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The Tutte Polynomial as a Growth Function

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Abstract. The 'dollar game' represents a kind of diffusion process on a graph. Under the rules of the game some configurations are both stable and recurrent, and these are known as critical configurations. The set of critical configurations can be given the structure of an abelian group, and it turns out that the order of the group is the tree-number of the graph. Each critical configuration can be assigned a positive weight, and the generating function that enumerates critical configurations according to weight is a partial evaluation of the Tutte polynomial of the graph. It is shown that the weight enumerator can also be interpreted as a growth function, which leads to the conclusion that the (partial) Tutte polynomial itself is a growth function.

Keywords: Tutte polynomial, chip-firing, critical group, growth function

1. The main result

The *Tutte polynomial* [13] of a graph G can be defined as a sum taken over the set $\Sigma(G)$ of spanning trees of G:

$$\mathcal{T}(G; x, y) = \sum_{T \in \Sigma(G)} x^{i(T)} y^{j(T)},$$

where $i(T)$ and $j(T)$ are non-negative integers associated with the spanning tree T. The fundamental property of $\mathcal{T}$ is that it satisfies a 'deletion-contraction' equation (see Section 7). Partial evaluations of the Tutte polynomial occur in a wide variety of seemingly unrelated situations: the graph-colouring polynomial and the Jones polynomial of a knot or link being just two examples [2, 14].

Recently it has been observed [1] that the set $\Sigma(G)$ is in bijective correspondence with several other sets of objects associated with G. In fact all these sets are instances of an abelian group $K(G)$, which has a natural presentation in terms of G. The main result of this paper is that the reciprocal polynomial of $\mathcal{T}(G; 1, z)$ is the growth function $\mathcal{L}(z)$ of $K(G)$ with respect to its natural presentation:

$$z^c \mathcal{T}(G; 1, z^{-1}) = \mathcal{L}(z) = \sum_{g \in K(G)} z^{L(g)},$$

where c is the cycle-rank of G and $L(g)$ is the length of g in $K(G)$. It should be noted [14] that this partial evaluation of the Tutte polynomial is precisely the one which measures the 'reliability' of a graph with respect to edge-failures, when the probability of an individual failure is $q = z^{-1}$.

The basis of the proof is the observation that manipulations involving the natural set of generators and relations for $K(G)$ correspond to moves in the so-called 'dollar game' on G [4]. The details are explained in Sections 9 and 10.

The dollar game is a version of the chip-firing game discussed by mathematicians [5, 10], and is closely related to a model developed by physicists which uses the terminology of 'sandpiles' and 'avalanches'. Gabrielov [6, 7] showed that several quantities associated with the avalanche model satisfy an equation related to the deletion-contraction equation and, in particular, he observed [7, p. 267] that a certain polynomial has this property. His arguments are based on geometrical ideas.

Using graph-theoretical methods, Merino Lopez [9] has shown that the generating function $\mathcal{C}(z)$ for critical configurations in the dollar game is equal to $\mathcal{T}(1, z)$ (Theorem A). We shall establish a correspondence between critical configurations and minimal representations of elements of the group $K(G)$, which leads to the result (Theorem B) that $\mathcal{L}(z) = z^c\mathcal{C}(z^{-1})$. The main result follows from these two theorems.

2. The dollar game

We shall consider a finite graph G consisting of a vertex-set V, an edge-set E, and an incidence relation such that each edge is incident with one or two vertices. Thus both loops and multiple edges are allowed. We denote by $\nu(v)$ the number of loops at a vertex v, and by $\nu(v, w)$ the number of edges joining the vertices v and w. For the avoidance of doubt, the *degree* of v is defined to be

$$deg(v) = 2\nu(v) + \sum_{w \neq v} \nu(v, w).$$

We shall assume, without always mentioning it explicitly, that *G is connected*.

Suppose that each vertex of G has a number of dollars, except for one vertex q, the 'government', which is in debt by the total amount of dollars held by the rest. The operation which we shall call *firing* a vertex v consists of transferring dollars from v along the edges incident with v. Two dollars are transferred around each loop at v (since we count a loop as being twice-incident with its vertex), and one dollar is transferred along each other edge incident with v. The former operation has no effect, since the two dollars return to v, but it is necessary to include it for the sake of consistency. We insist that a vertex $v \neq q$ can be fired if and only if v has at least as many dollars as incident edges. However, this restriction does not apply to q, because firing q merely increases its debt. This *dollar game* [4] is a variant of what is usually called a *chip-firing game* on the graph [5].

We describe briefly a few basic results from [4], which are in turn derived from [5]. The dollar game can be defined formally as follows. A *configuration* on (G, q) is an integer-valued function s defined on V such that

$$s(v) \geq 0 \ (v \neq q), \quad s(q) = -\sum_{v \neq q} s(v).$$

Let us say that vertex $v \neq q$ is *ready* in a configuration s if $s(v) \geq deg(v)$. If v is ready in s, then it can be *fired*, which results in a new configuration s' defined by

$$s'(x) = s(x) + \nu(x, v), \quad \text{if } x \neq v;$$

$$s'(v) = s(v) - deg(v) + 2\nu(v).$$

In particular, if the graph is simple (no loops or multiple edges), vertices adjacent to v gain one dollar, vertices not adjacent to v are unaffected, and v itself loses $deg(v)$ dollars. We denote by F_v the operator which takes s to s', and when $v \neq q$ we say that F_v is *legal* for s if and only if $s(v) \geq deg(v)$.

The first result we need is Lemma 3.1 in [4].

Lemma 1 *Any sequence of legal firings F_v, $v \neq q$ which starts from a given configuration on (G, q) has finite length.*

If no vertex $v \neq q$ is ready in s, then we say that s is a *stable* configuration. Lemma 1 says that, starting from any configuration and firing vertices other than q, we shall eventually reach a stable configuration. In that situation, and in that situation only, we allow the firing of q; in other words, F_q is defined to be *legal* for s if and only if s is stable.

