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Duality of Graded Graphs †

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Abstract. A graph is said to be graded if its vertices are divided into levels numbered by integers, so that the endpoints of any edge lie on consecutive levels. Discrete modular lattices and rooted trees are among the typical examples. The following three types of problems are of interest to us:

(1) path counting in graded graphs, and related combinatorial identities;
(2) bijective proofs of these identities;
(3) design and analysis of algorithms establishing corresponding bijections.

This article is devoted to (1); its sequel [7] is concerned with the problems (2)–(3). A simplified treatment of some of these results can be found in [8].

In this article, R.P. Stanley's [26, 27] linear-algebraic approach to (1) is extended to cover a wide range of graded graphs. The main idea is to consider *pairs* of graded graphs with a common set of vertices and common rank function. Such graphs are said to be *dual* if the associated linear operators satisfy a certain commutation relation (e.g., the "Heisenberg" one). The algebraic consequences of these relations are then interpreted as combinatorial identities. (This idea is also implicit in [27].)

[7] contains applications to various examples of graded graphs, including the Young, Fibonacci, Young-Fibonacci and Pascal lattices, the graph of shifted shapes, the r-nary trees, the Schensted graph, the lattice of finite binary trees, etc. Many enumerative identities (both known and unknown) are obtained.

Abstract of [7]. These identities can also be derived in a purely combinatorial way by generalizing the Robinson-Schensted correspondence to the class of graphs under consideration (cf. [5]). The same tools can be applied to permutation enumeration, including involution counting and rook polynomials for Ferrers boards. The bijective correspondences mentioned above are naturally constructed by Schensted-type algorithms. A general approach to these constructions is given. As particular cases we rederive the classical algorithm of Robinson, Schensted, and Knuth [20, 12, 21], the Sagan-Worley [17, 32] and Haiman [11] algorithms, the algorithm for the Young-Fibonacci graph [5, 15], and others. Several new applications are given.

Keywords: enumerative combinatorics, tableaux, Young diagram, differential poset, graded graph

1. Linear theory

1.1. Graded graphs

The terminology and notation used follows [25,26,18].

A *graded graph* is a triple $G = (P, \rho, E)$ where

(1) P is a discrete set of *vertices*,

(2) $\rho: P \to \mathbb{Z}$ is a *rank function*,

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(3) E is a multiset of *arcs/edges* (x, y) where $\rho(y) = \rho(x) + 1$.

Sometimes $|x|$ is written instead of $\rho(x)$. The set $P_n = \{x: \rho(x) = n\}$ is called a *level* of G. Finiteness of the levels is always assumed and some of them may be empty. The situation when

$$P_0 = \{\widehat{0}\}, \quad P_{-1} = P_{-2} = P_{-3} = \cdots = \emptyset$$

is typical (a graph with a *zero* $\widehat{0}$).

Examples of graded graphs can be found in [25, 2, 14, etc.].

G can be regarded either as an oriented or as an unoriented graph. The edges of an oriented graph define a *partial order* on P. If there are no multiple edges, G is the Hasse diagram of this poset. Therefore non-oriented paths (i.e., paths in the non-oriented graph) are often called *Hasse walks*.

Let $e(x \to y)$ denote the number of shortest non-oriented paths between x and y. In a graph with zero let $e(y) = e(\widehat{0} \to y)$. So $e(y)$ is the number of paths going from $\widehat{0}$ to y. Let

$$\alpha(n \to m) = \sum_{x \in P_n} \sum_{y \in P_m} e(x \to y). \tag{1.1.1}$$

In other words, $\alpha(n \to m)$ is the number of paths connecting the nth and mth levels. One can similarly define $\alpha(n \to m \to p)$, $e(x \to y \to z)$, etc. For instance, in a graph with zero

$$\alpha(0 \to n \to 0) = \sum_{x \in P_n} e(x)^2 \tag{1.1.2}$$

is the number of loops of length $2n$, i.e., pairs of oriented paths of length n starting at $\widehat{0}$ and having common endpoint).

The main results we are going to obtain in this article are combinatorial identities involving $e(x)$ and similar enumerative functions. A typical example is the Young-Frobenius identity

$$\sum_{x \in P_n} e(x)^2 = n! \tag{1.1.3}$$

or, equivalently,

$$\alpha(0 \to n \to 0) = n! \tag{1.1.4}$$

for Young's lattice. This is not an isolated result. Surprisingly, the same formula proves to be valid for the so-called Fibonacci lattices [24, 5, 26]. A similar identity (with additional coefficients) is known for the graph of shifted shapes (see, e.g., [18] or (2.2.7)). Another example is the lattice of binary trees where

$$\alpha(0 \to n) = n! \tag{1.1.5}$$

(see [24]). Each of these facts is known to have both computational and bijective proofs; however, these proofs use individual properties of the graphs (except for the proof of (1.1.3) in [26, 5]).

We develop combinatorial techniques for proving general results of this type for a wide class of graded graphs. Unified proofs of (1.1.4), (1.1.5), and many other enumerative identities, both known (see, e.g., [26, 18]) and unknown, are given. Then we develop, in [7], a general approach to Robinson-Schensted-type algorithms for the same class of graphs. This allows us to provide bijective proofs for all the enumerative results of this paper.

A few words about terminology and useful associations. The subject under consideration is related to the so-called *animal growth*, i.e., mathematical models of a growth of multi-cell patterns. From this point of view, vertices of G are "animals", and oriented paths are virtual scenarios of their growth.

Another convenient language is that of diagrams and tableaux. Vertices are referred to as *diagrams* or *shapes*. Monotone mappings of the form

$$g\colon \{0,1,\ldots,n\} \to P \tag{1.1.6}$$

are called *tableaux*. This terminology is especially natural when P is a distributive lattice of finite order ideals of a poset Q: then the elements of Q are drawn as boxes/cells forming the diagrams/shapes being the elements of P. A growth (1.1.6) of such a shape can be described by a tableau whose entries indicate the moments of discrete time when corresponding boxes get added to the shape. Formally, a number t is put into a box c if $g(t) = g(t-1) \cup \{c\}$. A tableau containing n boxes is said to be *standard* if the growth g is strictly monotone (bijective), i.e., a new box springs up at every moment till n. In other words, a tableau is standard if and only if it is filled with consecutive integers $1, 2, \ldots, n$. In the case of the Young graph this terminology coincides with the usual concepts of *Young tableau* and *standard Young tableau* (SYT).

1.2. The linear algebraic approach

We start with some basic concepts introduced by R.P. Stanley [26].

A graded graph is completely determined by the adjacency matrices of the bipartite graphs formed by its consecutive levels. Thus graded graphs can be studied as linear-algebraic objects.

Let $G = (P, \rho, E)$ be a graded graph, and K a field of characteristic zero. The finitary K-valued functions on P (or, equivalently, the formal linear combinations of vertices) form the vector space KP. For any $x, y \in P$ let $a(x,y)$ denote the multiplicity of (x,y), i.e., the number of edges joining x and y. The formulae

$$Ux = \sum_{y \text{ covers } x} a(x,y)y \tag{1.2.1}$$

and

$$(U^*)\, y = \sum_{y \text{ covers } x} a(x,y)\, x \tag{1.2.2}$$

define linear transformations U and U^* acting on KP.

It is well known that many path counting characteristics can be expressed in terms of operators U and U^*. For example, let $x \in P_n$. Then

$$U^k x = \sum_{y \in P_{n+k}} e(x \to y) y \tag{1.2.3}$$

and

$$(U^*)^k x = \sum_{y \in P_{n-k}} e(y \to x) y.$$

Hasse walks having arbitrary (but fixed) structure can be dealt with in the same manner. For example, the number of loops of length $2n$ starting at zero (cf. (1.1.2)) is the coefficient of $\widehat{0}$ in $(U^*)^n U^n \widehat{0}$.

The linear algebraic approach allows us to study not only ordinary graphs but also graded *networks*, i.e., graphs with arbitrary—say, rational, negative, complex, etc.—edge multiplicities or weights. The interpretation of $e(x)$ and similar quantities depends on the particular situation. For example, the weights can be defined as the transition probabilities of a certain Markov process. In this case the value of $e(x \to y)$ is the probability of going through y provided the process starts in x.

In what follows combinatorial terminology is used. However, most of the results remain valid for networks, and examples of the latter type will also be given.

The space KP can also be equipped with a Euclidean structure by declaring the vertices to form an orthonormal system. This structure makes the transforms U and U^* conjugate. However, we intentionally avoid exploiting this Euclidean structure; only purely linear techniques are used. So the asterisk in U^* is to be regarded as just a symbol.

Now we extend the linear algebraic approach to *pairs of graphs*.

Definition 1.2.1 Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of graded graphs with a common set of vertices and common rank function. Define an *oriented graded graph* $G = (P, \rho, E_1, E_2)$ by directing the G_1-edges up (in the direction of increasing rank) and the G_2-edges down (in the direction of decreasing rank). Now it is natural to introduce the *up* and *down operators* $U, D \in End(KP)$ associated with the graph G (or with the pair (G_1, G_2)) by

$$Ux = \sum_y a_1(x, y) y \tag{1.2.4}$$

and

$$Dy = \sum_x a_2(x, y) x \tag{1.2.5}$$

where $a_i(x, y)$ is the multiplicity (weight) of the edge (x, y) in G_i.

In the case $G_1 = G_2$ this definition coincides with (1.2.1)–(1.2.2); in other words, in this case $U^* = D$, an equation that may not hold in general. Also, note that an alternative construction $G^* = (P, \rho, E_2, E_1)$ leads to operators U^* and D^*, but we do not need to bring these operators into the picture.

Definition 1.2.2 Let G be as in Definition 1.2.1. Assume w is a word in the alphabet $\{U, D\}$ (or briefly a $\{U, D\}$-word). A path p in G is said to have *structure* w, or to be a *w-path*, if its consecutive arcs are directed up or down in accordance with the word w. So the U correspond to up-directed arcs of G_1 and the D to the down-directed arcs of G_2. We emphasize that in this definition a word w is to be read *from right to left* since it will be later interpreted as an operator.

For example, a U^2D-path is a path formed by three arcs

$$(x_1, x_0) \in E_2, (x_1, x_2) \in E_1, (x_2, x_3) \in E_1$$

where

$$\rho(x_1) = \rho(x_0) - 1, \;\; \rho(x_2) = \rho(x_1) + 1, \;\; \rho(x_3) = \rho(x_2) + 1.$$

Thus the number of w-paths in G from x to y is the coefficient of y in the expansion of wx where w is interpreted as an operator in KP.

1.3. Recurrent commutation relations

Throughout this section a graph G, the operators U and D, etc., are as in Definition 1.2.1. The restrictions of U and D onto "homogeneous" subspaces KP_n are denoted U_n and D_n, respectively. So two series of inversely directed linear homomorphisms arise:

$$\begin{array}{l} KP_0 \to^{U_0} KP_1 \to^{U_1} KP_2 \to \cdots \\ KP_0 \leftarrow_{D_1} KP_1 \leftarrow_{D_2} KP_2 \leftarrow \cdots \end{array} \tag{1.2.6}$$

We will study situations where the maps U and D (or the U_n and the D_n) satisfy some algebraic conditions. Formal algebraic consequences of those conditions (see Section 1.4) will then lead to enumerative combinatorial identities (Section 1.5).

Definition 1.3.1 Assume that, for any n,

$$D_{n+1}U_n = f_n(U_{n-1}D_n) \tag{1.3.1}$$

or, equivalently,

$$(DU - f_n(UD))|_{KP_n} = 0 \tag{1.3.2}$$

where the f_n are certain polynomials in one variable with coefficients in K. The equalities (1.3.1) or (1.3.2) are called *recurrent commutation relations*.

From a combinatorial point of view, (1.3.1) means that for any pair of vertices $x, y \in P_n$ the number of two-edged DU-paths from x to y (cf. Definition 1.2.2) can be expressed as a linear combination of the cardinalities of the I-paths, UD-paths, $UDUD$-paths, etc., joining x and y. (The I-path is a degenerate path in the case $x = y$.)

Remark 1.3.2 The relations (1.3.2) are symmetric (invariant) with respect to the interchange of the initial graphs G_1 and G_2. Indeed, replacing D and U by U^* and D^*, respectively, transforms (1.3.2) into an equivalent relation

$$(U^*D^* - f_n(D^*U^*))|_{KP_n} = 0.$$

This property allows us to regard the situation under consideration as, in a sense, a *duality* of the graphs G_1 and G_2 with respect to relations (1.3.2).

The relations (1.3.2) do not of course exhaust the whole class of symmetric in the above sense relations. We have chosen (1.3.2) because of the many interesting examples satisfying these relations and because in this case the relevant calculations can be done explicitly.

Definition 1.3.3 The main special case of (1.3.1) is the *linear commutation relations*

$$D_{n+1}U_n = q_nU_{n-1}D_n + r_nI_n. \tag{1.3.3}$$

These relations mean combinatorially that

(1) if x and y are different vertices of rank n then the number of DU-paths from x to y is q_n times the number of UD-paths joining x and y;
(2) if x is a vertex of rank n then the number of DU-loops (paths) from x to x equals the number of UD-loops times q_n plus r_n.

The most important special case of (1.3.3) is $q_n = 1$ (see Definition 1.3.4 below). However, there are various examples with $q_n \neq 1$, including infinite rooted t-nary trees, complete graded graphs, many combinatorial designs, etc. Note that in all mentioned examples $U^* = D$, which may be false in general.

There are several ways of constructing new graphs satisfying (1.3.3) from existing ones (cf. [27]). Some of these constructions will be described later on.

Definition 1.3.4 Graded graphs G_1 and G_2 with a common set of vertices and common rank function (see Definition 1.2.1) are said to be $\mathbf{r}$-*dual* if the following relations hold:

$$D_{n+1}U_n = U_{n-1}D_n + r_nI_n \tag{1.3.4}$$

where $\mathbf{r} = \{r_n\}$ is a sequence of constants, i.e., $r_n \in K$. In the case where all the r_n coincide, say, $r_n = r$, the graphs are called r-*dual*:

$$D_{n+1}U_n = U_{n-1}D_n + rI_n\ . \tag{1.3.5}$$

Finally, they are called simply *dual* when $r = 1$:

$$D_{n+1}U_n = U_{n-1}D_n + I_n. \tag{1.3.6}$$

Similarly, the conditions (1.3.3) may be referred to as a $(\mathbf{q}, \mathbf{r})$-duality.

These definitions are summarized and restated below in (1.4.9)–(1.4.13).

In the sequel we shall prove enumerative formulae for the general case (1.3.1) and then obtain their versions for the particular cases (1.3.3)–(1.3.6).

1.4. Graded algebras

Definition 1.4.1 Let K^∞ denote the commutative ring of infinite sequences of elements of our field K, with coordinatewise operations. Regarding K^∞ as an algebra over K, extend it by adding non-commuting generators U and D satisfying

$$\alpha U = U\alpha^{\downarrow} \tag{1.4.1}$$

and

$$\alpha^{\downarrow} D = D\alpha \tag{1.4.2}$$

for any $\alpha = \{\alpha_i\} \in K^\infty$ where $\alpha^{\downarrow} = \{\alpha_{i+1}\}$ is α "shifted down". Alternatively,

$$\alpha^{\uparrow} U = U\alpha \tag{1.4.3}$$

and

$$\alpha D = D\alpha^{\uparrow} \tag{1.4.4}$$

where $\alpha^{\uparrow} = \{\alpha_{i-1}\}$ is α "shifted up". The resulting non-commutative associative algebra $\mathfrak{A}$ can be graded:

$$\rho(U) = 1, \quad \rho(D) = -1, \quad \rho(\alpha) = 0 \quad \text{for all} \quad \alpha \in K^\infty;$$

the rank function extends to "homogeneous" elements by setting $\rho(AB) = \rho(A)\,\rho(B)$.

Definition 1.4.2 Let $\mathfrak{A}_0$ denote the subalgebra of $\mathfrak{A}$ formed by the elements of zero rank. This subalgebra is freely K^∞-spanned by the *balanced* $\{U, D\}$-words, i.e., those with an equal number of U's and D's. Since every balanced word commutes with any element of K^∞, we can regard as an algebra over K^∞.

