Journal of Algebraic Combinatorics 5 (1996), 83–86

A Classification of 2-Arc-Transitive Circulants

BRIAN ALSPACH*
Department of Mathematics and Statistics, Simon Fraser University, Burnaby, B.C., Canada, V5A 1S6

MARSTON D.E. CONDER
Department of Mathematics and Statistics, University of Auckland, Private Bag 92019, Auckland, New Zealand

DRAGAN MARUŠIČ†
IMFM, Oddelek za matematiko, Univerza v Ljubljani, Jadranska 19, 61111 Ljubljana, Slovenija

MING-YAO XU‡
Department of Mathematics, Peking University, Beijing 100871, People's Republic of China

Received August 19, 1993; Revised January 30, 1995

Abstract. A graph X is *k-arc-transitive* if its automorphism group acts transitively on the set of k-arcs of X. A *circulant* is a Cayley graph of a cyclic group. A classification of 2-arc-transitive circulants is given.

Keywords: 2-arc-transitive graph, circulant graph

1. Introductory remarks

Throughout this paper graphs are simple and undirected. Adopting the terminology of Tutte [3], a *k-arc* in a graph X is a sequence of $k+1$ vertices $v_1, v_2, \ldots, v_{k+1}$ of X, not necessarily all distinct, such that any two consecutive terms are adjacent and any three consecutive terms are distinct. We shall use the term *arc* rather than 1-arc. A graph X is said to be *k-arc-transitive* if the automorphism group of X, denoted $\mathrm{Aut}(X)$, acts transitively on the k-arcs of X. Of course, a connected k-arc-transitive graph is also j-arc-transitive for each $j \in \{1, 2 \ldots, k\}$. A graph X is said to be *exactly k-transitive* if it is k-arc-transitive but not $(k+1)$-arc-transitive.

Let n be a positive integer and $S \subseteq Z_n \setminus \{0\}$ satisfy $i \in S$ if and only if $n - i \in S$. Define a graph $X(n; S)$ as follows. The vertex set of $X(n; S)$ is labelled with the elements of Z_n, and there is an edge joining i and j if and only if $i - j \in S$. The graph $X(n; S)$ is called a *circulant* and the set S is called the *symbol of* $X(n; S)$.

*Partial support provided by the Natural Sciences and Engineering Research Council of Canada under Grant A-4792.

†Supported in part by "Ministrstvo za znanost in tehnologijo Slovenije", project no. P1-0208-0101-93.

‡Supported in part by the National Natural Sciences Foundation of China and the Doctoral Program Foundation of Institutions of Higher Education of China.

Our long-term goal is to classify all arc-transitive circulants (we use the term arc-transitive instead of 1-arc-transitive). It is not difficult to see that for any positive integer n and any even order subgroup S of Z_n^*, the graph $X(n; S)$ is an arc-transitive circulant. In fact, if $n = p$ is a prime the above construction exhausts the entire class of arc-transitive circulants on p vertices (see [1, 2]). With the exception of C_p and K_p, all of the above graphs are exactly 1-transitive circulants. On the other hand, there are circulants which are 2-arc-transitive. As a first step towards our long-term goal, we wish to determine which circulants are 2-arc-transitive.

The aim of this paper is to prove the following result.

Theorem 1.1 *A connected, 2-arc-transitive circulant of order n, $n \geq 3$, is one of the following graphs:*

(i) *the complete graph K_n, which is exactly 2-transitive;*
(ii) *the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$, $n \geq 6$, which is exactly 3-transitive;*
(iii) *the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ minus a 1-factor, $\frac{n}{2} \geq 5$ odd, which is exactly 2-transitive; and*
(iv) *the cycle C_n of length n, which is k-arc-transitive for all $k \geq 0$.*

The proof of this theorem is given in the next section. A few observations about the preceding theorem are in order. Of course, the cycles are something of an anomaly. They are the only connected circulants of degree 2 and the conclusion of (iv) is seen easily. This allows us to assume that all circulants under consideration have degree at least 3. The next result reduces the analysis to circulants of girth 4.