3. Critical configurations

The fact that we are allowed to use F_q if there is no alternative, means that firing can continue indefinitely. But Lemma 1 tells us that an infinite sequence of legal firings must contain F_q infinitely often, and consequently it must produce an infinite number of stable configurations. Since the number of distinct stable configurations is finite, there must be at least one, say r, which is recurrent. In other words, there is a non-empty finite sequence of legal firings which starts and ends with the same stable configuration r.

We say that a configuration on (G, q) is *critical* if it is stable and recurrent. The preceding remarks imply that, starting from any configuration and applying a sequence of legal firings (including F_q if necessary), we shall eventually reach a critical configuration.

In general, not every stable configuration is critical. For example, in the complete bipartite graph $K_{3,3}$ there are 5 vertices $v \neq q$ and each has degree 3, so there are 3^5 stable configurations. But only 3^4 of them are critical.

Suppose that $\mathcal{S}$ is a non-empty finite sequence of (not necessarily distinct) vertices of G, such that starting from s, the vertices can be fired legally in the order of $\mathcal{S}$. If v occurs $x(v)$ times, we shall refer to x as the *representative vector* for $\mathcal{S}$. The configuration s' after the sequence of firings $\mathcal{S}$ is given by

$$s'(v) = s(v) - x(v)[deg(v) - 2\nu(v)] + \sum_{w \neq v} x(w)\nu(v, w).$$

This is because each time v is fired it loses (effectively) $deg(v) - 2\nu(v)$ dollars, and each time a vertex $w \neq v$ is fired v gains $\nu(v, w)$ dollars. The relationship between s and s' can be written more concisely if we define the *Laplacian matrix* Q as follows:

$$(Q)_{vw} = \begin{cases} -\nu(v, w), & \text{if } v \neq w; \\ deg(v) - 2\nu(v), & \text{if } v = w. \end{cases}$$

In terms of Q the relationship between s and s' is then

$$s' = s - Qx.$$

The following lemma shows that there are severe restrictions on a sequence of firings under which a configuration recurs.

Lemma 2 *Any sequence of legal firings in which each vertex occurs the same number of times produces a final configuration which is the same as the initial one. Conversely, in any sequence of legal firings under which a configuration recurs, each vertex is fired the same number of times. If such a sequence exists for a given configuration, then there is a sequence in which every vertex is fired just once.*

Proof: See [4, Sections 2 and 3]. □

4. Critical sequences

Suppose that G has n vertices, and let $\pi : \{1, 2, \ldots, n\} \to V$ be a bijection. We shall say that π is a *critical sequence* on (G, q) if the sequence

$$\pi(1), \pi(2), \pi(3), \ldots, \pi(n)$$

of vertices of G has the following properties:

[C1]: $\pi(1) = q$;

[C2]: for each $j = 2, 3, \ldots, n$ there is an $i < j$ such that $\pi(i)$ is adjacent to $\pi(j)$.

A critical sequence may also be thought of as a *total order* on the vertex-set V, satisfying the conditions that q comes first [C1], and every other vertex is preceded by at least one neighbour [C2]. There is at least one critical sequence on (G, q), because any total order which is consistent with distance from q has these properties.

The relationship between critical sequences and critical configurations is clarified in the following lemma.

Lemma 3

(i) *If the configuration c is critical, so that it recurs under a sequence of firings in which every vertex occurs just once, then this sequence is a critical sequence.*

(ii) *For every critical sequence π there is a critical configuration c_π which recurs under π.*

Proof:

(i) Since c is stable q must be fired first, so [C1] holds. When a vertex v is fired, it must be ready at that stage. But initially v is not ready (because c is stable), so at least one neighbour of v must be fired before v. Thus [C2] holds.

(ii) Suppose that π is a critical sequence. Define $B_\pi(v)$ to be the set of edges which join v to vertices which come before v in the order defined by π, that is,

$$B_\pi(v) = \{e \in E \mid e \text{ has vertices } v, w \text{ such that } \pi^{-1}(w) < \pi^{-1}(v)\}.$$

Since condition [C2] is satisfied, $B_\pi(v)$ is not empty. Thus if we define

$$c_\pi(v) = deg(v) - |B_\pi(v)| \quad (v \neq q),$$

we have $c_\pi(v) \leq deg(v) - 1$, and c_π is a stable configuration.

Suppose we try to fire the vertices in the order π, starting from the configuration c_π. Firing $q = \pi(1)$ first is legal, since c_π is stable. Suppose all firings are legal until we come to fire $\pi(i) = v$. The total number of dollars held by v at that stage is the initial number, $c_\pi(v)$, plus the number of edges joining v to vertices which have been fired before v, $|B_\pi(v)|$. By the definition of $c_\pi(v)$ this number is equal to the degree of v, and so it is legal to fire v. Hence we have a legal sequence of firings, containing each vertex just once, and it follows that c_π is recurrent. □

The function from the set S of critical sequences to the set Γ of critical configurations defined by $\pi \mapsto c_\pi$ is neither a surjection nor an injection. If we are given a critical configuration c then, according to Lemma 2, there is at least one critical sequence π under which c recurs, but c may not be equal to c_π. This situation will be analysed in the next section.

Furthermore, there may be distinct critical sequences π and σ such that $c_\pi = c_\sigma$. In other words, if we partition S into equivalence classes by saying that π and σ are equivalent if and only if $c_\pi = c_\sigma$, then the classes may have more than one member. For example, if σ is obtained from π by transposing two vertices which are consecutive in π but not adjacent in G, then σ and π are in the same equivalence class. The following lemma is an important step towards the characterisation of the equivalence classes.