Since $\mathfrak{A}_0$ is a K^∞-algebra, we can operate in the usual way with expressions like $\mathbf{g}(w)$ where $w \in \mathfrak{A}_0$ (e.g., w is a balanced $\{U, D\}$-word) and $\mathbf{g}$ is a polynomial with coefficients in K^∞.

Lemma 1.4.3 *Let* $\mathbf{g}$ *be a polynomial with coefficients in* K^∞. *Then*

$$\mathbf{g}(UD)U = U\mathbf{g}^{\downarrow}(DU), \tag{1.4.5}$$
$$\mathbf{g}^{\uparrow}(UD)U = U\mathbf{g}(DU), \tag{1.4.6}$$
$$\mathbf{g}(DU)D = D\mathbf{g}^{\uparrow}(UD), \tag{1.4.7}$$
$$\mathbf{g}^{\downarrow}(DU)D = D\mathbf{g}(UD). \tag{1.4.8}$$

Proof: This follows from (1.4.1) – (1.4.4). □

Each oriented graded graph (pair of graphs) G provides a representation of the graded algebra $\mathfrak{A}$ by operators in $KP = \oplus KP_n$. Namely, U and D are represented by the up and down operators. A sequence $\alpha = \{\alpha_n\} \in K^\infty$ is interpreted as a "block-scalar" operator defined by $\alpha x = \alpha_n x,\ x \in KP_n$.

If the operators representing U and D satisfy some of the conditions (1.3.1)–(1.3.6), a representation of a quotient algebra of $\mathfrak{A}$ arises. To facilitate further references, let us restate (1.3.1) and (1.3.3)–(1.3.6) in terms which are natural for $\mathfrak{A}$:

recurrent commutation relations:

$$DU = \mathbf{f}(UD) \quad \text{or} \quad D_{n+1}U_n = f_n(U_{n-1}D_n) \tag{1.4.9}$$

linear commutation relations:

$$DU = \mathbf{q}UD + \mathbf{r}I \quad \text{or} \quad D_{n+1}U_n = q_nU_{n-1}D_n + r_nI_n \tag{1.4.10}$$

$\mathbf{r}$-duality:

$$DU = UD + \mathbf{r}I \quad \text{or} \quad D_{n+1}U_n = U_{n-1}D_n + r_nI_n \tag{1.4.11}$$

r-duality:

$$DU = UD + rI \quad \text{or} \quad D_{n+1}U_n = U_{n-1}D_n + rI_n \tag{1.4.12}$$

[standard] duality:

$$DU = UD + I \quad \text{or} \quad D_{n+1}U_n = U_{n-1}D_n + I_n \tag{1.4.13}$$

Here $\mathbf{f} = \{f_n\}$ is a polynomial with coefficients in K^∞; $\mathbf{q} = \{q_n\}$ and $\mathbf{r} = \{r_n\}$ are elements of K^∞; $I = \{\ldots, 1, 1, 1, \ldots\}$ is an identity element of K^∞ (so $\mathbf{r}I = \mathbf{r}$). Note that $\mathbf{f}$ is well-defined provided the degrees of the f_n are bounded.

Theorem 1.4.4 *The quotient algebra of* $\mathfrak{A}_0$ *under commutation relation* (1.4.9) *is commutative.*

Proof: A balanced $\{U, D\}$-word $w = w_1 \cdots w_n$ is called a *below-word* if for any $m \in \{1, \ldots, n\}$

$$\#\{i \le m \,:\, w_i = U\} \;\ge\; \#\{i \le m \,:\, w_i = D\}$$

and therefore

$$\#\{i > m \,:\, w_i = U\} \;\le\; \#\{i > m \,:\, w_i = D\}.$$

This means that any w-path lies entirely below its starting point.

Multiple use of (1.4.9) allows us to express any balanced $\{U, D\}$-word as a linear combination of below-words with coefficients in K^∞. Thus it suffices to prove that below-words commute pairwise. This will be done by double induction on the lengths of the words being multiplied. The base (one of the words is empty or both are UD) is trivial. Every nonempty below-word either is a concatenation of two nonempty below-words or has the form UwD where w is a below-word also. If at least one of the words under multiplication is really a concatenation of two below-words then the claimed commutation follows from the induction hypothesis. Otherwise the words are Uw_1D and Uw_2D where w_1 and w_2 are also below-words. Hence

$$(Uw_1D)(Uw_2D) = Uw_1\mathbf{f}(UD)w_2D = Uw_2\mathbf{f}(UD)w_1D = (Uw_2D)(Uw_1D),$$

as desired. □

Theorem 1.4.4 generalizes R.P. Stanley's result [26, Corollary 4.11 (a)] for the case (1.4.12).

In the case of the linear commutation relations (1.4.10) the statement of Theorem 1.4.5 can be strengthened.

Theorem 1.4.5 *Assume the relation* (1.4.10) *is invertible, i.e., all the coefficients* q_i *are nonzero. Then any element of the quotient algebra of* $\mathfrak{A}_0$ *under* (1.4.10) *can be expressed as a polynomial in* UD *with coefficients in* K^∞.

Proof: Any element of the quotient algebra can now be transformed, by multiple application of (1.4.10), into a combination of the U^nD^n. Hence it remains to prove that U^nD^n can be expressed as $\mathbf{g}(UD)$. By induction on n and (1.4.6),

$$U^{n+1}D^{n+1} = U\mathbf{g}(UD)D = U\mathbf{g}(\mathbf{q}^{-1}(DU - \mathbf{r}I))D = \mathbf{g}^{\uparrow}(\mathbf{q}^{-1\uparrow}(UD - \mathbf{r}^{\uparrow}I))UD$$

where $\mathbf{q}^{-1} = \{q_i^{-1}\}$. □

Note that, generally (i.e., for an arbitrary commutation relation (1.4.9)), a balanced word does not have to have a representation in terms of the U^nD^n. For example, if (1.4.9) is $DU = UDUD$ then $DU = UDUD = UUDUDD = UUUDUDDD = \cdots$—and that is all one can do.

Theorem 1.4.5 extends [26, Lemma 4.10] that treats the case (1.4.12).

Now we proceed to derive explicit algebraic identities from (1.4.9)–(1.4.13).

Lemma 1.4.6 *Assume* (1.4.9) *holds. Then, for any polynomial* $\mathbf{g}$ *over* K^∞,

$$\begin{aligned} \mathbf{g}(UD)U &= U\mathbf{g}^{\downarrow}(\mathbf{f}(UD)) \\ D\mathbf{g}(UD) &= \mathbf{g}^{\downarrow}(\mathbf{f}(UD))D. \end{aligned}$$

Proof: This follows from (1.4.5), (1.4.8), and (1.4.9). □

Definition 1.4.7 Inductively define

$$\mathbf{f}^{\downarrow 0} = \mathbf{f}, \qquad \mathbf{f}^{\downarrow(k+1)} = (\mathbf{f}^{\downarrow k})^{\downarrow}; \tag{1.4.14}$$

$$\mathbf{f}^{\Downarrow 1} = \mathbf{f}, \qquad \mathbf{f}^{\Downarrow (k+1)} = (\mathbf{f}^{\Downarrow k})^{\downarrow} \circ \mathbf{f}; \tag{1.4.15}$$

where $\circ$ stands for composition. In other words, the nth components of, e.g., $\mathbf{f}^{\downarrow k}$ and $\mathbf{f}^{\Downarrow k}$ are

$$\begin{aligned} f_n^{\downarrow k} &= f_{n+k}, \\ f_n^{\Downarrow k} &= f_{n+k-1} \circ f_{n+k-2} \circ \cdots \circ f_{n+1} \circ f_n. \end{aligned} \tag{1.4.16}$$

Lemma 1.4.8 *Assume* (1.4.9) *holds. Then, for* $k = 1, 2, 3, \ldots,$

$$DU^k = U^{k-1}\mathbf{f}^{\Downarrow k}(UD), \tag{1.4.17}$$
$$D^kU = \mathbf{f}^{\Downarrow k}(UD)D^{k-1}, \tag{1.4.18}$$
$$D^kU^k = \prod_{s=1}^{k} \mathbf{f}^{\Downarrow s}(UD) \tag{1.4.19}$$

This last equation is well-defined since all the factors commute.

Proof: All these identities can be proved by induction on k using Lemma 1.4.6 and (1.4.6). For example, (1.4.17) is obtained as follows:

$$DU^{k+1} = U^{k-1}\mathbf{f}^{\Downarrow k}(UD)U = U^{k-1}U(\mathbf{f}^{\Downarrow k})^{\downarrow}(\mathbf{f}(UD)) = U^k\mathbf{f}^{\Downarrow (k+1)}(UD). \qquad \square$$

The equalities (1.4.17) – (1.4.19) are particular cases of the following general formula for D^nU^m.

Theorem 1.4.9 *Assume* (1.4.9) *holds. Then*

$$D^lU^{k+l} = U^k\mathbf{f}^{k,l}(UD) \tag{1.4.20}$$

and

$$D^{k+l}U^l = \mathbf{f}^{k,l}(UD)D^k$$

where

$$\mathbf{f}^{k,l} = \prod_{s=k+1}^{k+l} \mathbf{f}^{\Downarrow s} \tag{1.4.21}$$

for k and l nonnegative integers.

Proof: Induct on k. The base is (1.4.19). The induction step follows by Lemma 1.4.6 and (1.4.15). $\square$

In cases (1.4.10) – (1.4.13) the vector functions $\mathbf{f}^{\Downarrow s}$ and $\mathbf{f}^{k,l}$ can be computed explicitly:

$$\mathbf{f}^{\Downarrow s}(t) = \mathbf{q}^{\downarrow (s-1)}(\cdots(\mathbf{q}^{\downarrow}(\mathbf{q}t + \mathbf{r}) + \mathbf{r}^{\downarrow})\cdots) + \mathbf{r}^{\downarrow (s-1)}; \tag{1.4.22}$$

$$\mathbf{f}^{\Downarrow s}(t) = t + \mathbf{r} + \mathbf{r}^{\downarrow} + \cdots + \mathbf{r}^{\downarrow(s-1)}; \tag{1.4.23}$$
$$f^{\Downarrow s}(t) = t + sr; \tag{1.4.24}$$
$$f^{\Downarrow s}(t) = t + s. \tag{1.4.25}$$

Formula (1.4.22) can be rewritten as

$$\mathbf{f}^{\Downarrow s}(t) = \mathbf{a}^{(s)}t + \mathbf{b}^{(s)} \tag{1.4.26}$$

where

$$\mathbf{a}^{(s)} = \prod_{j=0}^{s-1} \mathbf{q}^{\downarrow j} \tag{1.4.27}$$
$$\mathbf{b}^{(s)} = \sum_{i=0}^{s-1} \mathbf{r}^{\downarrow i} \prod_{j=i+1}^{s-1} \mathbf{q}^{\downarrow j}. \tag{1.4.28}$$

Thus we can use (1.4.26) and (1.4.23) – (1.4.25) in (1.4.21) to obtain respective versions of Theorem 1.4.9 for the cases (1.4.10) – (1.4.13). To save space we only state the versions of (1.4.20).

Theorem 1.4.10 *Let k and l be nonnegative integers. If* (1.4.10) *holds then*

$$D^l U^{k+l} = U^k \prod_{s=k+1}^{k+l} (\mathbf{a}^{(s)} UD + \mathbf{b}^{(s)})$$

where $\mathbf{a}^{(s)}$ *and* $\mathbf{b}^{(s)}$ *are given by* (1.4.27) – (1.4.28).

In particular, for the special cases (1.4.11) – (1.4.13) *one has, respectively:*

$$D^l U^{k+l} = U^k \prod_{s=k+1}^{k+l} (UD + \mathbf{r} + \mathbf{r}^{\downarrow} + \cdots + \mathbf{r}^{\downarrow(s-1)}); \tag{1.4.29}$$
$$D^l U^{k+l} = U^k (UD + (k+1)r) \cdots (UD + (k+l)r); \tag{1.4.30}$$
$$D^l U^{k+l} = U^k (UD + k + 1) \cdots (UD + k + l). \tag{1.4.31}$$

□

Some special cases of (1.4.29) – (1.4.31) were given in [27, 26].

Similarly, one can derive formulae expressing $U^n D^m$ in terms of UD. However, there exists another approach. Note that (1.4.1) – (1.4.4) remain valid when interchanging U and D, $^{\uparrow}$ and $^{\downarrow}$. The relation (1.4.11) is invariant with respect to replacement of U, D, and $\mathbf{r}$ by D, U and $-\mathbf{r}$, respectively. Therefore we can make corresponding substitutions in (1.4.29), thus obtaining, under the condition (1.4.11), the identity

$$\begin{aligned} U^l D^{k+l} &= D^k \prod_{s=k+1}^{k+l} (DU - \mathbf{r} - \mathbf{r}^{\uparrow} - \cdots - \mathbf{r}^{\uparrow(s-1)}) \\ &= D^k \prod_{s=k+1}^{k+l} (UD - \mathbf{r}^{\uparrow} - \cdots - \mathbf{r}^{\uparrow(s-1)}) \end{aligned}$$

where $\mathbf{r}^{\uparrow j}$ is defined as in (1.4.14). In particular, (1.4.12) implies

$$U^l D^{k+l} = D^k(UD - k) \cdots (UD - (k + l - 1))$$

that reduces to [26, (47)–(48)] when $k = 0$.

Now we state some useful particular cases of Lemma 1.4.8.

Corollary 1.4.11 [26] *Assume* (1.4.12) *holds. Then, for any nonnegative integer k,*

(i) $DU^k = U^k D + krU^{k-1}$,

(ii) $D^k U^k = (UD + r) \cdots (UD + kr)$.

Proof: See (1.4.30). □

Corollary 1.4.12 [27] *Assume* (1.4.11) *holds. Then, for any nonnegative integer k,*

$$\begin{aligned} DU^k &= U^k D + U^{k-1}(\mathbf{r} + \mathbf{r}^{\downarrow} + \cdots + \mathbf{r}^{\downarrow(k-1)}); \\ D^k U^k &= (UD + \mathbf{r})(UD + \mathbf{r} + \mathbf{r}^{\downarrow}) \ldots (UD + \mathbf{r} + \mathbf{r}^{\downarrow} + \cdots + \mathbf{r}^{\downarrow(k-1)}). \end{aligned} \tag{1.4.32}$$

Proof: See (1.4.29). □

1.5. Path enumeration

Each of the results of Section 1.4 can be regarded as an enumerative formula concerning path counting, keeping in mind the natural representation of the algebra $\mathfrak{A}$ in the space KP. Recall that, given a $\{U, D\}$-word w and vertices x and y, the coefficient of y in the expansion of wx is the number of w-paths from x to y (see Definition 1.2.2).

Consider the identity (1.4.32) as a typical example. Let x be a vertex of rank n, and y a vertex of rank $n + k - 1$. Applying both sides of (1.4.32) to x and equating the coefficients of y gives

$$\sum_{z \in P_{n+k}} e(x \to z \to y) = \sum_{z \in P_{n-1}} e(x \to z \to y) + e(x \to y) \sum_{s=n}^{n+k-1} r_n \tag{1.5.1}$$

since $x \in P_n$ and

$$(\mathbf{r} + \mathbf{r}^{\downarrow} + \cdots + \mathbf{r}^{\downarrow(k-1)})x = (r_n + r_{n+1} + \cdots + r_{n+k-1})x.$$

So the following enumerative theorem is obtained.

Theorem 1.5.1 *Assume the up and down operators in oriented graded graph G satisfy the relation* (1.4.11). *Then for any $x \in P_n$ and $y \in P_{n+k-1}$ the identity* (1.5.1) holds.

