Journal of Algebraic Combinatorics **9** (1999), 295–313

Murphy Operators and the Centre of the Iwahori-Hecke Algebras of Type A

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Received October 24, 1996

Abstract. In this paper we introduce a family of polynomials indexed by pairs of partitions and show that if these polynomials are self-orthogonal then the centre of the Iwahori-Hecke algebra of the symmetric group is precisely the set of symmetric polynomials in the Murphy operators.

Keywords: Hecke algebra, Murphy operator, symmetric group

1. Introduction

In [4] Murphy showed that for any field $\mathbb{F}$ the centre of the group algebra of the symmetric group $\mathfrak{S}_n$ on n symbols is the set of symmetric polynomials in the Murphy operators. One consequence of this result is a relatively easy proof of the Nakayama conjecture for $\mathbb{F}\mathfrak{S}_n$.

Given an invertible element q in a ring R let $\mathscr{H}=\mathscr{H}_{R,q}(\mathfrak{S}_n)$ be the associated Iwahori-Hecke algebra. Then $\mathscr{H}$ contains elements which are q-analogues of the Murphy operators of the symmetric group and once again the symmetric polynomials in these elements belong to the centre of $\mathscr{H}$. It is natural therefore to make the following conjecture.

Conjecture 1.1 [2] *For any ring R and any invertible element q in R the centre of the Iwahori-Hecke algebra $\mathscr{H}_{R,q}(\mathfrak{S}_n)$ is the set of symmetric polynomials in the Murphy operators.*

Dipper and James [2, Theorem 2.14] have proved this conjecture in the case where $\mathscr{H}$ is semisimple; unfortunately, there is a gap in their proof for the non-semisimple case. As for the symmetric group, one of the reasons why this conjecture is interesting is that as a corollary one can prove the Nakayama conjecture for $\mathscr{H}$ (this was proved by Gordon James and the author in [3], with one direction being done previously in [2]).

In this paper we reduce Conjecture 1.1 to a purely combinatorial problem of showing that certain polynomials are orthogonal. To the best of our knowledge these polynomials have not appeared in the elsewhere in the literature; it seems likely that they will be of independent interest.

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One of the reasons why the conjecture for $\mathscr{H}$ is more difficult to prove than in the symmetric group case is that the multiplication in $\mathscr{H}$ is much more complicated. In the first section of this paper we overcome this difficulty by proving a combinatorial result which allows us to rewrite an arbitrary product of Murphy operators as a linear combination of less complicated products (modulo "unimportant" terms). In the second section we apply this result towards Conjecture 1.1.

2. Rewriting rules for Murphy operators

Throughout this paper we fix a positive integer n and let $\mathfrak{S}_n = \mathfrak{S}(\{1, 2, \ldots, n\})$ be the symmetric group on $\{1, 2, \ldots, n\}$.

Let R be a commutative ring with 1 and q an invertible element of R. Then the Iwahori-Hecke algebra $\mathscr{H} = \mathscr{H}_{R,q}(\mathfrak{S}_n)$ is the unital associative R-algebra with generators T_1, $T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned} T_i^2 &= (q-1)T_i + q, \\ T_j T_{j+1} T_j &= T_{j+1} T_j T_{j+1}, \\ T_i T_j &= T_j T_i \quad \text{if } |i-j| > 2, \end{aligned}$$

for all $1 \le i, j < n$.

Given an integer i, where $1 \le i < n$, let $s_i = (i, i+1)$; then $\{s_1, s_2, \ldots, s_{n-1}\}$ is the (standard) set of Coxeter generators for the symmetric group $\mathfrak{S}_n$. Suppose that $w \in \mathfrak{S}_n$ and write $w = s_{i_1} s_{i_2} \ldots s_{i_k}$; this expression for w is reduced if k is minimal, in which case we say that w has length $\ell(w) = k$. Given such a reduced expression for w let $T_w = T_{i_1} T_{i_2} \ldots T_{i_k}$; the relations in $\mathscr{H}$ ensure that T_w is independent of the choice of reduced expression for w. Moreover, $\{T_w \mid w \in \mathfrak{S}_n\}$ is a basis for $\mathscr{H}$.