Lemma 4 *Let π and σ be critical sequences such that $c_\pi = c_\sigma$. Suppose there are vertices x and y such that x comes before y in π and x comes after y in σ. Then x and y are not adjacent.*

Proof: Suppose, for a contradiction, that x and y are the ends of an edge e. Then e is not in $B_\pi(x)$ but it is in $B_\sigma(x)$. Since $c_\pi = c_\sigma$, the two sets have the same size, and there must be an edge f which is not in $B_\sigma(x)$ but is in $B_\pi(x)$. In other words, there is a neighbour w of x which comes before x in π, and after x in σ.

Now we can repeat the argument with x, y, and e replaced by w, x, and f; and so on, indefinitely. This is clearly impossible, so x and y cannot be adjacent. □

5. The index of a configuration

For any configuration s denote by $M(s)$ the associated ‘money-supply’:

$$M(s) = \sum_{v \neq q} s(v) = -s(q).$$

If G has n vertices and m edges, and s is stable on (G, q), we have

$$M(s) = \sum_{v \neq q} s(v) \leq \sum_{v \neq q} [deg(v) - 1] = 2m - deg(q) - n + 1.$$

When c is critical, the following lemma provides a lower bound for $M(c)$. It is related to Theorem 3.3 in [5].

Lemma 5 *Let G be a connected graph with m edges, l of which are loops, and let q be any vertex of G. Then for any critical configuration c on (G, q) we have*

$$M(c) \geq m + l - deg(q).$$

Proof: Consider a critical sequence for c: in the course of this sequence some dollars are transferred. Think of the dollars as real dollar bills, and mark those that are transferred according to the following rule.

- Each edge e is incident with two vertices, say a and b, where if e is a loop then $a = b$. Suppose a is the first of these vertices to be fired. Mark a dollar bill which is transferred from a to b with the label e. If e is a loop at a, two dollars return immediately to a, both labelled e. If e is not a loop, the vertex b is fired subsequently, at which stage the dollar marked e is returned to a.

Since every vertex is fired just once, at the end of the process there are $2l$ dollar bills marked with the labels of loops, and $m - l$ dollar bills marked with the labels of the edges which are not loops. That is, there are $m + l$ marked dollar bills altogether. However, the dollars marked when q was fired (which was necessarily first) have returned to q, and there are $deg(q)$ of these. The remaining $(m + l) - deg(q)$ marked dollars are still in circulation. The final configuration is c, and so $M(c) \geq (m + l) - deg(q)$, as claimed. □

It is convenient to denote $m - deg(q)$ by M_0 and to define the *index* of a configuration s to be the integer

$$i(s) = M(s) - M_0 = M(s) - (m - deg(q)).$$

Lemma 5 shows that if c is a critical configuration then $i(c) \geq l$. Since c is stable, the calculation preceding the lemma shows that $i(c) = M(c) - M_0$ is at most $m - n + 1$. We shall refer to $\{l, l + 1, \ldots, m - n + 1\}$ as the *critical range* for the index.

Lemma 5 is a significant step towards identifying which stable configurations are critical. It says that a stable configuration whose index lies outside the critical range is not critical. On the other hand, it is not necessarily true that a stable configuration s is critical if $i(s)$ is in the critical range. We can make further progress towards characterising critical configurations by using the critical sequences defined in the previous Section. The following lemma is equivalent to Lemma 5 in that context.

Lemma 6 *Suppose that π is a critical sequence and c_π is the associated critical configuration defined in the proof of Lemma* 3. *Then $i(c_\pi) = l$, where l is the number of loops in G.*

Proof: By definition,

$$M(c_\pi) = \sum_{v \neq q} c_\pi(v) = \sum_{v \neq q} (deg(v) - |B_\pi(v)|).$$

In other words, $c_\pi(v)$ is the number of incidences between v and edges which do not join v to vertices preceding it in π. Suppose that e is an edge with distinct vertices x, y, labelled so that x comes before y in π. If $x \neq q$, the edge e contributes 1 to $M(c_\pi)$, by virtue of the term $c_\pi(x)$. If $x = q$ then e makes no contribution. So the non-loops contribute in all $(m - l) - deg(q)$. Every loop at a vertex v contributes 2 to $M(c_\pi)$ by virtue of the term $c_\pi(v)$, and so the contribution of the loops is $2l$. Thus the total is $m + l - deg(q)$, and we have

$$i(c_\pi) = M(c_\pi) - M_0 = (m + l - deg(q)) - (m - deg(q)) = l. \qquad \square$$

We can now throw some light on the structure of the set of all critical configurations.

Lemma 7 *Let c be a critical configuration and let s be a stable configuration such that, for all $v \neq q$, $s(v) \geq c(v)$. Then s is critical.*

Proof: Since c is critical, there is a critical sequence associated with it. The condition $s(v) \geq c(v)$ implies that the same sequence is legal for s, and so s is critical. □

Lemma 8 *If π is a critical sequence for c, then $c(v) \geq c_\pi(v)$ for all $v \neq q$; and if $i(c) = l$, then $c = c_\pi$.*

Proof: If π is a critical sequence for the configurations c_1 and c_2, it is also a critical sequence for the configuration c_m defined by

$$c_m(v) = \min\{c_1(v), c_2(v)\} \quad (v \neq q).$$

In this result, take $c_1 = c$ and $c_2 = c_\pi$. It follows that, since π is a critical sequence for c and c_π, π is also critical sequence for c_m. If $c(x) < c_\pi(x)$ for some vertex x, then c_m would be a critical configuration with $M(c_m) < M(c_\pi) = M_0 + l$, contradicting Lemma 5. Hence $c(v) \geq c_\pi(v)$ for all $v \neq q$. Finally, if $i(c) = l$ we must have $M(c) = M_0 + l = M(c_\pi)$ and so $c = c_\pi$. □