Most combinatorial reformulations of the identities of Section 1.4 are cumbersome. Fortunately, they get much simpler in case of graphs with zero. The reason is that $D\widehat{0} = 0$ and so

$$\mathbf{g}(UD)\widehat{0} = \mathbf{g}(0)\widehat{0} \tag{1.5.2}$$

for any polynomial $\mathbf{g}$ over K^∞. Thus, e.g., (1.4.20) implies

$$D^l U^{k+l}\widehat{0} = U^k \mathbf{f}^{k,l}(0)\widehat{0}. \tag{1.5.3}$$

The zero-rank component of the sequence $\mathbf{f}^{k,l}(0) \in K^\infty$ is only relevant in (1.5.3) since we apply it to $\widehat{0}$. So let us write the expression for $f_0^{k,l}(0)$ more explicitly. From (1.4.21) and (1.4.16) one has

$$f_0^{k,l} = \prod_{s=k+1}^{k+l} f_0^{\Downarrow s} = \prod_{s=k+1}^{k+l} f_{s-1} \circ f_{s-2} \circ \cdots \circ f_0$$

and then

$$f_0^{k,l}(0) = \prod_{s=k+1}^{k+l} f_{s-1}(f_{s-2}(\cdots(f_1(f_0(0)))\cdots)). \tag{1.5.4}$$

Combining together (1.5.3) – (1.5.4) results in the following theorem.

Theorem 1.5.2 *Assume the up and down operators in an oriented graded graph G with zero satisfy* (1.4.9). *Then, for any $x \in P_k$,*

$$\sum_{y \in P_{k+l}} e(\widehat{0} \to y \to x) = e(x) \prod_{s=k+1}^{k+l} f_{s-1}(f_{s-2}(\cdots(f_1(f_0(0)))\cdots)). \tag{1.5.5}$$

In particular, the conditions (1.4.10) – (1.4.12) *imply, respectively:*

$$\sum_{y \in P_{k+l}} e(\widehat{0} \to y \to x) = e(x) \prod_{s=k+1}^{k+l} \sum_{i=0}^{s-1} r_i \prod_{j=i+1}^{s-1} q_j \tag{1.5.6}$$

$$\sum_{y \in P_{k+l}} e(\widehat{0} \to y \to x) = e(x) \prod_{s=k+1}^{k+l} \sum_{i=0}^{s-1} r_i \tag{1.5.7}$$

$$\sum_{y \in P_{k+l}} e(\widehat{0} \to y \to x) = e(x)\, r^l (k+l)!/k! \tag{1.5.8}$$

Note that, in the above formulas, the notation $e(x)$ refers to paths in $G_1 = (P, \rho, E_1)$ (see Definition 1.2.1) since one is only allowed to move up along the E_1-arcs. Similarly,

$e(\widehat{0} \to y \to x)$ denotes the number of paths going in G_1 from $\widehat{0}$ to y and then in G_2 from y to x.

In the special case $l = 1$ and $G_1 = G_2$ the formulae (1.5.7)–(1.5.8) were given in [26,27]. Let us also state the case $l = 1$ separately, as it is the most important case for the applications.

Corollary 1.5.3 *Assume G has a zero,* (1.4.9) *holds, and G_2 has no multiple edges. Then, for any $x \in P_k$,*

$$\sum_{y \text{ covers } x \text{ in } G_2} e(y) = f_k(f_{k-1}(\cdots(f_1(f_0(0)))\cdots))e(x).$$

In particular, (1.4.11) *and* (1.4.12) *imply, respectively,*

$$\sum_{y \text{ covers } x \text{ in } G_2} e(y) = (r_0 + r_1 + \cdots + r_k)e(x);$$

$$\sum_{y \text{ covers } x \text{ in } G_2} e(y) = r(k+1)e(x).$$

Another interesting special case of Theorem 1.5.2 is $x = \widehat{0}$. It corresponds to counting loops of length $2l$.

Corollary 1.5.4 *Assume the condition* (1.4.9) *holds. Then, for any nonnegative l,*

$$\alpha(0 \to l \to 0) = \prod_{s=1}^{l} f_{s-1}(f_{s-2}(\cdots(f_1(f_0(0)))\cdots)).$$

In particular, (1.4.11) *and* (1.4.12) *imply, respectively,*

$$\alpha(0 \to l \to 0) = \prod_{s=1}^{l} \sum_{i=0}^{s-1} r_i;$$

$$\alpha(0 \to l \to 0) = r^l l!. \tag{1.5.9}$$

□

The last two identities were proved by R.P. Stanley [27, 26] in the case $G_1 = G_2$. For the Young lattice, (1.5.9) becomes the classical identity (1.1.4). Note that the left-hand side in (1.5.9) can be rewritten as

$$\sum_{y \in P_l} e_1(y)e_2(y)$$

where $e_1(y)$ and $e_2(y)$ denote the numbers of paths (saturated chains) connecting $\widehat{0}$ and y in G_1 and G_2, respectively.

Formulae similar to Theorems 1.4.9–1.4.10 and hence to Theorem 1.5.2 (though perhaps much more complicated) can be obtained for *any* $\{U, D\}$-word (cf. [26]). The essence of results of this type can be stated as follows.

Proposition 1.5.5 *Every element of the quotient algebra of* $\mathfrak{A}$ *by* (1.4.9) *can be expressed as a linear combination of terms of the form* $U^n\mathbf{g}(UD)$ *or* $\mathbf{g}(UD)D^n$ *where* $\mathbf{g}$ *is a polynomial over* K^∞. *Explicit formulae can be written in terms of* $\mathbf{f}$.

Proposition 1.5.6 *Let* w *be a* $\{U, D\}$*-word. Assume* G *is a graph with zero satisfying* (1.4.9). *Then the number of* w*-paths joining* $\widehat{0}$ *with an arbitrary vertex* x *of corresponding rank is proportional to* $e(x)$. *The coefficient depends only on* w *but not on* x. *It can be effectively computed from* $\mathbf{f}$.

Level sums. Certain enumerative formulae can be derived from the fact that, in addition to (1.4.9) – (1.4.13), some graphs satisfy linear-algebraic conditions involving the vectors

$$\mathbf{P}_n = \sum_{x \in P_n} x$$

or

$$\mathbf{P} = \sum_{x \in P} x = \sum_n \mathbf{P}_n$$

(cf. [26]). In particular, it is easy to verify that in the self-dual case one has, under (1.4.10),

$$q_n U_{n-1}\mathbf{P}_{n-1} + r_n \mathbf{P}_n = D_{n+1}\mathbf{P}_{n+1}.$$

Summing this equation over n we obtain, by virtue of (1.4.1),

$$(\mathbf{q}U + \mathbf{r} - D)\mathbf{P} = 0.$$

Multiply by D^n and use (1.4.2) and (1.4.18) to get

$$(\mathbf{q}^{\downarrow n}\mathbf{f}^{\Downarrow n}(UD)D^{n-1} + \mathbf{r}^{\downarrow n}D^n - D^{n+1})\mathbf{P} = 0. \tag{1.5.10}$$

Now suppose G has a zero and observe that the coefficient of $\widehat{0}$ in $D^n\mathbf{P}$ is just $\sigma_n = \alpha(0 \to n)$. Extract the coefficients of $\widehat{0}$ from (1.5.10) to obtain

$$q_n f_0^{\Downarrow n}(0)\sigma_{n-1} + r_n\sigma_n - \sigma_{n+1} = 0.$$

Recall that in the case under consideration $\mathbf{f}^{\Downarrow n}$ is given by (1.4.26) – (1.4.28) to get to the following result.

Corollary 1.5.7 *Assume* G *is a non-oriented graded graph (i.e.,* $G_1 = G_2$*) with zero, and* (1.4.10) *holds. Then the values* $\sigma_n = \alpha(0 \to n)$ *satisfy the following recurrence relation:*

$$\sigma_{n+1} = r_n\sigma_n + \left(\sum_{i=0}^{n-1} r_i \prod_{j=i+1}^{n} q_j\right)\sigma_{n-1}. \tag{1.5.11}$$

In case of r-differential posets, i.e., when $q_j = 1, r_i = r$, (1.5.11) reduces to the well known formula (cf. [26, (12)])

$$\sigma_{n+1} = r(\sigma_n + n\sigma_{n-1}).$$

1.6. *Characteristic polynomials*

The results of Section 1.4 show that the structure of representations of the graded algebras under consideration (namely, the quotients of $\mathfrak{A}_0$ and $\mathfrak{A}$ by (1.4.9)) is strongly dependent on the structure of an operator representing the word UD. This operator is a direct sum of operators $U_{n-1}D_n$ acting on invariant spaces KP_n. So our next step will be the computation of the characteristic polynomial of $U_{n-1}D_n$ (cf. [26, Section 4]).

Lemma 1.6.1 *Assume the up and down operators U and D in an oriented graded graph G satisfy the relation* (1.4.9). *If $D_nU_{n-1}x = \lambda x$ then*

$$U_{n-1}D_n(U_{n-1}x) = \lambda(U_{n-1}x);$$
$$D_{n+1}U_n(U_{n-1}x) = f_n(\lambda)U_{n-1}x.$$

Proof: The first implication is obvious; the second follows from (1.4.9). □

We see that applying U to the eigenvectors of D_nU_{n-1} results in eigenvectors of $D_{n+1}U_n$, though not all of them, in general.

Comments 1.6.2

1. It follows immediately from Lemma 1.6.1 that in a graph with zero, $U^n\hat{0}$ is an eigenvector of $D_{n+1}U_n$. This can also be seen as a special case $l = 1$ of Theorem 1.5.2.

2. When the conditions (1.4.9) are *invertible* (e.g., one has (1.4.10) with nonzero q_n), a corresponding version of Lemma 1.6.1 is valid:

$$D_{n+1}U_nx = \lambda x \text{ implies } D_nU_{n-1}(D_nx) = f_n^{-1}(\lambda)D_nx;$$

   i.e., applying D to the eigenvectors of $D_{n+1}U_n$ gives eigenvectors of D_nU_{n-1}—though maybe including zero ones.

3. Similar results hold for the operators $U_{n-1}D_n$.

Following [26], let $\mathrm{Ch}(M, \lambda)$ denote the characteristic polynomial of an operator or matrix M, normalized to be monic.

Lemma 1.6.3 [31, Ch. 1, Sec. 51] *Let $A : V \to W$ and $B : W \to V$ be linear homomorphisms, with dim $V = m$, dim $W = n$. Then $Ch(AB, \lambda) = \lambda^{n-m}Ch(BA, \lambda)$.*

Let

$$p_n = \#P_n, \quad \Delta_n = p_n - p_{n-1}. \tag{1.6.1}$$

Corollary 1.6.4 $Ch(U_{n-1}D_n, \lambda) = \lambda^{\Delta_n}Ch(D_nU_{n-1}, \lambda)$.

Note that (1.4.9) implies that the roots of $Ch(D_{n+1}U_n, \lambda)$ are the f_n-images of the roots of $Ch(U_{n-1}D_n, \lambda)$, with the same multiplicities.

The following theorem generalizes [26, Theorem 4.1].

Theorem 1.6.5 *Assume the up and down operators in an oriented graded graph G with zero satisfy* (1.4.9). *Suppose all the Δ_n are nonnegative. Then the characteristic polynomial $Ch(D_{n+1}U_n, \lambda)$ has the roots*

$$\lambda_{nk} = f_n(f_{n-1}(\dots(f_k(0))\dots)) \tag{1.6.2}$$

where $k = 0, 1, \dots, n$. The multiplicity of λ_{nk} is Δ_k.

The polynomial $Ch(U_{n-1}D_n, \lambda)$ has the roots $\lambda_{n-1,k}$ with the same multiplicities (including, by definition, $\lambda_{n-1,n} = 0$, with multiplicity Δ_n).

Note that, in terms of (1.4.16), $\lambda_{nk} = f_k^{\Downarrow(n-k+1)}(0)$.

Proof: Induct on n. The base is trivial. If

$$\mathrm{Ch}(D_nU_{n-1}, \lambda) = \prod_{k=0}^{n-1}(\lambda - \lambda_{n-1,k})^{\Delta_k}$$

then, by Corollary 1.6.4,

$$\mathrm{Ch}(U_{n-1}D_n, \lambda) = \lambda^{\Delta_n}\prod_{k=0}^{n-1}(\lambda - \lambda_{n-1,k})^{\Delta_k}$$

and therefore

$$\mathrm{Ch}(D_{n+1}U_n, \lambda) = (\lambda - f_n(0))^{\Delta_n}\prod_{k=0}^{n-1}(\lambda - f_n(\lambda_{n-1,k}))^{\Delta_k} = \prod_{k=0}^{n}(\lambda - \lambda_{n,k})^{\Delta_k}. \square$$

A calculation of characteristic polynomials of operators $D_{n+1}U_n$ and $U_{n-1}D_n$ can be carried out even in the absence of the restriction $\Delta_k \geq 0$ — though the latter usually holds in well known examples. Corresponding formulations are omitted.

The structure of up and down operators becomes perfectly clear in the diagonalizable case, e.g., when $U^* = D$ (equivalently, $G_1 = G_2$) and so $U_{n-1}D_n$ and $D_{n+1}U_n$ are self-adjoint. The condition $U^* = D$ is not essential. Some simple assumptions on the functions $\{f_n\}$ allow us to diagonalize the operators $U_{n-1}D_n$ and $D_{n+1}U_n$. Then we are able to describe the structure of U and D.

Theorem 1.6.6 *Assume the up and down operators in an oriented graded graph G with zero satisfy the relations* (1.4.9). *Suppose all the values λ_{nk} given by* (1.6.2) *are nonzero.*

Then there exist bases $\{v_{nj}\}$ *of the spaces* KP_n *such that*

$$\begin{aligned} Uv_{nj} &= v_{n+1,j} && \text{for}\quad j = 1, \ldots, p_n; && (1.6.3)\\ Dv_{n+1,j} &= \lambda_{nk} v_{nj} && \text{for}\quad j = 1, \ldots, p_n \quad \text{and } k \text{ depending on } j; && (1.6.4)\\ Dv_{n+1,j} &= 0 && \text{for}\quad j = p_n + 1, \ldots, p_{n+1}. && (1.6.5) \end{aligned}$$

Comments 1.6.7

1. The vectors v_{nj} are the eigenvectors of UD and DU.
2. The condition $\lambda_{nk} \neq 0$ is satisfied, e.g., if every f_n is positive on $[0, +\infty)$. For example, it is the case when $f_n(t) = q_n t + r_n$ (see (1.4.10)) with $q_n \geq 0$, $r_n > 0$. Stanley considers the special case $q_n = 1$, $r_n = r$, $U^* = D$ in [26, Section 4].
3. Regard G as a representation of the quotient algebra $\mathfrak{A}$/(1.4.9). Then Theorem 1.6.6 claims that such a representation is, in a sense, a direct sum of one-dimensional representations in spaces isomorphic to K^∞ (with a shifted grading).

Proof of Theorem 1.6.6: Bases $\{v_{nj}\}$ are constructed recursively, starting from $n = 0$. Set $v_{01} = \widehat{0}$ to form a basis for KP_0.

Assume $\{v_{mj}\}$ are defined for all $m \leq n$, and corresponding conditions (1.6.3)–(1.6.5) are satisfied. Then the operator $U_{n-1}D_n$ has eigenvectors v_{nj} and eigenvalues $\lambda_{n-1,k}$ including $\lambda_{n-1,n} = 0$ for $j = p_{n-1} + 1, \ldots, p_n$. By (1.4.9) and (1.6.2), $D_{n+1}U_n$ has the same eigenvectors and *nonzero* eigenvalues λ_{nk}. The vectors $U_n v_{nj}$ are linearly independent since the homomorphism D_{n+1} sends them to linear-independent vectors $\lambda_{nk} v_{nj}$. Set $v_{n+1,j} = U_n v_{nj}$ for $j = 1, \ldots, p_n$. Then (1.6.3) and (1.6.4) are guaranteed. Besides we have proved $p_{n+1} \geq p_n$, extending [26, Corollary 4.3]. Hence, by Corollary 1.6.4, the space KP_{n+1} falls into a direct sum of the image $U(KP_n)$ and an invariant space L of a dimension $\Delta_{n+1} = p_{n+1} - p_n$ that belongs to the zero eigenvalue of $U_n D_{n+1}$. Suppose $D_{n+1}y \neq 0$ for some $y \in L$. Then $U_n D_{n+1} y \neq 0$ since all λ_{nk} are nonzero. On the other hand, $U_n D_{n+1} y \in U(KP_n)$ and $U_n D_{n+1} y \in UD(L) = L$ (contradiction). Hence $L \subset Ker D_{n+1}$ (in fact, $L = Ker D_{n+1}$) and the system $\{v_{n+1,j}\}_{j=1}^{p_n}$ can be extended to a basis of KP_{n+1} by adding vectors satisfying (1.6.5). □

Theorem 1.6.6 provides effective techniques for computing vectors wx where w is a $\{U, D\}$-word and $x \in P_n$. (Recall that the y-component of wx is the number of w-paths from x to y). Calculations can be carried out as follows. Convert w into a standard form in the spirit of Theorem 1.4.9 (cf. Proposition 1.5.11). Expand x in the basis v_{nj}, i.e., in the UD eigenvectors, and perform the computation.