Proposition 1.2 *If X is neither C_n nor K_n and is a connected 2-arc-transitive circulant, then its girth is 4.*

Proof: If $X \neq C_n$, then its girth is either 3 or 4. Namely, let S be a symbol of X. For $s_1, s_2 \in S$ such that $s_2 \neq s_1, -s_1$ we have that $(0, s_1, s_1 + s_2, s_2, 0)$ is a 4-cycle in X. Suppose that C_3 is a 3-cycle of X. Any path of length 2 on C_3 has an edge joining its end vertices. Since X is 2-arc-transitive, the preceding is true of any path of length 2 in X. Since X is connected, X must be a complete graph. □

2. Proving Theorem 1.1

Let us start out with some of the tools and conventions that go into the proof of Theorem 1.1. If X is a graph and U and W are disjoint subsets of $V(X)$, we let $X[U, W]$ denote the bipartite graph induced by the edges of X having one end vertex in U and the other in W. Given $v \in V(X)$ we let $N^i(v)$ denote the set of all vertices of X at distance i from v. In particular, we let $N(v) = N^1(v)$. A bipartite graph X with a bipartition (U, W) is said to be *biregular* if the degrees of all vertices in U are equal and if the degrees of all vertices in W are equal, too.

Proposition 2.1 *Let X be a connected 2-arc-transitive graph and let $v \in V(X)$. Then $X[N(v), N^2(v)]$ is biregular.*

Proof: Let $v \in V(X)$. Since X is 2-arc-transitive, it follows that both $N(v)$ and $N^2(v)$ are orbits of the vertex stabilizer $\mathrm{Aut}(X)_v$. The result follows. □

Lemma 2.2 *Let $R \subseteq Z_n \setminus \{0\}$ and let $j \in Z_n \setminus \{0\}$. Let $\rho(j) = |R \cap (-R + j)|$. Then*

(i) *$j \notin 2R$ implies that $\rho(j)$ is even;*
(ii) *$j \in 2R$ and n odd imply that $\rho(j)$ is odd;*
(iii) *$j \in 2R$ and n even imply that $\rho(j)$ is even when $j = 2r$ for some $r \in R \cap (R + \frac{n}{2})$ but otherwise $\rho(j)$ is odd.*

Proof: Note that the elements belonging to the set $R \cap (-R + j)$ are paired off whenever $j \notin 2R$. Namely, if $r = -r' + j \in R \cap (-R + j)$, then $r' = -r + j \in R \cap (-R + j)$, too. This proves (i).

So assume that $j \in 2R$. Using the same argument as above, the parity of $\rho(j)$ now depends solely on whether the equation $j = 2x$ has one or two solutions in R. The latter occurs if and only if n is even and the two solutions differ by $\frac{n}{2}$. In other words when $j = 2r$ for some $r \in R \cap (R + \frac{n}{2})$. This proves both (ii) and (iii). □

Proof of Theorem 1.1: The proof that the transitivities of the graphs in Theorem 1.1 are as claimed is left to the reader. Also, note that the case $K_{m,m} - mK_2$, $m \geq$ even, is excluded as such graphs are not circulants. To see this, note that the underlying $2m$-cycle acting on the vertices of a connected bipartite circulant with $2m$ vertices must interchange the two bipartition sets. Hence, one bipartition set is the even integers and the other is the odd integers. This implies that the symbol contains only odd integers implying the degree must be even. This excludes $K_{m,m} - mK_2$.

Let $X \notin \{C_n, K_n\}$ be a connected 2-arc-transitive circulant of order n with symbol S. Then $n \geq 6$, the degree $d = |S|$ of X satisfies $2 < d < n - 1$, and the girth of X is 4 by Proposition 1.2. Also, as X is connected we have that $\langle S \rangle = Z_n$. Because of 2-arc-transitivity, each 2-arc of X is contained in the same number $q > 0$ of 4-cycles. Let $s, s' \in S$, where $s \neq s'$. Considering the 2-arc $(s, 0, s')$, we have that $|S \cap (S + s' - s)| = |(S + s) \cap (S + s')| = q + 1$ and so

$$|S \cap (S + t)| = q + 1 \text{ for each } t \in (S + S) \setminus \{0\}. \tag{1}$$

We are going to distinguish two cases.

Case 1. $q + 1$ is even.