Corollary *Suppose that G has no loops and c is a critical configuration of index* 0 *on (G, q). Then there is a vertex z such that $c(z) = 0$.*

Proof: By the lemma, $c = c_\pi$ for some critical sequence π. According to the definition of c_π, the vertex $z = \pi(n)$ has the required properties. □

Lemmas 7 and 8 provide a useful characterisation of the set Γ of all critical configurations. We can think of the configurations as points of the integer lattice $\mathbf{Z}^{n-1}$, with the natural partial order $\leq$ defined by $b \leq c$ if and only if $b(v) \leq c(v)$ for all $v \neq q$. With respect to this order there is a unique maximal element $c^\sharp$ of Γ, given by

$$c^\sharp(v) = deg(v) - 1 \quad (v \neq q).$$

The set of minimal elements of Γ is the set Γ_l of critical configurations with index l. The lemmas assert that if we know $c^\sharp$ and Γ_l, the entire set Γ is determined:

$$\Gamma = \{c \in \mathbf{Z}^{n-1} \mid b \leq c \leq c^\sharp \text{ for some } b \in \Gamma_l\}.$$

Thus we have a method of constructing all the critical configurations on (G, q). First, we write down the critical sequences π. Lemma 8 implies that every critical configuration with index l occurs as a c_π, although (for the reasons given at the end of Section 4), there may be repetitions. The critical configurations with index greater than l are then obtained by writing down the stable configurations which 'cover' the critical ones with index l. An example follows.

Example Let $K_{3,3}$ be the complete bipartite graph with two classes $A = \{v, w, x\}$ and $B = \{q, r, s\}$. Here $l = 0$, so the critical range is $0 \leq i(c) \leq 4$, corresponding to values of $M(c)$ between the minimum $M_0 = m - deg(q) = 9 - 3 = 6$ and the maximum, $M_0 + (m - n + 1) = 10$.

As usual we take q to be the 'government'. In any critical sequence q must come first. Of the remaining 5 vertices, 3 are in class A and 2 in class B, so there are $5!/3!2! = 10$ patterns (such as *AAABB*) for a sequence of these 5 vertices; and each pattern gives rise to 12 vertex-sequences. Condition [C2] for a critical sequence is satisfied if and only if q is followed by a class A vertex. So the number of allowable patterns is six: *AAABB*, *AABAB*, *AABBA*, *ABAAB*, *ABABA*, *ABBAA*. Transposing two vertices in the same class results in an equivalent critical sequence, and in this graph every A-vertex is adjacent to every B-vertex, so no other transpositions are allowed. Hence the number of equivalence classes of critical sequences corresponding to the six patterns is 1, 6, 3, 6, 12, 3, respectively. Thus we get $1 + 6 + 3 + 6 + 12 + 3 = 31$ critical configurations.

The following table contains one critical sequence π and the configuration c_π, for each of the six allowable patterns. The remaining ones can be obtained by permuting v, w, x and r, s.

		v	w	x	r	s
$AAABB$	$qvwxrs$	2	2	2	0	0
$AABAB$	$qvwrxs$	2	2	1	1	0
$AABBA$	$qvwrsx$	2	2	0	1	1
$ABAAB$	$qvrwxs$	2	1	1	2	0
$ABABA$	$qvrwsx$	2	1	0	2	1
$ABBAA$	$qvrswx$	2	0	0	2	2

Note that the complete list of 31 critical configurations with index 0 does not include every stable configuration with index 0. For example, 21111 is stable but does not occur in the list; the Corollary to Lemma 8 confirms that it is not critical (there is no 0).

Now we can construct recursively the critical configurations with index greater than 0, by increasing the numbers at each vertex. For example, the critical configuration 20022 is 'covered' by the critical configurations 21022, 20122, 21122, 22022, 20222, 22122, 21222, and 22222. Using this method it turns out that there are 29, 15, 5, 1 critical configurations of index 1, 2, 3, 4 respectively, giving 81 altogether.

6. Allowable orientations

Let E^- be the set of edges of G which are not loops. An *orientation* of G is a function h which assigns to each $e \in E^-$ one of its incident vertices $h(e)$. We call $h(e)$ the *head* of e. The other vertex of e is called the *tail* of e and is denoted by $t(e)$. Usually we think of e as being marked with an arrow which points from $t(e)$ to $h(e)$; it is worth stressing that a loop has no arrow. Given an orientation h of G, the *in-degree* of a vertex v is defined to be

$$in_h(v) = |\{e \in E^- \mid h(e) = v\}|.$$

We say that h is *acyclic* if there is no h-oriented cycle, that is, no sequence of edges $e_1, e_2, e_3, \ldots, e_r$ such that $h(e_1) = t(e_2), h(e_2) = t(e_3), \ldots, h(e_r) = t(e_1)$.

The relationship between acyclic orientations and chip-firing was noted in [5]. In a different context, the earlier paper of Greene and Zaslavsky [8] contains results equivalent to those stated below as Lemmas 9 and 10, and the proofs of those lemmas are therefore omitted.

Our motivation for considering orientations comes from the observation (Section 4) that $\pi \mapsto c_\pi$ is not an injection. Following the lead provided by Lemmas 4 and 6, we shall identify a set of orientations which is in bijective correspondence with the set of critical configurations of minimal index l. Specifically, we say that an orientation h is *allowable* on (G, q) if it satisfies the two following conditions.

[O1]: h is acyclic.

[O2]: $in_h(q) = 0$ and $in_h(v) \neq 0$ for all $v \neq q$.

The condition [O1] that h is acyclic implies that h defines a partial order $\prec$ on the vertex-set V, such that $t(e) \prec h(e)$ for all $e \in E^-$. We shall say that a total order π on V is an *extension* of h if and only if $t(e)$ comes before $h(e)$ in π, for all $e \in E^-$. It is a standard result that any partial order has an extension.