Example 1.6.8 *The Young graph.* In the Young graph (see, e.g., [25]) $f_n(t) = t + 1$, and the conditions of Theorem 1.6.6 hold. From (1.6.2) one has $\lambda_{nk} = n - k + 1$. Simple calculations give

$$v_{01} = \emptyset, \quad v_{11} = \langle 1 \rangle, \quad v_{21} = \langle 2 \rangle + \langle 1^2 \rangle, \quad v_{22} = \langle 2 \rangle - \langle 1^2 \rangle,$$

where $\langle 1 \rangle$, $\langle 2 \rangle$, and $\langle 1^2 \rangle$ are the Young diagrams corresponding to the respective partitions. Then, for example,

$$\begin{aligned}
D^n U^n v_{21} &= \left(\prod_{m=2}^{n+1} \lambda_{m0} \right) v_{21} = \tfrac{1}{2}(n+2)! v_{21}, \\
D^n U^n v_{22} &= \left(\prod_{m=2}^{n+1} \lambda_{m2} \right) v_{22} = n! v_{22}, \\
D^n U^n \langle 2 \rangle &= \tfrac{1}{4}(n+2)!(\langle 2 \rangle + \langle 1^2 \rangle) + \tfrac{1}{2} n!(\langle 2 \rangle - \langle 1^2 \rangle) \\
&= \tfrac{1}{4} n!((n^2 + 3n + 4)\langle 2 \rangle + (n^2 + 3n)\langle 1^2 \rangle).
\end{aligned}$$

In other words,

$$\sum_{x \in P_{n+2}} e(\langle 2 \rangle \to x \to \langle 2 \rangle) = \tfrac{1}{4}(n^2 + 3n + 4) n! \tag{1.6.6}$$

$$\sum_{x \in P_{n+2}} e(\langle 2 \rangle \to x \to \langle 1^2 \rangle = \tfrac{1}{4}(n^2 + 3n) n! \tag{1.6.7}$$

(Combining (1.6.6) and (1.6.7) results, of course, in $\frac{1}{2}(n+2)!$; cf. (1.1.4).) Now restate, e.g., (1.6.6) in terms of tableaux. Let $e_{\langle 2 \rangle}(x)$ denote the number of standard Young tableaux of shape x and with the "2" placed in the first row. Then

$$\sum_{x \in P_n} (e_{\langle 2 \rangle}(x))^2 = \tfrac{1}{4}(n^2 - n + 2)(n-2)! \tag{1.6.8}$$

A formula similar to (1.6.8) can be obtained for the number of paths in the Young graph which have any fixed structure and join any two fixed vertices. The simplest result is the following analogue of (1.6.6) – (1.6.7).

Theorem 1.6.9 *Let x and y be two Young diagrams each containing s boxes. Then for $n = 0, 1, 2, \ldots$*

$$\sum_{z \in P_{n+s}} e(x \to z \to y) = R_{xy}(n)\, n!$$

where R_{xy} is a polynomial (in n) of degree s with rational coefficients depending on x and y only. There exists an effective algorithm computing a polynomial R_{xy} from diagrams x and y. □

Results of this type can also be obtained for other graded graphs (pairs of graphs) satisfying the conditions of Theorem 1.6.6.

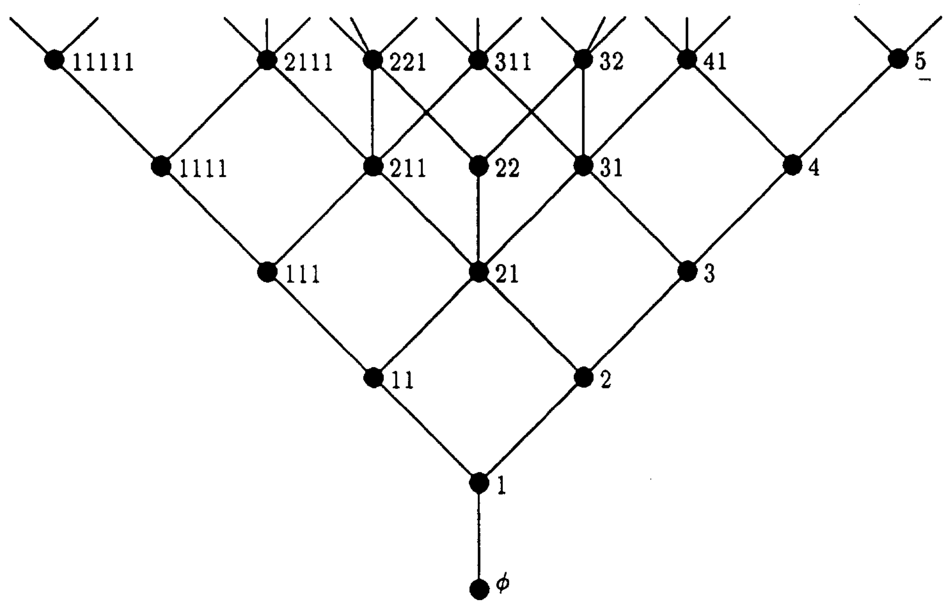


Figure 1. The Young graph.

2. Dual graphs: examples

In this chapter various examples of dual (r-dual, etc.) graded graphs are given. Section 2.8 contains an index of these examples. Enumerative applications of the results of Chapter 1 are given in Section 2.9.

2.1. Self-dual graphs

Definition 2.1.1 Let Q be a countable poset. $J(Q)$ will denote the set (the distributive lattice) of finite order ideals of Q (cf. [25, Section 3.1]). The lattice $J(Q)$ is a graded graph with zero; the rank of an ideal is its cardinality.

Example 2.1.2 *The Young graph* [1, 25, etc.]. Let

$$\mathbb{P}^2 = \{(i,j) : i > 0,\ j > 0;\ i,j \in \mathbb{Z}\}$$

be the two-dimensional integral quadrant with the usual (i.e., coordinatewise) partial order. The graph $\mathbb{Y} = J(\mathbb{P}^2)$ is called the *Young graph/lattice*. Its vertices are the Young diagrams, or Ferrers shapes; see Figure 1 where each Ferrers shape is represented by the sequence of its row lengths. The numbers $e(x)$ are the dimensions of irreducible representations of symmetric groups S_n (see, e.g., [22, Section 17]). The most well known identities involving $e(x)$ are (1.1.3) and

$$\alpha(0 \to n) = \sum_{x \in P_n} e(x) = \text{number of involutions in } S_n. \tag{2.1.1}$$

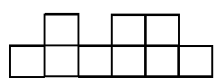
The Young graph is a *self-dual* graph or *differential poset* [26]. This means that its up and down operators U and $D = U^*$ satisfy (1.4.13); combinatorially,

(1) for any two shapes x and $y \neq x$ the number of shapes covered in the Young lattice by both x and y equals the number of shapes covering both x and y;
(2) for any shape x the number of shapes covered by x is exactly one less than the number of shapes covering x.

The first property is, of course, valid for every modular lattice.

Theorem 2.1.3 [26, Prop. 5.5] *The Young graph is the only distributive lattice among self-dual graded graphs with zero.*

Example 2.1.4 *The Young-Fibonacci graph* [5, 26]. Let $\{1,2\}^*$ denote the set of all the words in the alphabet $\{1,2\}$; we shall call such words *snakes*. A snake can be pictured as a shape; for example, 121221 becomes



A rank of a snake is the number of boxes in its shape; equivalently, rank is the sum of the entries of the corresponding $\{1,2\}$-word.

The Young-Fibonacci graph $\mathbb{YF}$ (see Figure 2) is defined as follows:

(1) the set of vertices is $\{1,2\}^*$;
(2) w' covers w if and only if $w' = 1w$ (concatenation) or $w' = 2v$ where w covers v.

We have already mentioned that (1.1.4) is known to hold for the Young-Fibonacci graph. This graph is self-dual since

(1) $\mathbb{YF}$ is a modular lattice [5,26];
(2) every vertex of $\mathbb{YF}$ has one more successor than predecessor.

The relation (1.4.13) can also be derived directly from the following alternative definition of this graph. Define the adjacency matrices between successive ranks, U_n, recursively by

$$U_0 = (1), \quad U_n = \begin{pmatrix} U_{n-1}^T \\ I_n \end{pmatrix} \tag{2.1.2}$$

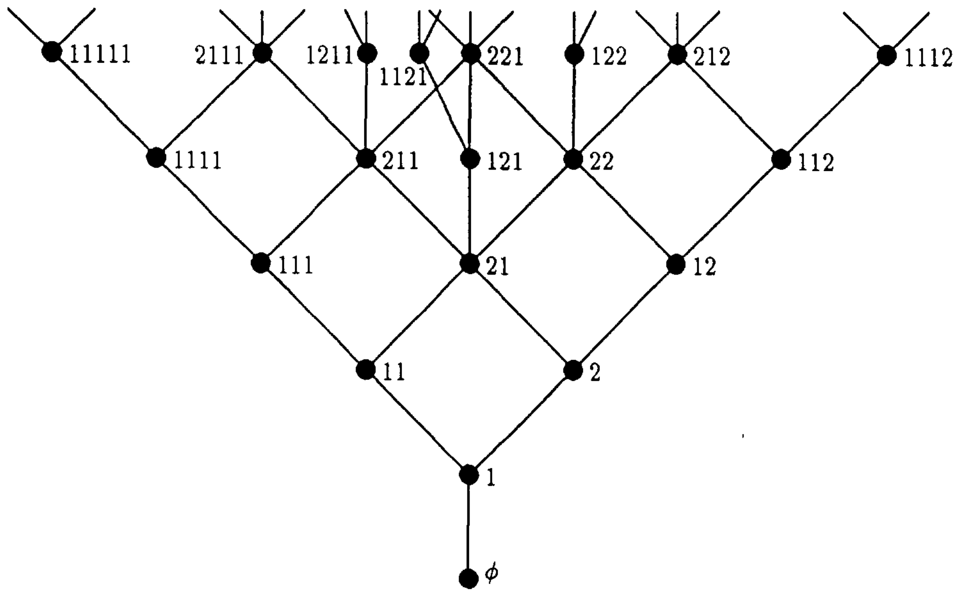


Figure 2. The Young-Fibonacci graph.

where ^{T} stands for transposition, and I_n is an identity matrix of appropriate size. Thus

$$U_0 = (1), \quad U_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 11 \\ 10 \\ 01 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 110 \\ 101 \\ 100 \\ 010 \\ 001 \end{pmatrix}$$

and so on. It is easy to see that this definition is equivalent to the original one. On the other hand, (2.1.2) immediately implies

$$U_n^T U_n = U_{n-1} U_{n-1}^T + I_n$$

which coincides with (1.4.13).

One can also construct other examples of self-dual graphs. However, they seem to be somewhat artificial. At the same time, there are a number of interesting examples of dual graphs with $G_2 \neq G_1$ as we will see.

2.2. *Weighted distributive lattices*

Assume $G_1 = (P, \rho, E_1)$ is a distributive lattice: $P = J(Q)$. Let us try to construct an r-dual graph G_2 as follows. Let $w: Q \to K$ be a weight function where K is our field. For any edge $(x, y) \in E_1$ take the element $q \in y$ such that $q \notin x$ and define $w(q)$ to be the multiplicity of the edge (x, y) in G_2. Thus G_2 is a *weighted distributive lattice.* (Strictly speaking, G_2 is not a graph but a graded network.)

We want G_1 and G_2 to satisfy (1.4.12) or, equivalently,

$$D_{n+1}U_n - U_{n-1}D_n = rI_n. \tag{2.2.1}$$

Note that whichever weight function w you choose the left-hand matrix in (2.2.1) is *diagonal.* Thus we only need the equality of diagonal elements on both sides of (2.2.1). Combinatorially, this means the following (cf. [27, Proposition 3.1]): for any finite order ideal x in Q, the total weight of the elements $q \in Q$ which can be added to x (i.e., such that $x \cup \{q\}$ is also an ideal) is r more than the total weight of the elements $q \in Q$ able to be deleted from x.

Thus a weight function $w: Q \to K$ defines a weighted distributive lattice G_2 which is dual to $G_1 = J(Q)$ if and only if the condition

$$\sum_{\substack{q \in Q \setminus x \\ x \cup \{q\} \in P}} w(q) = r + \sum_{\substack{q \in x \\ x \setminus \{q\} \in P}} w(q). \tag{2.2.2}$$

holds for any vertex x of G_1 and G_2.

The condition (2.2.2) is very restrictive. However, we can give a number of examples where it is satisfied. The Young graph is the first one, with $r = 1$ and $w = 1$ on $Q = \mathbb{P}^2$.

Example 2.2.1 *The chain* $\mathbb{N} = \{0, 1, 2, \ldots\}$. This graph can be treated as the lattice of ideals of $\mathbb{P} = \{1, 2, 3, \ldots\}$ with the usual ordering. The only weight function w on $\mathbb{P}$ satisfying (2.2.2) with $r = 1$ is $w(q) = q$. Thus the graph dual to $\mathbb{N}$ is the same chain having its nth edge of multiplicity n; see Figure 3.

Another interpretation of this example is the following. Consider the vector space of polynomials in one variable t. Define U to be the multiplication by t and D to be the derivation $\frac{d}{dt}$. This approach can be applied to any algebra with derivation; see Section 2.3.

Example 2.2.2 *Pascal graphs.* Let $Q = r\mathbb{P}$, i.e., Q is a disjoint union of r copies of $\mathbb{P}$. Then $\mathbb{N}^r = J(Q)$ is an r-dimensional *Pascal graph.* Properly speaking, $\mathbb{N}^r$ is the lattice of r-dimensional vectors with nonnegative integer coordinates; see Figure 4. The numbers $e(x)$ are the ordinary r-nomial coefficients.

To construct an r-dual graph for $\mathbb{N}^r$ (the coincidence of these two r's is not accidental) let us make the following general observation. Assume $Q = Q_1 \cup Q_2 \cup \cdots$ is a finite disjoint union of posets. Let w_i be a weight function on Q_i satisfying (2.2.2) with $r = 1$. Then the function $w: Q \to K$ defined by

$$w(q) = \alpha_i w_i(q) \quad \text{if} \quad q \in Q_i \tag{2.2.3}$$

satisfies (2.2.2) provided $\sum \alpha_i = r$.

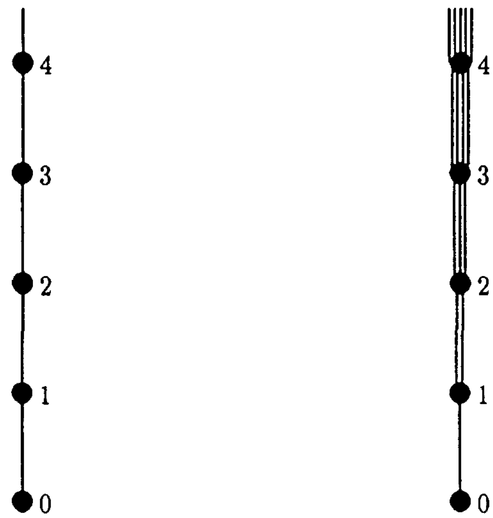


Figure 3. The chain.

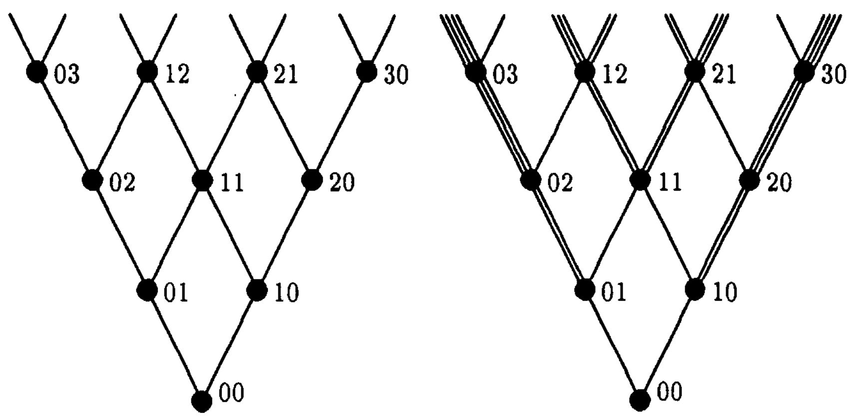


Figure 4. The 2-dimensional Pascal graph.