Since $2S \subseteq S + S$ and $S = -S$, part (ii) of Lemma 2.2 implies that n is even. Moreover, part (iii) of Lemma 2.2 implies that $s \in S + \frac{n}{2}$ for each $s \in S \setminus \{\frac{n}{2}\}$. Therefore $S \cap (S + \frac{n}{2}) \neq \emptyset$. This implies that $\frac{n}{2} \notin S$ for otherwise X would have girth 3. Hence $S = S + \frac{n}{2}$. Also, $\frac{n}{2} \in N^2(0)$ and an application of Proposition 2.1 gives us that each vertex in $N^2(0)$ has degree d in $X[N(0), N^2(0)]$ and so $d|N^2(0)| = (d - 1)|N(0)|$ from which we infer that

$|N^2(0)| = d - 1$. More precisely, $(S + s) \setminus \{0\} = N^2(0)$ for each $s \in S$. Clearly, each 2-arc in X is contained in $d - 1$ cycles of length 4 which implies that $d = q + 1$. Therefore, $X[N(0), N^2(0)] = K_{q+1,q}$ and so $X = K_{q+1,q+1} = K_{\frac{n}{2},\frac{n}{2}}$.

Case 2. $q + 1$ is odd.

Combining Lemma 2.2 and (1) we can deduce that

$$(S + S) \setminus \{0\} = 2S \setminus \{0\} \tag{2}$$

and moreover if n is even, $S \cap (S + \frac{n}{2}) = \emptyset$. Let us first assume that $0 \in 2S$. Then n must be even and furthermore $\frac{n}{2} \in S$. Also, by (2), $S + S = 2S$ and so $S + s = 2S$ for each $s \in S$. Since X has no triangles, we must have that $N(0) = S$ and $N^2(0) = 2S \setminus \{0\}$. It follows that a 2-arc of the form $(s, 0, s')$, where $s, s' \in S$, is contained in $|2S \setminus \{0\}| = |S| - 1 = d - 1$ cycles of length 4. Thus $d = q + 1$ and so $X[N(0), N^2(0)] = K_{q+1,q}$. Hence $X = K_{q+1,q+1} = K_{\frac{n}{2},\frac{n}{2}}$.

Assume now that $0 \notin 2S$. Then we have that $(S + S) \setminus \{0\} = 2S$ by (2). Since X has no triangles it follows that $N(0) = S$ and $N^2(0) = 2S$. Clearly, $0 \in S + s$ for each $s \in S$. Using Proposition 2.1, we conclude that for each $s \in S$ there exists a unique $s' \in S \setminus \{s\}$ such that $2s' \notin S + s$. It follows that the number of 4-cycles containing the 2-arc $(0, s, 2s)$, where $s \in S$, must be $d - 2$ and so $|S| = d = q + 2$. Hence $X[N(0), N^2(0)] = K_{q+2,q+2} - (q + 2)K_2$.

The regularity of X implies that the set $V(X) \setminus (\{0\} \cup N(0) \cup N^2(0))$ contains at least one vertex. Let $u \in N^3(0)$. Suppose u has a neighbour $y \in N^2(u)$ and a neighbour $z \notin N^2(u)$. Since y has no neighbours, other than u, outside of $N(0)$, the 2-arc (z, u, y) is in no 4-cycles. We conclude that there is a unique vertex in $V(X) \setminus (\{0\} \cup N(0) \cup N^2(0))$. In particular, n is even and this vertex is $\frac{n}{2}$ with $N^2(0)$ its set of neighbours. Thus, $X = K_{q+3,q+3} - (q + 3)K_2 = K_{\frac{n}{2},\frac{n}{2}} - \frac{n}{2}K_2$. □

Acknowledgment

The second and third author thank Simon Fraser University for hospitality during the two visits that led to the completion of this work.

References

1. J.L. Berggren, "An algebraic characterization of symmetric graphs with p points, p an odd prime," *Bull. Austral. Math. Soc.* **7** (1972), 131–134.
2. C.Y. Chao, "On the classification of symmetric graphs with a prime number of vertices," *Trans. Amer. Math. Soc.* **158** (1971), 247–256.
3. W.T. Tutte, "On the symmetry of cubic graphs," *Canad. J. Math.* **11** (1959), 61–64.