Given an allowable orientation h, define

$$c_h(v) = deg(v) - in_h(v) \quad (v \neq q).$$

Lemma 9 *The total ordering π is an extension of an allowable orientation h if and only if it is a critical sequence for c_h. Furthermore, c_h is a critical configuration with index l.*

Lemma 10 *The map $h \mapsto c_h$ is a bijection from the set of allowable orientations to the set of critical configurations with index l on (G, q).*

Let e be an edge of G which is neither a loop nor a *co-loop* (its removal does not result in a disconnected graph). Suppose the vertices incident with e are x and y. Define $G - e$ to be the graph obtained by *deletion* of e, that is, the graph with the same vertices as G and all its edges except e. Define G/e to be the graph obtained by *contraction* of e, that is, the graph whose vertex-set is obtained by replacing x and y by a new vertex $*$, and replacing every edge in G incident with x or y by an edge incident with $*$. Note that the edge e does not correspond to any edge of G/e, but if there are other edges in G joining x and y (that is, if $\nu(x, y) \geq 2$) then these edges become loops incident with the vertex $*$ in G/e.

Lemma 11 *Suppose that $G - e$ and G/e are the graphs formed by the deletion and contraction of an edge e of G. Let $\alpha(G, q)$ be the number of allowable orientations on (G, q). Then*

$$\alpha(G, q) = \alpha(G - e, q) + \alpha(G/e, q).$$

Proof: Using Lemma 10, this follows from a more general result given in the following section. □

7. Counting critical configurations

For each $i \geq 0$ let $\Gamma_i(G, q)$ denote the set of critical configurations on (G, q) which have index i, and let

$$\gamma_i(G, q) = |\Gamma_i(G, q)|, \quad \Gamma(G, q) = \bigcup_{i \geq 0} \Gamma_i(G, q).$$

We define the generating function $\mathcal{C}(z) = \mathcal{C}(G; z)$ as follows:

$$\mathcal{C}(G; z) = \sum_{c \in \Gamma(G,q)} z^{i(c)} = \sum_{i=l}^{m-n+1} \gamma_i(G, q) z^i.$$

For example, the calculations for $K_{3,3}$ given in Section 5 yield the result

$$\mathcal{C}(K_{3,3}; z) = 31 + 29z + 15z^2 + 5z^3 + z^4.$$

Merino Lopez [9] has established an alternative characterisation of $\mathcal{C}(G; z)$ (Theorem A below). In order to state it we need to recall the definition of the Tutte polynomial.

Let G be a connected graph and T a spanning tree of G. For each edge $g \in T$ there is a unique cut consisting of all the edges which have one in end in each of the components obtained by deleting g from T. We denote this by $\mathrm{cut}(T, g)$; it contains g itself and edges which are not in T. For each edge h which is not in T there is a unique cycle consisting of h and edges which are in T; we denote this by $\mathrm{cyc}(T, h)$.

Assume that the edges of G are given a fixed ordering $e_1, e_2, \ldots, e_m$. Suppose $e_i \in T$. Then we say that e_i is *internally active* if i is the least index of any edge in $\mathrm{cut}(T, e_i)$. Similarly, if $e_j \notin T$, we say that e_j is externally active if j is the least index of any edge in $\mathrm{cyc}(T, e_j)$. The *internal* (*external*) *activity* of T is defined to be the number of edges which are internally (externally) active. Denoting these quantities by $int(T)$ and $ext(T)$ respectively, we define a polynomial in two variables

$$\sum_T x^{int(T)} y^{ext(T)}.$$

It can be shown that this polynomial is independent of the edge-ordering used in its definition, and it is known as the *Tutte polynomial* of G, denoted by $\mathcal{T}(G; x, y)$. In other words,

$$\mathcal{T}(G; x, y) = \sum t_{ij} x^i y^j,$$

where t_{ij} is the number of spanning trees with internal activity i and external activity j, in any fixed ordering of the edges.

We can now state the theorem of Merino Lopez [9].

Theorem A *Let G be a connected graph with Tutte polynomial $\mathcal{T}(x, y)$. Then for any vertex q of G the generating function $\mathcal{C}(z)$ for critical configurations on (G, q) is given by*

$$\mathcal{C}(z) = \mathcal{T}(1, z).$$

It follows that $\sum_i t_{ij}$, the total number of spanning trees with external activity j, is equal to the number of critical configurations with index j. Since the set of spanning trees which contribute to $\sum_i t_{ij}$ depends on the chosen ordering of the edges, we cannot expect a 'bijective' proof of this fact.

In Tutte's original paper [13] it is shown that $\mathcal{T}$ satisfies the deletion-contraction equation:

$$\mathcal{T}(G; x, y) = \mathcal{T}(G - e; x, y) + \mathcal{T}(G/e; x, y).$$

It follows from Theorem A that the polynomial $\mathcal{C}(z)$ and its coefficients γ_i also satisfy this equation. In fact, Merino Lopez [9] proves the following result.

Lemma 12 *Let e be an edge incident with q, and let q^* be the vertex obtained by contracting e. Then for all $i \geq 0$, the set $\Gamma_i(G, q)$ has a natural decomposition into two parts, which are in bijective correspondence with $\Gamma_i(G - e, q)$ and $\Gamma_i(G/e, q^*)$ respectively. Thus*:

$$\gamma_i(G, q) = \gamma_i(G - e, q) + \gamma_i(G/e, q^*).$$

Note added in Proof: The author is grateful to a referee for pointing out that the characterisation of the minimal elements of Γ obtained in Section 5, together with Theorem A, resolves a conjecture of Stanley [12, p. 59] in the case of graphs. Roughly speaking, the coefficients of the Tutte polynomial $\mathcal{T}(1, z)$ can be represented by the cardinalities of certain sets with nice properties.