Definition 2.1 [2] The Murphy operators of $\mathscr{H}$ are the elements $L_1 = 0$ and

$$L_i = q^{1-i} T_{(1,i)} + q^{2-i} T_{(2,i)} + \cdots + q^{-1} T_{(i-1,i)},$$

for $i = 2, \ldots, n$.

Using the defining relations in $\mathscr{H}$ it is a straightforward exercise to show that the following holds.

Lemma 2.2 [2, Section 2] *Suppose that $1 \le i < n$ and $1 \le j \le n$. Then*

(i) *T_i and L_j commute when $i \ne j-1, j$.*

(ii) *T_i commutes with $L_i L_{i+1}$ and $L_i + L_{i+1}$.*

(iii) *L_i and L_j commute.*

By (i) and (ii) the symmetric polynomials in the Murphy operators belong to the centre of $\mathscr{H}$.

Now that we have amassed sufficient notation to explain Conjecture 1.1 in the general case we restrict our attention to the generic Iwahori-Hecke algebra (where we renormalise everything).

Let $q^{\frac{1}{2}}$ be an indeterminate over $\mathbb{Z}$ and let $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ be the ring of Laurent polynomials in $q^{\frac{1}{2}}$ with integer coefficients. Then $\mathscr{H} = \mathscr{H}_{\mathcal{A},q}(\mathfrak{S}_n)$ is the generic Iwahori-Hecke algebra of $\mathfrak{S}_n$.

For any $w \in \mathfrak{S}_n$ we let $\tilde{T}_w = q^{-\frac{1}{2}\ell(w)} T_w$. Define $\alpha = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$; then the multiplication in $\mathscr{H}$ is completely determined by

$$\tilde{T}_i \tilde{T}_w = \begin{cases} \tilde{T}_{s_i w} & \text{if } \ell(s_i w) > \ell(w), \\ \tilde{T}_{s_i w} + \alpha \tilde{T}_w & \text{if } \ell(s_i w) < \ell(w), \end{cases}$$

for all $i = 1, 2, \ldots, n-1$ and $w \in \mathfrak{S}_n$.

We define $\tilde{L}_i = \tilde{T}_{(1,i)} + \tilde{T}_{(2,i)} + \cdots + \tilde{T}_{(i-1,i)}$ for $m = 2, \ldots, n$. Because $\tilde{L}_i = q^{\frac{1}{2}} L_i$ we also call $\tilde{L}_i$ a Murphy operator; this should cause no confusion. In fact, the Murphy operators are quite hard to work with directly; instead we work with the elements $\mathscr{L}_1 = 1$ and

$$\mathscr{L}_i = \tilde{T}_{i-1}\tilde{T}_{i-2}\ldots\tilde{T}_1\tilde{T}_1\ldots\tilde{T}_{i-2}\tilde{T}_{i-1}, \quad \text{for } i = 2, \ldots, n,$$

which we call $\mathscr{L}$-Murphy operators. This terminology is justified by the well-known lemma.

Lemma 2.3 *Suppose that* $1 \le i \le n$. *Then* $\mathscr{L}_i = \alpha \tilde{L}_i + 1$.

Proof: When $i = 1$ there is nothing to prove. Therefore, by induction and using the fact that $\tilde{T}_i \tilde{T}_{(i,j)} \tilde{T}_i = \tilde{T}_{(i+1,j)}$ when $j = 1, 2, \ldots, i-1$, we find

$$\begin{aligned} \mathscr{L}_{i+1} &= \tilde{T}_i \mathscr{L}_i \tilde{T}_i = \tilde{T}_i(\alpha \tilde{L}_i + 1)\tilde{T}_i \\ &= \alpha\big(\tilde{T}_{(i+1,i-1)} + \tilde{T}_{(i+1,i-2)} + \cdots + \tilde{T}_{(i+1,1)}\big) + \tilde{T}_i^2 \\ &= \alpha \tilde{L}_{i+1} + 1. \end{aligned}$$

□

We remark that the easiest way to verify Lemma 2.2 is to first prove the corresponding result for the $\mathscr{L}$-Murphy operators.