In case of the Pascal graph $\mathbb{N}^r$ each Q_i is isomorphic to $\mathbb{P}$; so the w_i can be taken from Example 2.2.1. The most natural option for α_i is $\alpha_i = 1$; thus the r in $\mathbb{N}^r$ and in (1.4.12) are the same. As a result, a weight function $w\colon \mathbb{P} \to \mathbb{Z}$ defined by

$$w(q^{(i)}) = q \tag{2.2.4}$$

is obtained where $q^{(i)}$ denotes the element q taken from the ith component of $r\mathbb{P}$. Thus the multiplicity of an edge

$$((x_1,\ldots,x_r),(x_1,\ldots,x_i+1,\ldots,x_r)) \tag{2.2.5}$$

in the r-dual graph for $\mathbb{N}^r$ is x_i+1; see Figure 4. This graph is an rth cartesian power of the dual graph for $\mathbb{N}$ (cf. Example 2.2.1). This is a special case of the following observation.

Lemma 2.2.3 *Assume graphs G_1 and G_2 are r-dual, and H_1 and H_2 are s-dual. Then $G_1 \times H_1$ and $G_2 \times H_2$ are $(r+s)$-dual.*

By means of Lemma 2.2.3, one can easily construct new pairs of r-dual graphs from old ones. For example, the graph $\mathbb{Y}^r$, the rth cartesian power of the Young graph, is self-r-dual, as is $\mathbb{YF}^r$. See [9] for other examples.

Example 2.2.4 *Bernoulli networks.* These are graded networks obtained by assigning weights to the edges of the Pascal graph $\mathbb{N}^r$, namely, an edge (2.2.5) has weight $\alpha_i \geq 0$ where $\sum_{i=1}^r \alpha_i = 1$. Then the values $e(x)$ are the probability weights of the corresponding multinomial distribution. In other words,

$$e(x) = \mathrm{Prob}\{\xi(n) = x\}$$

where $\xi(n)$ is an r-dimensional Bernoulli-type Markov process with transition probabilities α_i. Details are omitted.

Example 2.2.5 *Diagrams with $\leq r$ rows.* Let Q be a direct product of the infinite chain $\mathbb{P}$ and a finite chain $[r] = \{1,\ldots,r\}$. Then $J(Q) = J(\mathbb{P}\times[r])$ is the distributive lattice of Young diagrams containing $\leq r$ rows. It is a sublattice and an order ideal of Young's lattice. So the values $e(x)$ are the same as in $\mathbb{Y}$.

In a particular case $r=2$ the lattice $J(\mathbb{P}\times[2])$ is known as the *SemiPascal* graph [14]; see Figure 5. The numbers $e(x)$ for the vertices x of *SemiPascal* lying in the main diagonal (i.e., for the $(2\times n)$-rectangular diagrams) are the *Catalan numbers* $\frac{1}{n+1}\binom{2n}{n}$ (cf. [30], Section 3.1]).

Lemma 2.2.6 *The weight function $w\colon \mathbb{P}\times[r]\to K$ defined by*

$$w((q_1,q_2)) = r + q_1 - q_2 \tag{2.2.6}$$

satisfies (2.2.2). □

Thus we have constructed a weighted lattice that is r-dual to $J(\mathbb{P}\times[r])$. Note that the r involved in $\mathbb{P}\times[r]$, (2.2.6), and (2.2.2) do coincide.

Example 2.2.7 *The Young graph: non-standard weights.* Unexpectedly, in the case of the Young graph the constant weight function on $\mathbb{P}^2$ is not the only possible one. The general form of a weight function on $\mathbb{P}^2$ obeying (2.2.2) is

$$w((q_1,q_2)) = r + \alpha(q_1 - q_2)$$

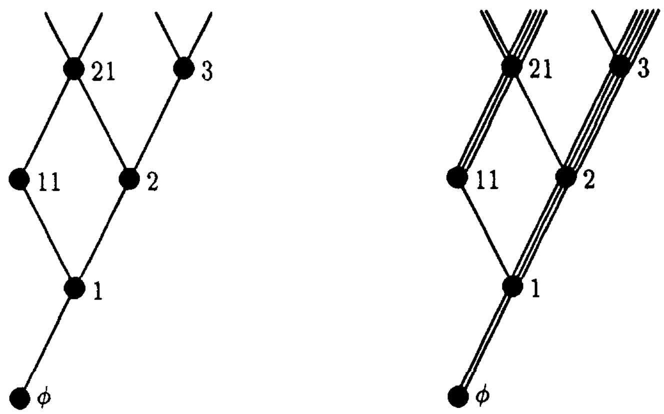


Figure 5. SemiPascal.

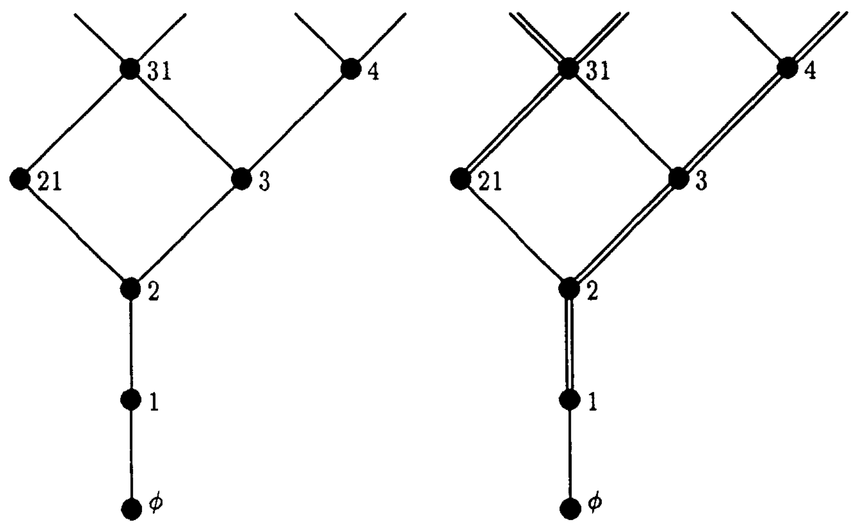


Figure 6. The graph of shifted shapes.

where α is an arbitrary constant. However, $w(q) = r$ is the only *nonnegative* suitable weight function.

Example 2.2.8 *Shifted shapes* [17, 32, 27]. Let $Q = SemiPascal$ (see Example 2.2.5). The graph $\mathbb{SY} = J(Q)$ is the *graph of shifted shapes* which are the Young diagrams with *strictly* decreasing row lengths; see Figure 6. Since $\mathbb{SY}$ is *not* an order ideal of Young's lattice, the values $e(x)$ in $\mathbb{SY}$ differ from those in $\mathbb{Y}$. The main identity involving the $e(x)$'s in $\mathbb{SY}$ is the following [19, 16]:

$$\sum_{x\in P_n} e(x)^2 2^{n-h(x)} = n! \tag{2.2.7}$$

where $h(x)$ is the *height* (= number of rows) of the shape x.

The only weight function on *SemiPascal* satisfying (2.2.2) with $r = 1$ is

$$w(q) = \begin{cases} 1, & \text{if } q \text{ is a diagonal element} \\ 2, & \text{otherwise.} \end{cases} \tag{2.2.8}$$

Thus the dual graph for $\mathbb{SY}$ is the same graph but with the edges which correspond to adding non-diagonal boxes doubled; see Figure 6.

Unfortunately, the construction of weighted distributive lattices does not work in several interesting cases, e.g., for the lattice $J(\mathbb{P}^3)$ of three-dimensional shapes, for the distributive Fibonacci lattice $\mathbb{F}$ (see Example 2.3.7), for the lattice of binary trees $J(\mathbb{T}_2)$ (see Section 2.4), etc. Later on we will construct dual graphs for $\mathbb{F}$ and $J(\mathbb{T}_2)$ using other approaches.

Now consider the special case $r = 0$. So we are interested in pairs of graded graphs, with a common set of vertices and a common rank function, satisfying the commutation relation

$$DU = UD. \tag{2.2.9}$$

The relation (2.2.9) can hardly be valid in a graph with $\widehat{0}$ since $UD\widehat{0} = 0 \neq DU\widehat{0}$. A natural class of graded graphs to be studied in this context is the class of distributive lattices without zero. So let us extend Definition 2.1.1 to posets Q which are not bounded from below. Namely, define $J(Q)$ to be the [distributive] lattice of the order ideals of Q which are bounded from above. We avoid now discussing when such a lattice satisfies the level finiteness condition.

Example 2.2.9 *The chain* $\mathbb{Z}$. This lattice can also be regarded as $J(\mathbb{Z})$. Condition (2.2.9) obviously holds. Thus $\mathbb{Z}$ is a self-0-dual graph.

Example 2.2.10 *SkewStrips*. Let $t \in \mathbb{P}$. Define the poset *t-Plait* as follows:

(i) The vertices are $\ldots, -1, 0, 1, 2, \ldots$ and $\ldots, -1', 0', 1', 2', \ldots$.
(ii) The order relation is

(a) $n \leq m$, $n' \leq m'$, and $n < m'$ in *t-Plait* whenever $n \leq m$ in $\mathbb{Z}$;
(b) $n' < m$ whenever $n + t \leq m$ in $\mathbb{Z}$.

The distributive lattice *t-SkewStrip* $= J(t\text{-}Plait)$ (see Figure 7) can be easily seen to satisfy (2.2.9); so it is self-0-dual. Note that the case $t = 1$ reduces to Example 2.2.9.

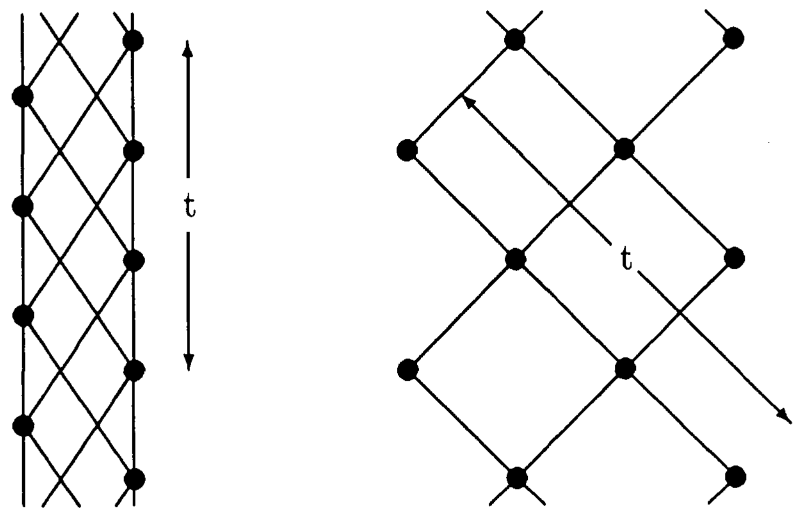


Figure 7. t-Plait and t-SkewStrip.

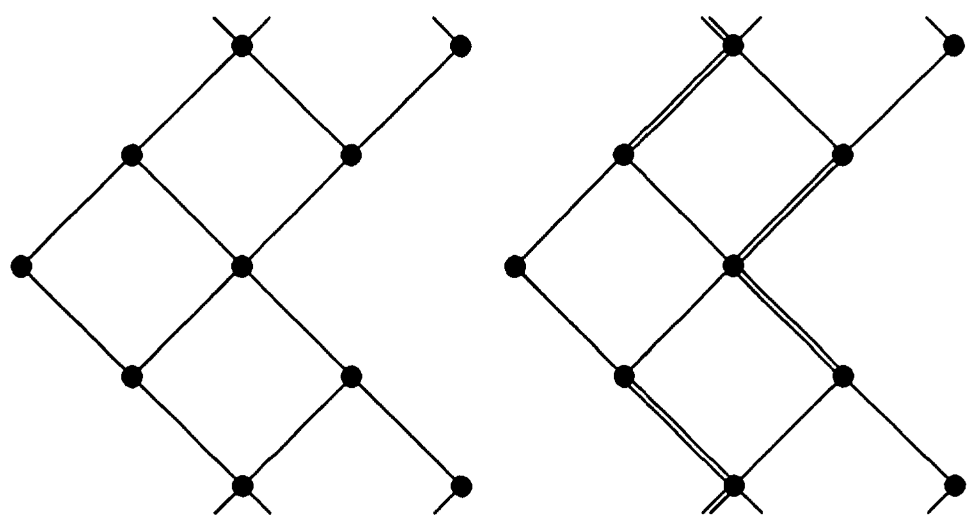


Figure 8. The lattice of skewstrip shapes.

Example 2.2.11 *SkewStrip shapes.* These are the elements of the distributive lattice $J(t\text{-}SkewStrip)$ (see Figure 8). To build a 0-dual graph, define an appropriate weight function on *t-SkewStrip* and use the weighted lattice construction. One can check that the weight function w defined on *t-SkewStrip* by

$$w(q) = \begin{cases} 1, & \text{if } q \text{ covers a single element} \\ 2, & \text{otherwise} \end{cases} \tag{2.2.10}$$

(cf. (2.2.8)) satisfies (2.2.2) with $r = 0$.

2.3. Derivations in graded algebras

Let A be a graded associative K-algebra with identity; so A (as a vector space) is a direct sum of "homogeneous" subspaces A_n, $n \in \mathbb{Z}$ (usually $n \in \mathbb{N}$), and if $a \in A_n$ and $b \in A_m$ then $ab \in A_{n+m}$. Assume $D\colon A \to A$ is a derivation, i.e., a linear endomorphism satisfying $D(ab) = aD(b)+D(a)b$. Assume, in addition, that if $a \in A_n$ then $D(a) \in A_{n-1}$. Suppose there exists an element $t \in A$ such that

$$D(t) = r \cdot id \tag{2.3.1}$$

where id stands for an identity element of A; of course $t \in A_1$. (Informally, D is the derivative with respect to t/r.) Hence the operator $U \in \mathrm{End}(A)$ defined by

$$U(a) = ta \tag{2.3.2}$$

and the derivation operator D satisfy the condition (1.4.12) where $I \in \mathrm{End}(A)$ is the identity transformation. (One can also take the right multiplication $U(a) = at$ instead of (2.3.2).) When the homogeneous subspaces A_n are finite-dimensional we can fix arbitrary bases in them to obtain a pair of r-dual graphs or networks.

Now we reconstruct several examples of Sections 2.1–2.2 using this tool.

Example 2.3.1 *The chain* $\mathbb{N} = \{0, 1, 2, \ldots\}$ (cf. Example 2.2.1). Let $A = K[t]$ be the algebra of polynomials in the variable t with the natural grading $\rho(t^n) = n$. Then U is the operator of multiplication by t, and D is $\frac{d}{dt}$. Now take the basis $\{t^n\colon n \in \mathbb{N}\}$ to get Example 2.2.1.

Example 2.3.2 *The Young graph* (cf. Example 2.1.2). Let A be the algebra of symmetric functions in commuting variables $u_1, u_2, \ldots$. Alternatively, A is the algebra of polynomials in the variables

$$t_n = \sum_j u_j^n, \quad n = 1, 2, \ldots$$

where

$$\rho(t_n) = n. \tag{2.3.3}$$

Now let D be the derivation with respect to t_1 and U the multiplication by t_1. Since $D(t_1) = 1$, we have a pair of dual graded graphs; to make the construction explicit, bases for the A_n's should be chosen. One choice is the basis of functions $t_1^{\alpha_1} \cdots t_s^{\alpha_s}$; this results in a set of disjoint chains and corresponding dual weighted chains (cf. Example 2.3.1). On the other hand, the bases formed by the Schur functions give rise to the Young graph of Example 2.1.2. See [13, 26, 21] for more details.