8. The critical group

Let G be a connected graph and let $C^0 = C^0(G; \mathbf{Z})$ denote the abelian group of integer-valued functions defined on V. Associated with the matrix Q defined in Section 3 we have the *Laplacian homomorphism* $Q : C^0 \to C^0$ defined by

$$(Qf)(v) = (deg(v) - 2\nu(v))\, f(v) - \sum_{x \in V} \nu(v, x)\, f(x).$$

If $\sigma : C^0 \to \mathbf{Z}$ is the homomorphism defined by

$$\sigma(f) = \sum_{v \in V} f(v),$$

then is it easily verified that $\sigma Q = 0$, so Im Q is a subgroup of Ker σ. The quotient group

$$K(G) = \text{Ker } \sigma / \text{Im } Q$$

will be called the *critical group* of G. It has also been referred to as the *Jacobian* group [1, 11].

Let q be a vertex of G. Denoting by δ_u the function which takes the value 1 at u and 0 at every other vertex, we see that for each $u \neq q$ the function $\delta_u - \delta_q$ is in Ker σ. Let $g_u = [\delta_u - \delta_q]$, the coset of this function with respect to Im Q.

It can be shown [4, Theorem 8.1] that $\{g_u \mid u \neq q\}$ is a set of generators for $K(G)$. Furthermore, these generators satisfy a canonical set of relations, which we shall call the *Picard presentation*. The reason for this name is the analogy with the Picard group in Algebraic Geometry [1, 3]. Since the presentation of the group (but not the group itself) depends on the choice of q, we shall denote it by $K(G, q)$ in this context.

Specifically, in the Picard presentation $K(G, q)$ there is a relation R_v for each $v \neq q$:

$$R_v : deg(v) \cdot g_v = 2\nu(v) \cdot g_v + \sum_{w \neq q} \nu(v, w) \cdot g_w.$$

Adding all these relations we obtain an important consequence, which we shall call R_q.

$$R_q : \sum_{u \neq q} \nu(q,u) \cdot g_u = 0.$$

Any configuration s on (G, q) corresponds to a *representation* of an element g of $K(G, q)$, defined by $g = \sum s(u)g_u$. We say that this is a *minimal representation* of g if any other representation $\sum s'(u)g_u$ of the same element g satisfies $\sum s'(u) \geq \sum s(u)$. The *length* $L(g)$ of g is defined to be $\sum s(u)$, where $\sum s(u)g_u$ is a minimal representation.

We define the *growth function* of $K(G, q)$ to be the polynomial function $\mathcal{L}$ given by the formula

$$\mathcal{L}(z) = \sum_{g \in K(G,q)} z^{L(g)}.$$

Example Let $K_{3,3}$ be the complete bipartite graph with the notation as in Section 5. The Picard presentation has 5 generators which (writing r instead of g_r and so on) are r, s, v, w, x. The relations are:

$$R_r: 3r = v + w + x$$
$$R_s: 3s = v + w + x$$
$$R_v: 3v = r + s$$
$$R_w: 3w = r + s$$
$$R_x: 3x = r + s.$$

The additional relation is $R_q: v + w + x = 0$.

In Section 11 we shall describe a general method for listing minimal representations. In a small case like this, elementary algebra and dogged persistence are sufficient. In the following list, only one of each type of minimal representation is listed—that is, v stands for any one of v, w or x, and so on. Each type is followed by the number of minimal representations of that type, in square brackets.

Length 0: 0 [1];
Length 1: r [2], v [3];
Length 2: $2r$ [2], $r + s$ [1], $r + v$ [6], $2v$ [3], $v + w$ [3];
Length 3: $2r + s$ [2], $2r + v$ [6], $r + s + v$ [3], $r + 2v$ [6], $r + v + w$ [6], $2v + w$ [6];
Length 4: $2r + 2s$ [1], $2r + s + v$ [6], $2r + v + w$ [6],
$r + v + 2w$ [12], $r + s + 2v$ [3], $2v + 2w$ [3].

Observe that, for example, the type $v + w + r + s$ does not appear in the list, because it reduces to $2x$. Counting the types we obtain:

$$\mathcal{L}(z) = 1 + 5z + 15z^2 + 29z^3 + 31z^4.$$

In general, an element g of $K(G, q)$ may have more than one representation of length $L(g)$.

9. The dollar game and the Picard presentation

A representation $g = \sum s(u)g_u$ of an element of $K(G, q)$ is associated with a configuration s for the dollar game (where the definition of s is extended to q by defining $s(q) = -\sum s(u)$, so that s is in Ker σ). The coset $[s] \in K(G, q)$ is just g, since

$$g = \sum_{u \neq q} s(u)g_u = \sum_{u \neq q} s(u)[\delta_u - \delta_q] = \left[\sum_{u \neq q} s(u)\delta_u - \left(\sum_{u \neq q} s(u)\right)\delta_q\right] = [s].$$

There is an obvious connection between applying a relation R_v to the representation $\sum s(u)g_u$ and firing the vertex v in the configuration s. In this context it is helpful to think of R_v as a *rewriting rule*, rather than an identity:

$$R_v\colon \mathit{deg}(v) \cdot g_v \;\longmapsto\; 2\nu(v)g_v + \sum_{w \neq q} \nu(v, w)g_w.$$

If $s(v) \geq \mathit{deg}(v)$ we can apply the rewriting rule R_v and collect up the terms using the abelian group laws. The result is a representation $\sum t(u)g_u$, and the associated configuration t is the result of applying F_v to s.