Define the inner product $\langle , \rangle : \mathscr{H} \otimes \mathscr{H} \to \mathcal{A}$ to be the $\mathcal{A}$-linear extension of the map

$$\langle \tilde{T}_x, \tilde{T}_y \rangle = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise,} \end{cases}$$

where $x, y \in \mathfrak{S}_n$. An important property of the inner product, which we shall apply without mention, is that

$$\langle h_1 h_2, h_3 \rangle = \langle h_2, h_1^* h_3 \rangle, \quad \text{for all } h_1, h_2, h_3 \in \mathscr{H},$$

where $* : \mathscr{H} \to \mathscr{H}$ is the unique $\mathcal{A}$-linear anti-isomorphism of $\mathscr{H}$ such that $\tilde{T}_w^* = \tilde{T}_{w^{-1}}$ for all $w \in \mathfrak{S}_m$. This is easy to check because $\langle h_1, h_3 \rangle$ is the coefficient of 1 in $h_1^* h_3$.

Following Dipper and James [2], if $x \in \mathfrak{S}_n$ we say that appears in $h \in \mathscr{H}$ if $\langle \tilde{T}_x, h \rangle \neq 0$; equivalently, if $h = \sum_{w \in \mathfrak{S}_n} a_w \tilde{T}_w$ then $a_x \neq 0$ ($a_w \in \mathcal{A}$).

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n is a weakly decreasing sequence of non-negative integers such that $\sum_i \lambda_i = n$. Write $\lambda = (\lambda_1, \ldots, \lambda_k)$, if $\lambda_i = 0$ only if $i > k$, and let

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_k}$$

be the corresponding Young or parabolic subgroup of $\mathfrak{S}_n$.

Definition 2.4 [2] Let λ be a partition of n and suppose that $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ are the standard Coxeter generators for $\mathfrak{S}_\lambda$ where $n > i_1 > i_2 > \cdots > i_k > 0$. Then $u_\lambda \in \mathfrak{S}_\lambda$ is the permutation $s_{i_1} s_{i_2} \ldots s_{i_k}$.

Example 2.5 Suppose that $\lambda = (3, 3, 2, 1)$, a partition of 9. Then

$$\mathfrak{S}_\lambda = \mathfrak{S}_3 \times \mathfrak{S}_3 \times \mathfrak{S}_2 \times \mathfrak{S}_1 = \mathfrak{S}(\{1, 2, 3\}) \times \mathfrak{S}(\{4, 5, 6\}) \times \mathfrak{S}(\{7, 8\}) \times \mathfrak{S}(\{9\})$$

is generated by the set of simple transpositions $\{s_1, s_2, s_4, s_5, s_7\}$. Therefore u_λ is the element $s_7 s_5 s_4 s_2 s_1 = (1, 2, 3)(4, 5, 6)(7, 8)$.

Remark 2.6 The elements u_λ are elements of minimal length in their conjugacy class (in fact they are Coxeter elements in $\mathfrak{S}_\lambda$). The reason why they are of interest to us is that Dipper and James [2, Theorem 2.12] have shown that if c is in the centre of $\mathscr{H}$ then there exists a partition λ such that u_λ appears in c.

Our initial aim is to prove the following result.

Theorem 2.7 *Suppose that λ is a partition of n and that $1 \leq i_1, i_2, \ldots, i_k \leq n$. Then u_λ appears in $\tilde{L}_{i_1} \tilde{L}_{i_2} \ldots \tilde{L}_{i_k}$ only if $\ell(u_\lambda) \leq k$.*

Because the Murphy operators commute we may assume that $i_1 \geq i_2 \geq \cdots \geq i_k$. Furthermore, by Lemma 2.3, we may replace the Murphy operator $\tilde{L}_i$ in the statement of Theorem 2.7 with the $\mathscr{L}$-Murphy operator $\mathscr{L}_i$. We shall need quite a few lemmas before we can approach the summit.

Definition 2.8 An element w of $\mathfrak{S}_n$ is decreasing if it has a reduced expression of the form $w = s_{i_1} s_{i_2} \ldots s_{i_r}$ for some $n > i_1 > i_2 > \cdots > i_r > 0$.

In particular, each u_λ is decreasing. To prove Theorem 2.7 we actually give a recursive formula for calculating the inner products $\langle \tilde{T}_w, \tilde{L}_{i_1} \tilde{L}_{i_2} \ldots \tilde{L}_{i_k} \rangle$ whenever w is a decreasing element of length k.

