepsilon)^{-\gamma-1}\, dH(\varepsilon). \tag{B.6}$$

It follows from (B.4)–(B.6) that $R^{\psi}(\theta) > R^{\hat{\psi}}(\theta)$ is equivalent to

$$\frac{\gamma \int (\theta + \varepsilon)^{-\gamma-1}\, dH(\varepsilon)}{\int (\theta + \varepsilon)^{-\gamma}\, dH(\varepsilon)} > \frac{\gamma \int (k\theta + \varepsilon)^{-\gamma-1}\, dH(\varepsilon)}{\int (k\theta + \varepsilon)^{-\gamma}\, dH(\varepsilon)}, \tag{B.7}$$

or equivalently,

$$\int (k\theta+\varepsilon)^{-\gamma}\,dH(\varepsilon)\int(\theta+\varepsilon)^{-\gamma-1}\,dH(\varepsilon)$$
$$> \int (k\theta+\varepsilon)^{-\gamma-1}\,dH(\varepsilon)\int(\theta+\varepsilon)^{-\gamma}\,dH(\varepsilon). \tag{B.8}$$

We next recast this inequality in terms of absolute risk premia. Observe that

$$-\int(\theta+\varepsilon)^{-\gamma}\,dH(\varepsilon) = -[\theta-\pi_1(k\theta)]^{-\gamma}, \tag{B.9}$$

where $\pi_1(\theta)$ is the absolute risk premium for the fair risk $H(\varepsilon)$ and the utility function $u(\theta) = -\theta^{-\gamma}$. Hence, we also have

$$-\int(k\theta+\varepsilon)^{-\gamma}\,dH(\varepsilon) = -[k\theta-\pi_1(k\theta)]^{-\gamma}. \tag{B.10}$$

Similarly,

$$-\int(\theta+\varepsilon)^{-\gamma-1}\,dH(\varepsilon) = -[\theta-\pi_2(\theta)]^{-\gamma-1} \tag{B.11}$$

and

$$-\int(k\theta+\varepsilon)^{-\gamma-1}\,dH(\varepsilon) = -[k\theta-\pi_2(k\theta)]^{-\gamma-1}, \tag{B.12}$$

where $\pi_2(\theta)$ is the absolute risk premium for the fair risk $H(\varepsilon)$ and the utility function $u(\theta) = -\theta^{-\gamma-1}$.

Using (B.9)–(B.12), we can restate (B.8) as

$$[k\theta-\pi_1(k\theta)]^{-\gamma}[\theta-\pi_2(\theta)]^{-\gamma-1} > [k\theta-\pi_2(k\theta)]^{-\gamma-1}[\theta-\pi_1(\theta)]^{-\gamma}, \tag{B.13}$$

which we can rewrite as

$$\left(\frac{\theta-\pi_1(\theta)}{k\theta-\pi_1(k\theta)}\right)^{\gamma} > \left(\frac{\theta-\pi_2(\theta)}{k\theta-\pi_2(k\theta)}\right)^{1+\gamma}. \tag{B.14}$$

Since $u(\theta) = -\theta^{-\gamma-1}$ exhibits decreasing absolute risk aversion, the term within brackets on the right hand side of (B.14) is less than one. Hence, the inequality in (B.14) is satisfied if

$$\frac{\theta-\pi_1(\theta)}{k\theta-\pi_1(k\theta)} > \frac{\theta-\pi_2(\theta)}{k\theta-\pi_2(k\theta)}, \tag{B.15}$$

or equivalently,

$$[\pi_2(\theta) - \pi_1(\theta)]k\theta - [\pi_2(k\theta) - \pi_1(k\theta)]\theta \\ + \pi_1(\theta)\pi_2(k\theta) - \pi_1(k\theta)\pi_2(\theta) > 0. \tag{B.16}$$

Since $H(\varepsilon)$ is a small fair risk and the index of absolute risk aversion for $u(\theta) = -\theta^{-\gamma}$ is $(1+\gamma)\theta^{-1}$, while the index for $u(\theta) = -\theta^{-\gamma-1}$ is $(2+\gamma)\theta^{-1}$, we have $\pi_1(\theta) \cong (1/2)\sigma_\varepsilon^2(1+\gamma)/\theta$ and $\pi_2(\theta) \cong (1/2)\sigma_\varepsilon^2(2+\gamma)/\theta$ where σ_ε^2 is the variance of the risk $H(\varepsilon)$. Substituting into the left-hand side of (B.16), we arrive at

$$(1/2)\sigma_\varepsilon^2\{k[(2+\gamma) - (1+\gamma)] - (1/k)[(2+\gamma) - (1+\gamma)]\} \\ + \left[(1/2)\sigma_\varepsilon^2\right]^2\left(\frac{1+\gamma}{\theta}\frac{2+\gamma}{k\theta} - \frac{1+\gamma}{k\theta}\frac{2+\gamma}{\theta}\right) \\ = (1/2)\sigma_\varepsilon^2(k - 1/k), \tag{B.17}$$

which is positive as desired, since k is greater than one.