Similarly, we can express the additional relation R_q in the following way (chosen to conform with our definition of the firing F_q):

$$R_q\colon 0 \;\longmapsto\; \sum_{u \neq q} \nu(q, u)g_u.$$

Lemma 13 *Each element of $K(G, q)$ has a representation $\sum s(u)g_u$ for which s is stable*; *that is*, $0 \leq s(u) \leq \mathit{deg}(u) - 1$ *for all $u \neq q$*.

Proof: Let $g = \sum f(u)g_u$ be any element of $K(G, q)$, remembering that the values of f may be negative, and define $f(q) = -\sum f(u)$. Let l be the configuration defined on vertices $u \neq q$ by

$$l(u) = \begin{cases} \mathit{deg}(u) - 1 & \text{if } f(u) \geq 0, \\ \mathit{deg}(u) - 1 - f(u) & \text{if } f(u) < 0. \end{cases}$$

Although l is not necessarily stable, it follows from Lemma 1 that there is a finite sequence of legal firings which reduces l to a stable configuration k. If this sequence has representative vector x, we have $k = l - Qx$. Let $z = f + l - k$, so that $z = f + Qx$. Then $[z] = [f]$, and

$$z(u) = f(u) + l(u) - k(u) \geq \mathit{deg}(u) - 1 - k(u) \geq 0.$$

Hence $\sum z(u)g_u$ is a representation of the given element g.

If z is stable, we are finished. If not, it follows from Lemma 1 again that we can apply the rules R_v $(v \neq q)$ until we are forced to stop. At this stage we have a representation for which the associated configuration is stable. □

10. The unique critical representative

Recall that in Section 3 we established the following result. If we start from any configuration of the dollar game, and carry out a sequence of legal firings, we must eventually arrive at a critical configuration—that is, a stable configuration which recurs. A fundamental result about the dollar game is the following.

Lemma 14 *Let s be a configuration on (G, q). Then there is a unique critical configuration which can be reached by a legal sequence of firings, starting from s.*

Proof: [4, Theorem 3.8]. □

This result has a simple interpretation in terms of the group $K(G, q)$. Any configuration s satisfies $\sigma(s) = 0$, and so defines a coset $[s]$ in $K(G, q)$. If s' is obtained from s by a sequence of legal firings, it is of the form $s' = s - Qx$, and so it belongs to the same coset $[s]$. Lemma 14 asserts that each coset $[s]$ has a unique critical representative.

This explains the name 'critical group' for $K(G, q)$. Indeed, we can think of the critical configurations themselves as the elements of a group. In that case, we must define an abelian group operation $\bullet$ on the set of critical configurations so that the coset of $c_1 \bullet c_2$ is the sum of the cosets $[c_1]$ and $[c_2]$ in $K(G, q)$. This implies that we must take $c_1 \bullet c_2$ to be the unique critical representative of $[c_1 + c_2]$—in other words, the unique critical configuration which can be obtained by applying a sequence of legal firings to $c_1 + c_2$. More details can be found in [4].

Our purpose here is to consider the index of the critical representative c of $[s]$ when s is stable. The configuration s may itself be critical, in which case $c = s$. If s is not critical, then there is a sequence of legal firings, starting with F_q, which leads from s to c. If $i(s) < 0$ then, because $i(c) \geq 0$, it follows that $i(s) < i(c)$. This may still be true even if $i(s)$ is in the critical range. For example, it was pointed out in Section 5 that the stable configuration $s = 21111$ on $K_{3,3}$ is not critical, even though its index is 0, which is in the critical range. In this case the critical representative $c = 02222$ is obtained as follows:

$$21111 \xrightarrow{q} 32211 \xrightarrow{x} 02222,$$

and we have $i(c) > i(s)$ again. However, it is possible for there to be a stable (but not critical) element s whose critical representative c is such that $i(s) = i(c)$.

Example Let G consist of a 5-cycle q, x, t, u, z together with a vertex y and two edges joining y to q and x. So $M_0 = 7 - 3 = 4$. Denote by $abcde$ the values of a configuration at the vertices x, y, z, t, u. Then we have a sequence of legal firings

$$s = 21100 \xrightarrow{q} 32200 \xrightarrow{x} 03210 \xrightarrow{y} 11210 \xrightarrow{z} 11011 = c.$$

It is easy to check that c is recurrent, and clearly $i(s) = i(c) = 0$.

Lemma 15 *Let s be a stable configuration and c the unique critical representative of the coset $[s]$. Then $i(c) \geq i(s)$.*

Proof: The general theory asserts that there is a sequence of legal firings

$$s \xrightarrow{q} s' \longrightarrow \cdots \longrightarrow c.$$

Suppose that the set X of vertices other than q which are fired more than once in the complete sequence is not empty, and let $x \in X$ be the vertex whose second firing occurs first in the sequence. If t is the configuration immediately before this second firing, and $n(w)$ is the number of firings of w up to this point, we have

$$t(x) = s(x) - deg(x) + 2\nu(x) + \Sigma, \quad \text{where } \Sigma = \sum_{w \neq x} n(w)\nu(w, x).$$

Since $s(x) < deg(x)$, and $t(x) \geq deg(x)$, we must have $2\nu(x) + \Sigma > deg(x)$. It follows that $n(w) > 1$ for at least one $w \neq x$ and, by the definition of x, we must have $w = q$. We have shown that no vertex can be fired a second time until q has been fired twice.

In the course of the sequence of firings M initially increases by $deg(q)$ when q is fired, and decreases by $\nu(q, y)$ every time a neighbour y of q is fired. However, as we have shown, no neighbour of q can be fired more than once unless q is fired again. Hence the index cannot fall below its initial value unless q is fired again. But then we can repeat the same argument. Hence $M(c) \geq M(s)$, and consequently $i(c) \geq i(s)$ as claimed. □

11. Counting minimal representations

In Section 5 we noted that the configuration $c^\sharp$ defined by $c^\sharp(v) = deg(v) - 1$ for all $v \neq q$ is the unique maximal critical configuration; in fact $M(c^\sharp) = M_0 + (m - n + 1)$, and so the index of $c^\sharp$ is $m - n + 1$.