Example 2.3.3 *Pascal graphs* (cf. Example 2.2.2). Let $A = K[t_1, \dots, t_r]$ be the algebra of polynomials in r commuting variables. Define

$$t = t_1 + \cdots + t_r. \tag{2.3.4}$$

Thus

$$Uf(t_1, \dots, t_r) = (t_1 + \cdots + t_r) f(t_1, \dots, t_r) \tag{2.3.5}$$

and

$$D = \frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_r}. \tag{2.3.6}$$

Now condition (2.3.1) holds, and we get the pair of Example 2.2.2.

Example 2.3.4 *Diagrams with $\leq r$ rows* (cf. Example 2.2.5). Here A is the algebra of symmetric polynomials in r commuting variables $t_1, \dots, t_r$. As in Example 2.3.3, define U and D by (2.3.5) – (2.3.6). The result coincides with that of Example 2.2.5.

The remaining examples in this section are different from those considered previously.

Example 2.3.5 *Infinite r-nary trees*. Let A be a free associative algebra with r noncommuting generators $t_1, \dots, t_r$. In other words, A is the algebra of linear combinations of words in the alphabet $\{t_1, \dots, t_r\}$, including the empty word. Multiplication is concatenation. All the t_i's have rank 1. Let $t = t_1 + \cdots + t_r$, as before. The derivation D is defined in a natural way (cf. (2.3.6)): if $w = w_1 \cdots w_k$ is a word in the alphabet $\{t_1, \dots, t_r\}$ then

$$D(w) = w_2 \cdots w_k + w_1 w_3 \cdots w_k + \cdots + w_1 \cdots w_{k-1}. \tag{2.3.7}$$

Thus $D(t) = r\emptyset$ as in (2.3.1). Now let U be the right multiplication by t, so that

$$U(w) = wt_1 + wt_2 + \cdots + wt_r, \tag{2.3.8}$$

and (1.4.12) holds. The natural basis in A is the basis of words. With respect to this basis, the operator U defines the *infinite r-nary tree* $\mathbb{T}_r$. The r-dual graph for $\mathbb{T}_r$ is defined by the operator D (see (2.3.7)). In this graph a word w of rank k is linked with a word v of rank $k-1$ by $a(v, w)$ edges where $a(v, w)$ is the number of ways to obtain v by deleting a single letter from w. See Figure 9 for the case $r = 2$: the infinite binary tree and its 2-dual graph.

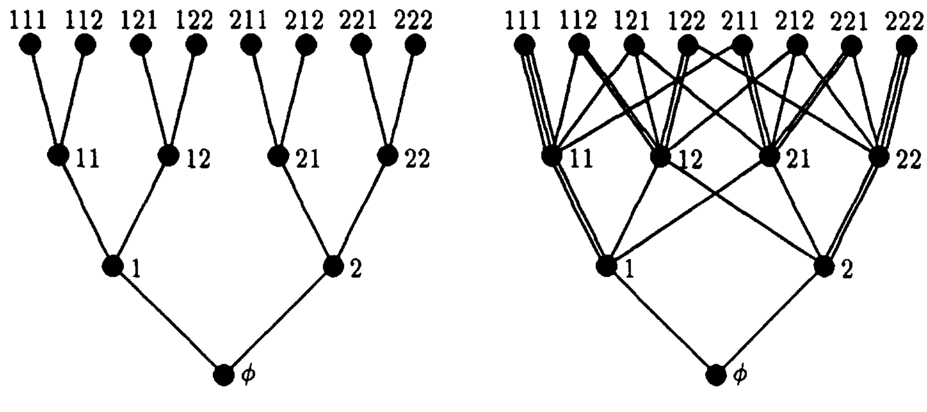


Figure 9. The binary tree.

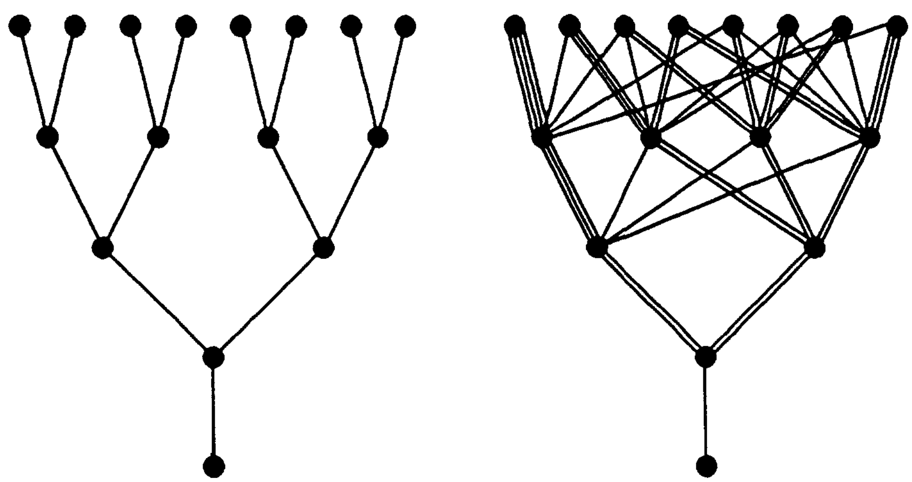


Figure 10. The lifted binary tree.

Example 2.3.6 *The lifted binary tree, first construction* (cf. Example 2.4.1). Let A be the algebra of symmetric polynomials in two *non-commuting* variables t_1 and t_2; the rank of both t_1 and t_2 is 1. For any $\{t_1, t_2\}$-word w, let $\bar{w}$ denote the word obtained from w by replacing all the t_1 entries by t_2, and vice versa. Then the elements of the form $w + \bar{w}$ provide a linear basis for A. Now define U and D as in Example 2.3.5 (cf. (2.3.7) and (2.3.8) with $r = 2$). Thus we have a pair of 2-dual graphs. The operator U, with respect to the basis $\{w + \bar{w}\}$, defines the *lifted binary tree*, i.e., the infinite binary tree $\mathbb{T}_2$ with an additional edge attached below the old zero. The operator D defines the corresponding 2-dual graph. See Figure 10.

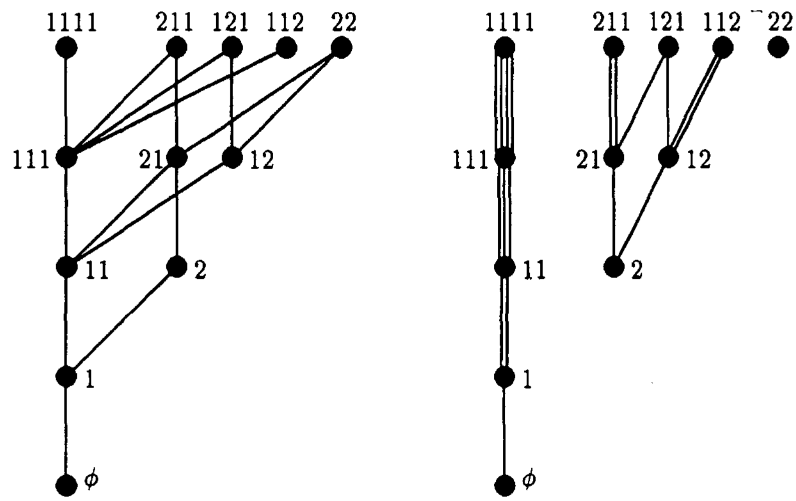


Figure 11. The Fibonacci graph.

Example 2.3.7 *The Fibonacci graph* [24]. Recall the definition of the Young-Fibonacci graph from Example 2.1.4. There exists another remarkable graph with the same set of vertices $\{1,2\}^*$. In fact, this graph was historically the first of the two. Its vertices are also snakes though ordered in another way.

Let Q be a poset *comb* defined as follows:

(i) the vertices are $1, 1', 2, 2', 3, 3', \ldots$;
(ii) the order relation is

  (a) $n \leq m$ in Q whenever $n \leq m$ in $\mathbb{P}$;
  (b) $n \leq m'$ whenever $n \leq m$.

Now define the *Fibonacci graph* $\mathbb{F}$ as the distributive lattice $J(Q)$. R.P. Stanley's notation for this graph is *Fib*(1); see Figure 11. Finite order ideals of the comb are just the snakes or, equivalently, the $\{1,2\}$-words. The rank function is of course the same as before (i.e., rank = number of boxes = sum of digits).

The following surprising result holds.

Lemma 2.3.8 [5, 26, 29b] *The values $e(x)$ in $\mathbb{F}$ and $\mathbb{YF}$ are the same.*

In other words, the number of ways of growing a snake does not depend on whether it is considered as an element of the Fibonacci or the Young-Fibonacci graph. Thus all identities involving the $e(x)$'s of $\mathbb{YF}$ (e.g., (1.1.4)) are also valid for $\mathbb{F}$.

Now let us describe the algebra with derivation associated with the Fibonacci graph. A first try is to consider the free algebra with two non-commuting generators having ranks 1 and 2, respectively. A basis of this algebra can be formed by the snakes, and the ranks are just those we need. Unfortunately, this construction does not result in the Fibonacci graph. The right approach is different.

Let $u, v \in \{1, 2\}^*$ be two snakes. The *shuffle product* uv is the formal sum of the snakes which can be obtained by shuffling v with u. For example,

$$12 \cdot 21 = \underline{12}21 + \underline{1}2\underline{2}1 + \underline{1}21\underline{2} + 2\underline{12}1 + 2\underline{1}1\underline{2} + 21\underline{12}$$

(the digits corresponding to I, i.e., to the first multiplier are underlined). This multiplication is associative and commutative. So we have defined a graded commutative algebra A for which snakes form a linear basis.

Lemma 2.3.9 *Let D be the down operator of the Fibonacci graph, i.e., the operator which changes any* 2 *into* 1 *or deletes the last digit if it is* 1. *Then D is a derivation in the shuffle algebra A.*

Definition 2.3.10 A graph dual to the Fibonacci graph $\mathbb{F}$ can be determined by the operator of multiplication by the snake 1. In other words, a snake $w \in \{1, 2\}^*$ of rank k is linked with a snake v of rank $k - 1$ by $a(v, w)$ edges where $a(v, w)$ is the number of ways to obtain v by deleting a single 1 from w (cf. Example 2.3.5). See Figure 11. Note that this graph is not connected.

Another examples of dual graphs associated with graded algebras will be given in the sequel.

2.4. Trees

The next three sections are devoted to constructing dual graphs for some rooted trees.

Example 2.4.1 *The lifted binary tree, second construction* (cf. Example 2.3.6). The vertices of this tree can be naturally labelled by the bit strings for the nonnegative integers: $0, 1, 10, 11, 100, 101, 110, \ldots$ so that 0 is the root; 1 is the only vertex of rank 1; 10 and 11 are the vertices of rank 2, etc. (see Figure 12). Generally, the rank is the length of the string, except for the case $\rho(0) = 0$.

Now define the graph *BinWord* (a *lifted binary subword order*; cf. [3]; see Figure 12) with the same set of vertices, the same rank function, and the following covering relation: x covers y if and only if x can be obtained by deleting a single symbol from y. In addition, 1 covers 0. For example, 101001 covers 11001, 10001, 10101, and 10100. We emphasize that *BinWord* is a graph without multiple edges.

The following observation is surprising but easily proved.

Lemma 2.4.2 *BinWord and the lifted binary tree are dual.*

Comments 2.4.3 The graph *BinWord* is invariant with respect to the reflection

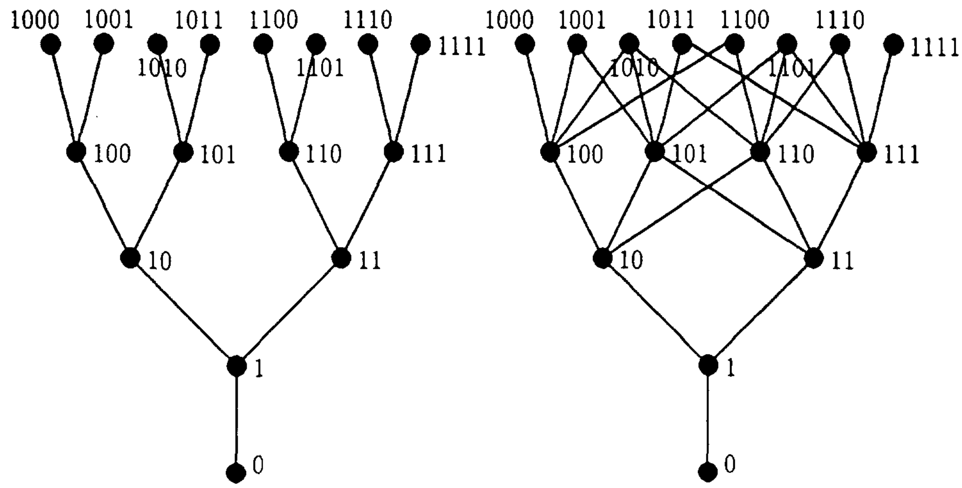


Figure 12. The lifted binary tree and BinWord.

$$w_1 w_2 \cdots w_n \mapsto w_1 w_n \cdots w_2.$$

Simultaneously, this map transforms the binary tree into another (though isomorphic) tree related to inserting new symbols between w_1 and w_2. Thus there are at least two dual graphs for *BinWord*—both without multiple edges.

Example 2.4.4 *The lifted* $(r+1)$*-nary tree.* This is a generalization of the previous example. Now the vertices are the words in the alphabet $\{0, 1, \ldots, r\}$. Two words are linked in the tree whenever one of them can be obtained by adding a single symbol to the end of another. The corresponding analogue of *BinWord* is a graph $(r+1)$*Word* where two words are linked if one can be obtained by inserting a single symbol into another (cf. [3]).

Lemma 2.4.5 *The graph* $(r+1)$*Words and the lifted* $(r+1)$*-nary tree satisfy the relations*

$$D_{n+1}U_n = U_{n-1}D_n + rI_n, \quad n = 1, 2, \ldots$$

and

$$D_1 U_0 = U_{-1} D_0 + I_0, \quad (n = 0).$$

Thus these graphs are not exactly r-dual. They are $\mathbf{r}$-dual (cf. (1.4.11)) with $\mathbf{r} = (1, r, r, r, \ldots)$.

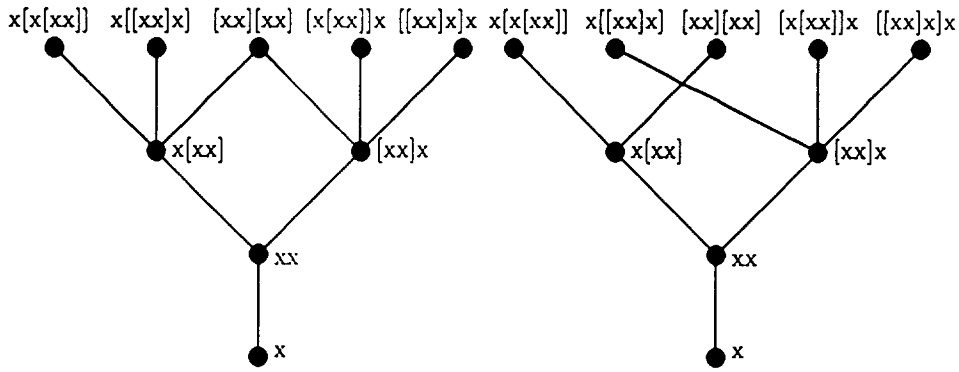


Figure 13. The lattice of binary trees and BracketTree.

Example 2.4.6 *The Bracket Tree.* This tree (see Figure 13) is defined as follows. The vertices of rank n are the syntactically correct formulae defining different versions of calculation of *non-associative* product $x \cdot x \cdot \cdots \cdot x$ containing $n + 1$ entries of x. In other words, any vertex of rank n is a valid sequence of $n - 1$ opening and $n - 1$ closing brackets inserted into $x \cdot x \cdot \cdots \cdot x$. We call such sequences the *bracket schemes*. Two schemes are linked in the tree if one of them results from another by deleting the first entry of x and subsequent removing the pair of unnecessary brackets. For example, the scheme [[xx][xx]][xx] covers [x[xx]][xx]. Clearly, this defines a tree.