Notes

1. When a limit of integration is unspecified, it is to be understood that the unstated limit is the appropriate extreme value of the associated support.
2. This concept of small risk is consistent with, but stronger than the concept employed by Pratt (1964), who requires only that the variance of the risk becomes arbitrarily small. As a consequence, the concept of small that we employ is not exploited through the use of Taylor series approximations, but rather through compensated increases in risk.
3. Following Diamond and Stiglitz, we confine attention to foreground risks with no probability mass at their support limits either before or after a compensated increase in risk. Thus, in particular, $F_{\bar{r}}(\theta, \bar{r})$ equals zero at both limits of the support of $F(\theta, \bar{r})$.
4. Exponential utility provides a knife-edge example. It is easy to verify that the degree of risk aversion for $U(\theta) = -\exp(-\gamma\theta)$ remains equal to γ after the introduction of an additive background risk. The effect on expected exponential utility of a compensated increase in risk accompanied by a small fair background risk is

$$-\iint \exp[-\gamma(\theta+\varepsilon)]\, dF_{\bar{r}}(\theta, \bar{r})\, d[H(\varepsilon) - \tilde{H}(\varepsilon)] \\ = \gamma \iint \exp[-\gamma(\theta+\varepsilon)]F_{\bar{r}}\, d\theta\, d(H - \tilde{H}) \\ = \gamma \iint \exp(-\gamma\theta)\exp(-\gamma\varepsilon)F_{\bar{r}}\, d\theta\, d(H - \tilde{H}) \\ = \gamma \int \exp(-\gamma\theta)F_{\bar{r}}\, d\theta \int \exp(-\gamma\varepsilon)\, d(H - \tilde{H}).$$

   This expression vanishes, indicating correctly that the decision maker is not vulnerable, since the first integral on the last line equals zero given that $F_{\bar{r}}$ is a compensated increase in risk for $U(\theta)$.
5. Logarithmic utility provides an example with constant relative risk aversion. One finds that the derivative in part (c) can be written $2(\theta+\varepsilon)^{-4}(2\theta - \varepsilon)$, which is positive for all θ and ε as long as the maximum value for ε is sufficiently small.
6. To see that the utility distributions for each member of the family Φ undergoes the same mean preserving spread, note that the integral conditions (3) are equivalent to those for a mean preserving spread of the distribution of utility $\hat{F}(U(\theta), \bar{r}) \equiv F(\theta, \bar{r})$, since $\hat{F}_{\bar{r}} dU \equiv F_{\bar{r}} U_\theta\, d\theta$. For each member $U(\theta, \nu)$ of the family

Φ, a compensated increase in risk $F_{\bar{r}}(\theta, \bar{r}, \nu)$ satisfying (3′) yields the same value for $G(y, \bar{r})$ and hence for $\int^{U(y)} \hat{F}_{\bar{r}}(U, \bar{r})\, dU$, and therefore the utility distribution for each member of Φ undergoes the same mean preserving spread.

7. Theorems 1 and 2 can be extended to *local risk vulnerability* as defined by Gollier and Pratt by incorporating unfair (negative-mean) as well as fair (zero-mean) background risks. One finds that a decision maker is locally risk vulnerable if and only if $V(\theta)$ is positive and $R_\theta(\theta)$ is negative, confirming the result obtained by Gollier and Pratt. One also finds that greater local risk vulnerability is associated with higher values for the indices of local vulnerability, $V(\theta)$, and decreasing risk aversion, $-R_\theta(\theta)$.
8. The conditions invoked by Eeckhoudt and Kimball are that risk aversion be decreasing ($R_\theta < 0$) and that the index of prudence, $P(\theta) \equiv -U_{\theta\theta\theta}/U_{\theta\theta}$, introduced by Kimball [1990], also be decreasing ($P_\theta < 0$). Gollier and Pratt show that these conditions are sufficient for local vulnerability.

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