We define an equivalence relation $\overset{\text{dec}}{=}$ on $\mathscr{H}$ whereby $h_1 \overset{\text{dec}}{=} h_2$ if and only if $\langle \tilde{T}_w, h_1\rangle = \langle \tilde{T}_w, h_2\rangle$ for all decreasing elements $w \in \mathfrak{S}_n$ ($h_1, h_2 \in \mathscr{H}$). A note of warning: $\overset{\text{dec}}{=}$ does not respect the multiplication in $\mathscr{H}$.

We need one more set of definitions before we can start to work. Henceforth, we fix an integer i where $1 \le i < n$ and let $\mathfrak{S}_i = \mathfrak{S}(\{1, 2, \dots, i\})$ and $\mathscr{H}_i = \mathscr{H}_{\mathcal{A},q}(\mathfrak{S}_i)$, which we consider as a subalgebra of $\mathscr{H}$ in the natural way. We also let $\mathscr{H}_i^+ = \sum_{w\in\mathfrak{S}_i} \mathbb{N}[\alpha]\tilde{T}_w$ and $\mathscr{H}^+ = \mathscr{H}_n^+$. Note that $\mathscr{H}^+$ is closed under multiplication and that $\mathscr{L}_i \in \mathscr{H}_i^+$ for $i = 1, 2, \dots, n$.

Lemma 2.9 *Suppose that $h \in \mathscr{H}^+$ and that w appears in $\tilde{T}_i h$ for some $w \in \mathfrak{S}_n$ such that $\ell(s_i w) > \ell(w)$. Then $s_i w$ appears in $\tilde{T}_i h$.*

Proof: By assumption, $\tilde{T}_i \tilde{T}_w = \tilde{T}_{s_i w}$ so

$$\left\langle \tilde{T}_{s_i w}, \tilde{T}_i h\right\rangle = \langle \tilde{T}_i \tilde{T}_w, \tilde{T}_i h\rangle = \left\langle \tilde{T}_w, \tilde{T}_i^2 h\right\rangle = \langle \tilde{T}_w, h\rangle + \alpha\langle \tilde{T}_w, \tilde{T}_i h\rangle.$$

Now h and $\tilde{T}_i h$ belong to $\mathscr{H}^+$, so $\langle \tilde{T}_w, h\rangle$ and $\langle \tilde{T}_w, \tilde{T}_i h\rangle$ are both polynomials in α with non-negative coefficients. Therefore, no cancellation can occur and $\langle \tilde{T}_{s_i w}, \tilde{T}_i h\rangle \neq 0$ because $\langle \tilde{T}_w, \tilde{T}_i h\rangle \neq 0$. □

The following easy but useful lemma is from [2].

Lemma 2.10 *Let Y be a Young subgroup of $\mathfrak{S}_n$ and suppose that $w \in \mathfrak{S}_n$ appears in $\tilde{T}_x h$ or $h\tilde{T}_x$ for some $x \notin Y$ and some $h \in \mathscr{H}(Y)$. Then $w \notin Y$.*

Proof: It suffices to consider the case where $h = \tilde{T}_y$ for some $y \in Y$. The lemma now follows easily by induction on $\ell(y)$. □

The next seemingly innocuous result will get us half way to Theorem 2.7.

Proposition 2.11 *Suppose that $w = s_i v$ for some $v \in \mathfrak{S}_i$ and $1 < i < n$. Let $x \in \mathfrak{S}_i$ and suppose that w appears in $\tilde{T}_i \tilde{T}_x \tilde{T}_i h$ for some $h \in \mathscr{H}_i$. Then $x \in \mathfrak{S}_{i-1}$.*