We define the *conjugate* of a stable configuration s to be $s^* = c^\sharp - s$. It follows from the definition of stability that s^* is also a stable configuration, and clearly the conjugate of s^* is s.

Lemma 16 *Every element g of $K(G, q)$ has a unique representation $\sum t(u)g_u$ such that the conjugate configuration t^* is critical.*

Proof: According to Lemma 13, g has a representation $\sum s(u)g_u$ such that s is stable. Furthermore, $[s] = g$.

The conjugate configuration s^* is also stable and, by the theory outlined above, there is a sequence of legal firings which leads from s^* to a critical configuration c. Let $t = c^*$, so that $\sum t(u)g_u$ is a representation of some element of $K(G, q)$.

Since $c = t^*$ is obtained from s^* by a legal sequence of firings, we have $[t^*] = [s^*]$. It follows from the definition of conjuacy that $[t] = [s]$, and $[s] = g$, so that $\sum t(u)g_u$ is a representation of g, and by its definition $t^* = c$ is critical.

Suppose we are given any representation $\sum y(u)g_u$ of g with $y^* = c'$ critical. Then we have $g = [t] = [y]$, so $[t^*] = [y^*]$, and $[c] = [t^*] = [y^*] = [c']$. But each coset has a unique critical representative, hence $c = c'$, and it follows that $y(u) = t(u)$ for all vertices u. □

Lemma 17 *The unique representation $g = \sum t(u)g_u$ for which t^* is critical is a minimal representation of g.*

Proof: Suppose first that we have a representation $g = \sum z(u)g_u$ in which z is not stable. Then applying the rewriting rules R_v $(v \neq q)$ must eventually produce a representation $g = \sum s(u)g_u$ in which s is stable. The rules imply that $\sum s(u) \leq \sum z(u)$ (since R_q is not used), so it is sufficient to assume that we have a representation with s stable.

In that case, applying Lemma 15 with s^* as the stable configuration and t^* as the critical one leads to the conclusion that $i(s^*) \leq i(t^*)$. This means that $M(s) \geq M(t)$, that is, $\sum s(u) \geq \sum t(u)$, and hence $\sum t(u)g_u$ is a minimal representation. □

Lemmas 16 and 17 define a two-stage procedure for reducing a representation $\sum z(u)g_u$ to a minimal one.

- *Stage 1*: If necessary, reduce $\sum z(u)g_u$ to $\sum s(u)g_u$, where s is stable. This may be done by applying the rewriting rules R_v $(v \neq q)$ only.
- *Stage 2*: Apply the rewriting rules (including R_q if and only if it is needed) to $\sum s^*(u)g_u$ until a representation $\sum t^*(u)g_u$ with t^* critical is obtained. Then $\sum t(u)g_u$ is a minimal representation.

Example Consider again the graph $K_{3,3}$, with the notation as in Sections 5 and 8. A minimal representation of the element $g = 4v + w$ can be found as follows.

Stage 1: Applying R_v reduces g to $v + w + r + s$, where the corresponding configuration 11011 is stable.
Stage 2: The conjugate configuration is $22222 - 11011 = 11211$. Applying legal firings (or the equivalent rewriting rules), we get

$$11211 \xrightarrow{q} 22311 \xrightarrow{x} 22022.$$

The last configuration is critical. Its conjugate is 00200, so a minimal representation of $g = 4v + w$ is $2x$.

We now turn to the theoretical consequences of Lemmas 16 and 17. First we note that, since $\sum t(u)g_u$ is a representation of g, the coset $[t]$ is g. If h denotes the coset $[c^\sharp]$, then

$$[t^*] = [c^\sharp - t] = h - g.$$

So the lemmas tell us that an element g in $K(G, q)$ has length $M(t)$, where t^* is the unique critical representative of $h - g$. We have

$$\begin{aligned} L(g) = M(t) &= M(c^\sharp) - M(t^*) \\ &= i(c^\sharp) - i(t^*) \\ &= (m - n + 1) - i(t^*). \end{aligned}$$

Theorem B *Let G be a connected graph and q a vertex of G. The growth function $\mathcal{L}$ of the critical group $K(G, q)$ is related to the $\mathcal{C}$ function as follows*:

$$\mathcal{L}(z) = z^{m-n+1}\mathcal{C}(z^{-1}).$$

Proof: We must check first that the mapping $g \mapsto t^*$ is a bijection from the group $K(G, q)$ to the set $\Gamma(G, q)$ of critical configurations. Clearly, the mapping $\beta : g \mapsto h - g$ of $K(G, q)$ into itself is a bijection. But the fact that each coset has a unique critical representative means that the mapping which takes $h - g = [t^*]$ to t^* is a bijection, and so $g \mapsto t^*$ is a bijection.

The calculation given above shows that $L(g) = (m - n + 1) - i(t^*)$. Hence the number of elements of $K(G, q)$ which have length i is equal to the number of critical configurations with index $(m - n + 1) - i$. The result follows from the definitions of $\mathcal{L}$ and $\mathcal{C}$. □

12. The growth function and the Tutte polynomial

Combining Theorems A and B we have the main result.

Theorem C *Let G be a connected graph with n vertices and m edges, and let q be any vertex of G. Then the Tutte polynomial $\mathcal{T}$ of G and the growth function $\mathcal{L}$ of the Picard presentation $K(G, q)$ are related as follows*:

$$\mathcal{T}(1, z) = z^{m-n+1}\mathcal{L}(z^{-1}).$$

Corollary *The maximum length of an element in $K(G, q)$ is $m - n + 1$.*

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