Lemma 2.4.7 *The Bracket Tree is dual to the distributive lattice $J(\mathbb{T}_2)$ of finite order ideals of the infinite binary tree.*

Proof: This statement needs explanation. First of all, we must show that the *Bracket Tree* and $J(\mathbb{T}_2)$ have the same set of vertices (see, e.g., [30, Section 3.1] for another proof). A vertex of the *Bracket Tree* is a bracket scheme. Any bracket scheme is associated with an appropriate parsing tree. Remove the leaves of this tree (they correspond to the x's) together with the edges incident to them. As a result we obtain a binary tree which is an order ideal of the infinite binary tree $\mathbb{T}_2$. Thus we have constructed a desired rank-preserving bijection

$$\text{bracket schemes} \longleftrightarrow \text{order ideals of } \mathbb{T}_2.$$

The statement of the lemma can now be easily verified. □

See [24] for other results concerning $J(\mathbb{T}_2)$.

Comments 2.4.8 Again, the dual graph is not unique (cf. Comments 2.4.3). Each automorphism of the binary tree $\mathbb{T}_2$ induces an automorphism of the lattice $J(\mathbb{T}_2)$. However, the construction of the *Bracket Tree* is not invariant with respect to these automorphisms

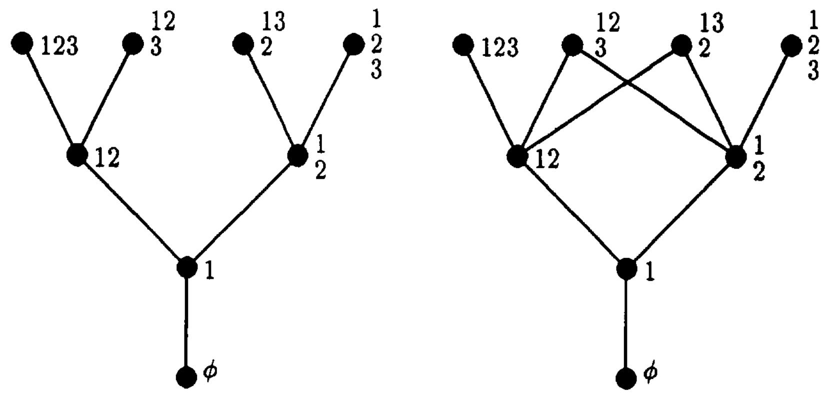


Figure 14. The SYT-Tree and the Schensted graph.

since it involves a specific path in the binary tree, formed by its leftmost edges. Since there is a continuum of paths from the root to infinity, this gives a continuum of different (though isomorphic) graphs dual to $J(\mathbb{T}_2)$.

This example can also be interpreted in terms of a well-known bijective correspondence [30] between the bracket schemes of rank n and the standard Young tableaux having a rectangular diagram of shape $2 \times n$—or, equivalently, the paths in *SemiPascal* joining $(0, 0)$ and (n, n) (the Dyck paths).

2.5. Tableaux

In this section we examine examples of tableaux trees, i.e., trees of paths in graded graphs with zero.

Definition 2.5.1 Let $G_1 = (P, \rho, E_1)$ be a graded graph with zero $\widehat{0}$. Let T_n denote the set of paths (= saturated chains) joining $\widehat{0}$ and the vertices of rank n. Define $\mathbb{T}(G_1)$ to be the graded rooted tree with vertices $\bigcup_{n=0}^{\infty} T_n$ and covering relation

$$\tau' \in T_{n+1} \text{ covers } \tau \in T_n \text{ if and only if } \tau' \text{ is a prolongation of } \tau.$$

In [7] a universal scheme will be given that allows us to construct a dual (r-dual, etc.) graph for the tree $\mathbb{T}(G_1)$ from G_1 and its dual graph G_2. For now, let us consider the main two cases: the Young and the Young-Fibonacci graphs.

Example 2.5.2 *The SYT-Tree and the Schensted graph.* The *SYT-Tree* (see Figure 14) is defined as $\mathbb{T}(\mathbb{Y})$; the vertices are the standard Young tableaux which are linked if one is obtained from another by deleting a box with the maximal entry. The dual graph for

SYT-Tree is the graph we call *Schensted* (see Definition 2.5.5 below). To construct it, we use the Schensted insertion [20] and the following definition.

Definition 2.5.3 Let τ be a saturated multichain, i.e.,

$$\tau = (x_0, x_1, x_2, \ldots, x_n) \tag{2.5.1}$$

where $x_i \in P$, $x_0 = \widehat{0}$, and for each i either x_{i+1} covers x_i or $x_{i+1} = x_i$. Let stand(τ) denote the "standard tableau" (saturated chain) obtained by deleting repeated entries from τ. For example, if τ is a partial Young tableau with different entries then stand(τ) is an SYT having the same shape (say, of n boxes) filled with the numbers $1, 2, \ldots, n$ so that the induced order of boxes remains the same. Tableaux τ and τ' are said to be *equivalent* if stand$(\tau) =$ stand(τ').

Definition 2.5.4 *Schensted insertion* [20]. Let τ be a Young tableau. Assume $a_0 \in \mathbb{P}$ is not an entry of τ. In the first row of τ, find the minimal entry which is greater than a_0, say a_1, and replace it by a_0. Then insert a_1 into the second row in the same manner, replacing the minimal entry $a_2 > a_1$; a_2 goes to the third row, etc., until some a_i is greater than or equal to all the elements of the $(i+1)$st row. Then a_i is placed into a new box added to this row. The resulting tableau is denoted Ins(τ, a_0) and is said to be obtained by *Schensted inserting* a_0 into τ. Details can be found in [12, 21, 18, etc.].

Definition 2.5.5 *The Schensted graph.* The vertices of this graph (see Figure 14) are the standard Young tableaux. Given two SYT σ and τ, σ covers τ in the Schensted graph if and only if there exists a partial Young tableau τ' and a number k such that stand$(\tau') = \tau$ and Ins$(\tau', k) = \sigma$. In other words, one can obtain all the tableaux covering τ by increasing its entries $k, k+1, \ldots, n$ (where $n = \rho(\tau)$) by 1 and then inserting k; $k = 1, 2, \ldots, n+1$.

Lemma 2.5.6 *The Schensted graph and the SYT-Tree are dual.*

This statement can be verified directly. However, it is a particular case of a more general result (see [7]) on Schensted-type constructions, so we will not bother to prove it here.

Example 2.5.7 *The tree of standard Young-Fibonacci tableaux.* Since the Young-Fibonacci graph $\mathbb{YF}$ is also a self-dual lattice, it is natural to have a corresponding analogue of Schensted insertion. Such an analogue was introduced in [5] and is reformulated here. Another description of the same algorithm can be found in [15].

The vertices of $\mathbb{YF}$ are the snakes which can be written as $\{1, 2\}$-words or pictured as diagrams. The growth of a snake can be represented as a *Young-Fibonacci tableau*, i.e., a filling of such a shape by integers indicating when respective boxes get added. This process can also be treated as a growth of a tableau. At the moment n, a new box filled with n can be added to a current tableau. Recall from Example 2.1.4 that this new box is either placed into the leftmost available position in the second row (in the first column of length 1) or inserted into the first row at any place to the left of this position, thus dividing the tail of the snake. As an example, the growth process

$$\emptyset \to 1 \to 11 \to 21 \to 22 \to 212 \to 2112$$

corresponds to the tableau

$$\begin{array}{|c|c|c|c|} \cline{1-1} \cline{4-4} 3 & \multicolumn{2}{c|}{} & 4 \\ \hline 2 & 6 & 5 & 1 \\ \hline \end{array}\,.$$

Note that this construction has an essential distinction from the classical Young's case: when a new box is inserted into a snake, the old boxes move. Thus the position of a box can change in the course of the growth. This could not be otherwise since the lattice $\mathbb{YF}$ is not distributive.

An arrangement of positive integers within a "snakeshape" ought to obey certain conditions in order to represent a valid Young-Fibonacci tableau.

Lemma 2.5.8 *A filling of a snakeshape with different positive integers is a valid Young-Fibonacci tableau (standard YFT provided these are* $1, \ldots, n$*) if and only if*

(i) *for any pair* $\begin{array}{|c|}\hline B\\\hline A\\\hline\end{array}$ *the inequality* $A \le B$ *holds;*

(ii) *to the right of any* $\begin{array}{|c|}\hline B\\\hline A\\\hline\end{array}$ *there are no numbers from the interval* $(A, B]$ *in either the lower or upper rows);*

(iii) *if the position above* $\boxed{A}$ *is not occupied yet then to the right of* $\boxed{A}$ *there are no numbers greater than* A *in either the lower or upper rows.*

Note that the conditions (ii) and (iii) are quite similar: (iii) converts into (ii) if the position above $\boxed{A}$ is regarded as occupied by $\boxed{\infty}$.

Definition 2.5.9 The standard Young-Fibonacci tableaux form a tree *SYFT-Tree* $= \mathbb{T}(\mathbb{YF})$ shown in Figure 15.

Definition 2.5.10 *Young-Fibonacci insertion* [5]. As in Definition 2.5.4, assume τ is a partial Young-Fibonacci tableaux. Then τ satisfies the conditions of Lemma 2.5.8.

Assume $a_0 \in \mathbb{P}$ is not an entry of τ. Attach a new box $\boxed{a_0}$ just to the left of τ (in the lower row). Then find all the entries of τ which are greater than a_0 and sort them:

$$a_0 < a_1 < a_2 < \cdots < a_k.$$

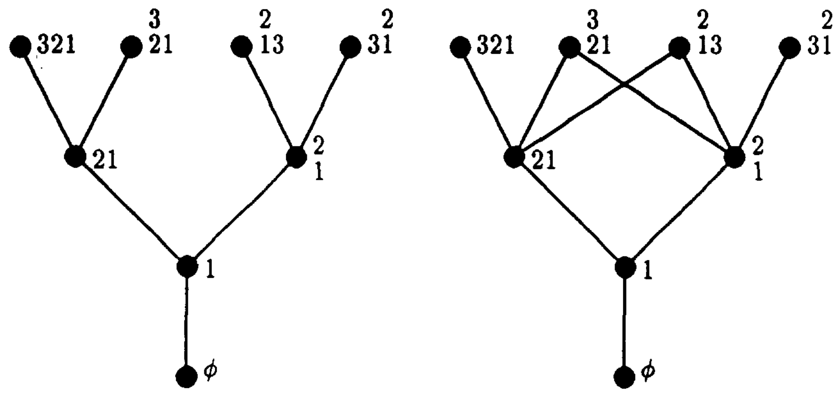


Figure 15. The SYFT-Tree and InsYF.

Now put a box $\boxed{\bullet}$ just above $\boxed{a_0}$ and move the a_i chainwise according to the following rule: $\boxed{\bullet} \leftarrow \boxed{a_1} \leftarrow \boxed{a_2} \leftarrow \cdots \leftarrow \boxed{a_k}$; that is, a_1 moves into the new box, a_2 moves to the old place of a_1, etc. The box that was occupied by a_k disappears; if it was situated in the lower row, the left and right parts of the snake are concatenated. For example:

$$a_0 = 2, \qquad \tau = \begin{array}{|c|c|c|c|} \cline{2-2}\cline{4-4} \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{c|}{} & 3 \\ \hline 7 & 4 & 6 & 1 \\ \hline \end{array} \qquad \mathrm{Ins}(\tau, a_0) = \begin{array}{|c|c|c|c|} \cline{1-2}\cline{4-4} 3 & 6 & \multicolumn{1}{c|}{} & 4 \\ \hline 2 & 5 & 7 & 1 \\ \hline \end{array}$$

As before, $\mathrm{Ins}(\tau, a_0)$ denotes the resulting tableau.

Now let us construct the Young-Fibonacci analogue of the Schensted graph (cf. Definition 2.5.5).

Definition 2.5.11 The graph *Ins YF* (see Figure 15) is defined exactly as the Schensted graph, replacing "Young" by "Young-Fibonacci" everywhere in Definition 2.5.5.

Lemma 2.5.12 *The graphs Ins YF and SYFT-Tree are dual.*

The proof of this lemma is omitted for the same reasons as in the case of Lemma 2.5.6. Actually, it is not difficult to verify the duality directly.

The insertion procedure of Definition 2.5.10 has many nice properties which are similar to those of the Schensted insertion. In particular, it allows to establish a one-to-one correspondence between permutations and the pairs of standard Young-Fibonacci tableaux having the same shape. We will return to this subject in [7].

2.6. *Permutation trees*

This section is devoted to a single example of dual graphs that plays a fundamental role in the sequel [7]. Both graphs in this example are trees.

Definition 2.6.1 A finite subset $M = \{(x_1, y_1), \dots, (x_n, y_n)\}$ of the coordinate plane $\mathbb{R}^2$ is said to be *diagonal* if all the x_i are distinct and all the y_j are distinct (though $x_i = y_j$ is allowed). Sorting the x_i and the y_j produces bijective numberings

$$\nu\colon \{x_1, \dots, x_n\} \to \{1, \dots, n\}$$
$$\mu\colon \{y_1, \dots, y_n\} \to \{1, \dots, n\}$$

such that $x_i < x_j$ if and only if $\nu(x_i) < \nu(x_j)$; $y_i < y_j$ if and only if $\mu(y_i) < \mu(y_j)$. The map

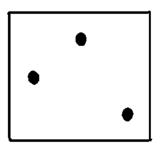
$$\sigma = \mu \circ \nu^{-1}\colon \{1, \dots, n\} \to \{1, \dots, n\}$$

is of course a permutation. The permutation σ is called the *type* of the diagonal set M. For example, if

$$M = \{(6,7),\ (0,10),\ (1,15)\}$$

then the type of M is

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{or} \quad 231 \quad \text{or}$$



;

The latter notation is the most natural in this context. Diagonal sets of the same type are said to be *equivalent*. The set of all the types is denoted *Perm*. Thus *Perm* contains the permutations of $\{1, \dots, n\}$ for all $n \in \mathbb{N}$. For $n = 0$, the permutation of rank zero is the type of the empty set.

Definition 2.6.2 *Permutation algebras*. We define a graded associative algebra $\mathit{Perm}^{\sqcup}$ as follows:

(i) the elements of $\mathit{Perm}^{\sqcup}$ are the formal finite linear combinations of permutations. Thus $\mathit{Perm}^{\sqcup}$ coincides with $K\mathit{Perm}$ as a set where K is our field;
(ii) grading is defined by letting Perm_n be all permutations of $\{1, \dots, n\}$;
(iii) the product $\sigma_1 \sqcup \sigma_2$ of two permutations $\sigma_1 \in \mathit{Perm}_{n_1}$ and $\sigma_2 \in \mathit{Perm}_{n_2}$ is a formal sum of the permutations $\sigma \in \mathit{Perm}_{n_1+n_2}$ satisfying the following property: there exists a diagonal set M of type σ and a vertical line l such that l cuts M into parts having types σ_1 (to the left) and σ_2 (to the right). For instance,

$$12 \sqcup 21 = 1243 + 1342 + 1432 + 2341 + 2431 + 3421.$$

The definition of the dual algebra *Perm*$^{\sqsubset}$ is completely analogous: use horizontal cuts instead of vertical ones. In other words, $\sigma_1 \sqsubset \sigma_2$ is the shuffle product of σ_1 and $n_1 + \sigma_2$. For example,

$$12 \sqsubset 21 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312.$$

The algebras *Perm*$^{\sqcup}$ and *Perm*$^{\sqsubset}$ are isomorphic. They have the same set of elements and the same rank function. The multiplications $\sqcup$ and $\sqsubset$ satisfy

$$\sigma_1 \sqsubset \sigma_2 = (\sigma_1^T \sqcup \sigma_2^T)^T$$

where ^{T} stands for the transposition (= inversion of permutations).

Lemma 2.6.3 *Perm*$^{\sqcup}$ *and Perm*$^{\sqsubset}$ *are graded associative algebras with identity.*

Definition 2.6.4 Let $\partial^{\uparrow}$, $\partial^{\downarrow}$, $\partial^{\rightarrow}$, and $\partial^{\leftarrow}$ denote four linear endomorphisms $KPerm \to KPerm$ acting as follows. Assume that $\sigma \in KPerm$ and M is a diagonal set of type σ. Delete the uppermost (lowermost, rightmost, leftmost, respectively) element from M. The resulting set has, by definition, the type $\partial^{\uparrow}\sigma$ ($\partial^{\downarrow}\sigma$, $\partial^{\rightarrow}\sigma$, $\partial^{\leftarrow}\sigma$, respectively). For example,

$$\partial^{\uparrow}(3142) = 312,\ \partial^{\downarrow}(3142) = 231,\ \partial^{\rightarrow}(3142) = 213,\ \partial^{\leftarrow}(3142) = 132.$$

In other words, delete the maximal (the minimal, the last, the first) element from the permutation and standardize the result.

Lemma 2.6.5 *The operators* $\partial^{\uparrow}$ *and* $\partial^{\downarrow}$ *are derivations in the algebra Perm*$^{\sqcup}$; *the operators* $\partial^{\rightarrow}$ *and* $\partial^{\leftarrow}$ *are derivations in Perm*$^{\sqsubset}$.

Let 1 denote the only permutation of rank 1 (not to be confused with the identity element $\emptyset$ having zero rank). Then

$$\partial^{\uparrow}(1) = \partial^{\downarrow}(1) = \partial^{\rightarrow}(1) = \partial^{\leftarrow}(1) = \emptyset\,. \tag{2.6.1}$$

Thus we can use the main construction of Section 2.3 to obtain the following.

Lemma 2.6.6 *Define the operators* $U^{\uparrow}$, $U^{\downarrow}$, $U^{\rightarrow}$, *and* $U^{\leftarrow}$ *on KPerm by*

$$\begin{aligned} U^{\uparrow}&: \sigma \mapsto \sigma \sqsubset 1,\\ U^{\downarrow}&: \sigma \mapsto 1 \sqsubset \sigma,\\ U^{\rightarrow}&: \sigma \mapsto \sigma \sqcup 1,\\ U^{\leftarrow}&: \sigma \mapsto 1 \sqcup \sigma\,. \end{aligned} \tag{2.6.2}$$

Let I denote the identity transformation. Then the equality $DU = UD + I$ *holds in each of the following cases*:

(i) $U = U^{\uparrow}$ *or* $U = U^{\downarrow}$, $D = \partial^{\rightarrow}$ *or* $D = \partial^{\leftarrow}$;

(ii) $U = U^{\rightarrow}$ *or* $U = U^{\leftarrow}$, $D = \partial^{\uparrow}$ *or* $D = \partial^{\downarrow}$.

Proof: Follows from Lemma 2.6.5 and (2.6.1). □

Note that there are *eight* dual pairs in the lemma.

So we have four graded graphs associated with the up operators (2.6.2), and four graded graphs (each dual to two of the first four) corresponding to the down operators $\partial^{\uparrow}$, $\partial^{\downarrow}$, $\partial^{\rightarrow}$, and $\partial^{\leftarrow}$. It turns out that these quadruples coincide.

Lemma 2.6.7 *Suppose KPerm is made into a Hilbert space by declaring Perm to be an orthonormal system. Then the operators in the following pairs are conjugate to each other*:

$$U^{\uparrow} \text{ and } \partial^{\uparrow},\ U^{\downarrow} \text{ and } \partial^{\downarrow},\ U^{\rightarrow} \text{ and } \partial^{\rightarrow},\ U^{\leftarrow} \text{ and } \partial^{\leftarrow}.$$

Proof: For instance, $U^{\uparrow}\sigma$ is the sum of permutations σ' obtained by inserting $n+1$ into $\sigma \in Perm_n$; these are exactly those satisfying $\partial^{\uparrow}\sigma' = \sigma$. □

Thus we have actually *four* graphs associated with the symbols $^{\uparrow}$, $^{\downarrow}$, $^{\rightarrow}$, and $^{\leftarrow}$. Lemma 2.6.6 shows that among these graphs there are four dual pairs. Let us describe the situation explicitly.

Example 2.6.8 *Permutation trees.* These are the trees $PT^{\uparrow}$, $PT^{\downarrow}$, $PT^{\rightarrow}$, and $PT^{\leftarrow}$ (see Figure 16) defined as follows:

(i) the vertices of rank n are the permutations of $\{1, \ldots, n\}$ (i.e., all the four graphs have the common set of vertices *Perm*);
(ii) any permutation σ covers the permutation $\partial^{\uparrow}\sigma$ ($\partial^{\downarrow}\sigma$, $\partial^{\rightarrow}\sigma$, $\partial^{\leftarrow}\sigma$, respectively).

The permutation trees have the up and down operators U and ∂ with respective superscripts. Below we state again the duality relations among permutation trees.

Theorem 2.6.9 *Each of the trees* $PT^{\uparrow}$ *and* $PT^{\downarrow}$ *is dual to both* $PT^{\rightarrow}$ *and* $PT^{\leftarrow}$.

2.7. Miscellaneous

Example 2.7.1 *The Hex* [14]. This graph (see Figure 17 for its definition) is interesting because its $e(x)$ for ranks $2n$ and $2n+1$ are the ordinary binomial coefficients $\binom{n}{k}$, $1 \le k \le n$. Thus they coincide with their analogues for the Pascal graph $\mathbb{N}^2$. Recall that the construction of Example 2.2.2 provides a 2-dual for $\mathbb{N}^2$; a 1-dual graph for $\mathbb{N}^2$ is necessarily disconnected. However, this is not the case for the *Hex*. The latter has a connected dual graph with positive integer multiplicities; see Figure 17.

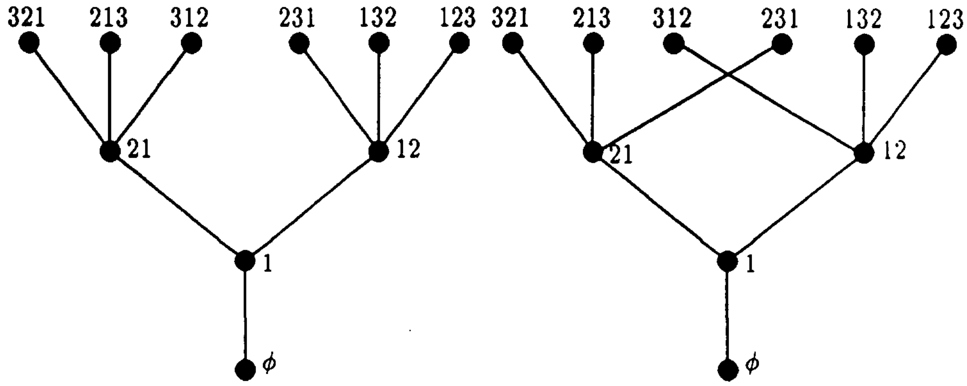


Figure 16. Dual permutation trees.

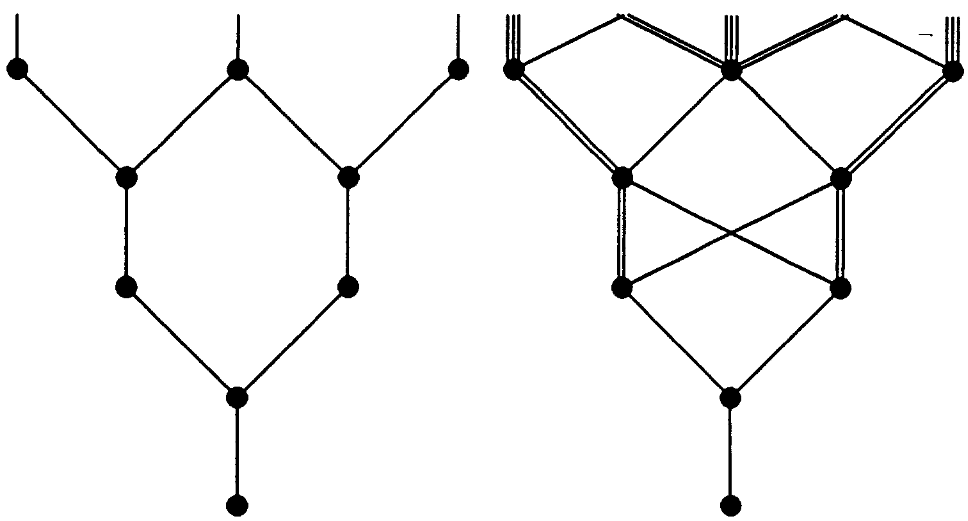


Figure 17. Hex.

Example 2.7.2 *Fan*$_t$. Amalgamate t disjoint infinite chains by gluing their zeros together to create the graph *Fan*$_t$. Define a *complete graded graph* with the same set of vertices by connecting any two vertices lying on consecutive levels with a simple edge. These two graphs satisfy (1.4.11) with $r_0 = t$, $r_1 = r_2 = \cdots = 0$. There are similar constructions with other values of the r_i.

Example 2.7.3 *The Reflected Young graph.* Take the levels $P_0, P_1, \dots, P_n$ of the Young graph and reflect them in the top rank level to obtain a new P_{n+1} which is a copy of P_{n-1}, new P_{n+2} which is a copy of P_{n-2}, etc. The resulting finite graph *RefYoung* has a height $2n$; see Figure 18. This graph is a sequentially differential poset (in the terminology of [27]), i.e., it is self-**r**-dual with

$$r_0 = r_1 = \cdots = r_{n-1} = 1,\ r_n = 0,\ r_{n+1} = r_{n+2} = \cdots = r_{2n} = -1.$$

Paths in this graph joining $\widehat{0}$ and $\widehat{1}$ can also be regarded as loops in the usual Young graph.

The reflection construction of Example 2.7.3 can be applied to any pair of dual (r-dual, etc.) graphs G_1 and G_2. Details are left to the reader.

2.8. Example index

Table 1 summarizes the examples of dual and r-dual graded graphs given in the paper.

2.9. Enumerative corollaries

The identities of Section 1.5 can be formulated for each of the preceding examples of dual graphs. Listing all of them would take too much room. So we only apply Corollaries 1.5.3 and 1.5.4 to our main examples.

Table 1.

G_1	G_2	r	Example	Figure
Young graph $\mathbb{Y}$	G_1	1	2.1.2, 2.3.2	1
	G_1, weighted	$\forall$	2.2.7	
Young-Fibonacci graph $\mathbb{YF}$	G_1	1	2.1.4	2
The chain $\mathbb{N}$	G_1, weighted	1	2.2.1, 2.3.1	3
Pascal graph $\mathbb{N}^r$	G_1, weighted	r	2.2.2, 2.3.3	4
Diagrams with $\leq r$ rows	G_1, weighted	r	2.2.5, 2.3.4	
SemiPascal	G_1, weighted	2	2.2.5	5
Graph of shifted shapes $\mathbb{SY}$	G_1, weighted	1	2.2.8	6
The chain $\mathbb{Z}$	G_1	0	2.2.9	
t − *SkewStrip*	G_1	0	2.2.10	7
t − *SkewStrip* shapes	G_1, weighted	0	2.2.11	8
Infinite r-nary tree $\mathbb{T}^r$	See Ex. 2.3.5	r	2.3.5	9
Lifted binary tree	See Ex. 2.3.6	1	2.3.6	10
	BinWord	1	2.4.1	12
Fibonacci graph	See Def. 2.3.10	1	2.3.7	11
Bracket Tree	$J(\mathbb{T}^2)$	1	2.4.6	13
SYT-Tree	Schensted graph	1	2.5.2	14
SYFT-Tree	*InsYF*	1	2.5.7	15
Permutation tree (either $PT^{\uparrow}$ or $PT^{\downarrow}$)	*Permutation tree* (either $PT^{\rightarrow}$ or $PT^{\leftarrow}$)	1	2.6.8	16
Hex	See Fig. 17	1	2.7.1	17

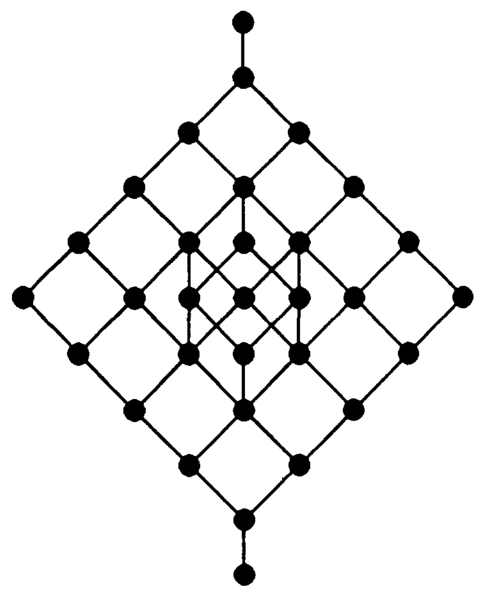


Figure 18. The reflected Young graph.

Most of the results stated below are well-known. However, their common origin was not previously realized.

Corollary 2.9.1 *Assume G is a self-dual graph with zero. Then*

(1) $\alpha(0 \to n \to 0) = n!$;

(2) *for any vertex x of rank k*

$$\sum_{y \text{ covers } x} e(y) = (k+1)e(x) .$$

In particular, these identities hold for the Young graph $\mathbb{Y}$ and the Young-Fibonacci graph $\mathbb{YF}$.

Proof: See (1.5.9) and (1.5.8) with $l = 1$. □

Corollary 2.9.2 *Assume G_1 and G_2 are dual graphs with zero, and G_2 is a tree. Then in G_1*

(1) $\alpha(0 \to n) = n!$;

(2) *every vertex of rank k is covered by exactly $k+1$ vertices (taking into account the edge multiplicities).*

In particular, this is the case when G_1 is one of the following: BinWord, $J(\mathbb{T}^2)$, Schensted graph, InsYF, Permutation Tree.

Proof: The proof is the same as for Corollary 2.9.1, using the fact that $e_2(y) = 1$ where the index 2 refers to G_2. Note that the first statement of the lemma follows from the second one. □

Corollary 2.9.3 *Assume a weight function w defined on a poset Q satisfies (2.2.2). Then in the distributive lattice $P = J(Q)$ the following identities hold:*

$$\sum_{x \in P_n} e(x)^2 \prod_{q \in x} w(q) = r^n n! \tag{2.9.1}$$

and if x is of rank k,

$$\sum_{\substack{y \text{ covers } x \\ y = x \cup \{q\}}} w(q) e(y) = r(k+1) e(x). \tag{2.9.2}$$

In particular, these identities are valid in the following distributive lattices with the weight functions w defined as indicated below:

(1) *The graph of shifted shapes* $\mathbb{SY}$*:*

$$w(q) = \begin{cases} 1, & \text{if } q \text{ is a diagonal box} \\ 2, & \text{otherwise;} \end{cases}$$

(2) *The graph of the diagrams with* $\leq r$ *rows:* $\quad w(q) = w(q_1, q_2) = r + q_1 - q_2;$

(3) *SemiPascal:* $\quad w(q) = w(q_1, q_2) = 2 + q_1 - q_2;$

(4) *The Young graph* $\mathbb{Y}$*:*

$$w(q) = w(q_1, q_2) = r + \alpha(q_1 - q_2) \tag{2.9.3}$$

where α is an arbitrary constant;

(5) *The Pascal graph* $\mathbb{N}^r = J(r\mathbb{P})$ *: w is the natural projection $r\mathbb{P} \to \mathbb{P}$.*

Proof: Similar to the proof of Corollary 2.9.1. □

Comments 2.9.4

1. In the case of the graph of shifted shapes $\mathbb{SY}$ the identity (2.9.1) reduces to Schur's formula (2.2.7).

2. In the case of the graph of r-row diagrams (2.9.1) turns into

$$\sum_{x=(x_1,\ldots,x_r)\in P_n} e(x)^2 \prod_{j=1}^{r} \frac{(x_j+r-j)!}{(r-j)!} = r^n n!$$

3. In the case of the Young graph we have a generalization of the Young-Frobenius formula:

$$\sum_{x\in P_n} e(x)^2 \prod_{(q_1,q_2)\in x} (1+\beta(q_1-q_2)) = n!$$

(combine (2.9.1), (2.9.3), and $\beta = \alpha/r$). In turn, (2.9.2)–(2.9.3) with $r = 0$ result in

$$\sum_{y:\, y \text{ covers } x} e(y)(q_1-q_2) = 0$$

where $y\backslash x = (q_1, q_2)$.

4. In the case of the Pascal graph $\mathbb{N}^r$ (2.9.1) becomes the classical $\sum e(x) = r^n$, where $e(x)$ denotes the appropriate multinomial coefficient.

Further applications are given in the sequel [7].

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