Proof: Because w appears in $\tilde{T}_i \tilde{T}_x \tilde{T}_i h$ there exists some $z \in \mathfrak{S}_i$ such that w appears in $\tilde{T}_i \tilde{T}_x \tilde{T}_i \tilde{T}_z$. By assumption, $\ell(w) = \ell(v) + 1$ so $\tilde{T}_i \tilde{T}_w = \tilde{T}_v + \alpha \tilde{T}_w$; therefore,

$$0 \neq \langle \tilde{T}_w, \tilde{T}_i \tilde{T}_x \tilde{T}_i \tilde{T}_z\rangle = \langle \tilde{T}_i \tilde{T}_w, \tilde{T}_x \tilde{T}_i \tilde{T}_z\rangle = \langle \tilde{T}_v, \tilde{T}_x \tilde{T}_i \tilde{T}_z\rangle + \alpha\langle \tilde{T}_w, \tilde{T}_x \tilde{T}_i \tilde{T}_z\rangle.$$

However, $\langle \tilde{T}_v, \tilde{T}_x \tilde{T}_i \tilde{T}_z\rangle = \langle \tilde{T}_v \tilde{T}_{z^{-1}}, \tilde{T}_x \tilde{T}_i\rangle = 0$ by Lemma 2.10; so

$$0 \neq \langle \tilde{T}_w, \tilde{T}_i \tilde{T}_x \tilde{T}_i \tilde{T}_z\rangle = \alpha\langle \tilde{T}_w, \tilde{T}_x \tilde{T}_i \tilde{T}_z\rangle = \alpha\langle \tilde{T}_i \tilde{T}_v \tilde{T}_{z^{-1}}, \tilde{T}_x \tilde{T}_i\rangle = \alpha\left\langle \tilde{T}_i \tilde{T}_v \tilde{T}_{z^{-1}}, \tilde{T}_{x s_i}\right\rangle.$$

Therefore, $x s_i$ appears in $\tilde{T}_i \tilde{T}_v \tilde{T}_{z^{-1}}$ and so there exists some $y \in \mathfrak{S}_i$ such that $s_i y = x s_i$. Hence, $x \in s_i \mathfrak{S}_i s_i \cap \mathfrak{S}_i = \mathfrak{S}_{i-1}$ as required. □

The relevance of the proposition to Theorem 2.7 is revealed by the following corollaries.

Corollary 2.12 *Suppose that $i \geq 1$ and let $h \in \mathscr{H}_i$ and suppose that w is decreasing. Then*

$$\langle \tilde{T}_w, \mathscr{L}_{i+1}h\rangle = \begin{cases} \alpha\big\langle \tilde{T}_{s_i w}, h\big\rangle, & \text{if } s_i w \in \mathfrak{S}_i, \\ \langle \tilde{T}_w, h\rangle, & \text{if } w \in \mathfrak{S}_i, \\ 0, & \text{otherwise}. \end{cases}$$

In particular, w appears in $\mathscr{L}_{i+1}h$ if and only if either
(i) *$w \in \mathfrak{S}_i$ and w appears in h, or*
(ii) *$w = s_i v$ where $v \in \mathfrak{S}_i$ is decreasing and v appears in h.*

Proof: If $w \notin \mathfrak{S}_{i+1}$ then w does not appear in $\mathscr{L}_{i+1}h$. In general, by Lemma 2.3 we have that

$$\begin{aligned}&\langle \tilde{T}_w, \mathscr{L}_{i+1}h\rangle \\ &\quad = \alpha\big\langle \tilde{T}_w, \tilde{T}_{(i+1,i)}h\big\rangle + \alpha\big\langle \tilde{T}_w, \tilde{T}_{(i+1,i-1)}h\big\rangle + \cdots + \alpha\big\langle \tilde{T}_w, \tilde{T}_{(i+1,1)}h\big\rangle + \langle \tilde{T}_w, h\rangle.\end{aligned}$$

If $w \in \mathfrak{S}_i$ then w does not appear in $\tilde{T}_{(i+1,j)}h$ for any $1 \leq j \leq i$ by Lemma 2.10, so $\langle \tilde{T}_w, \mathscr{L}_{i+1}h\rangle = \langle \tilde{T}_w, h\rangle$ as claimed. On the other hand, if $w = s_i v$ for some $v \in \mathfrak{S}_i$ then by Proposition 2.11, w does not appear in $\tilde{T}_{(i+1,j)}h$ for $1 \leq j < i$ and by Lemma 2.10 w does not appear in h so $\langle \tilde{T}_w, \mathscr{L}_{i+1}h\rangle = \alpha\langle \tilde{T}_{s_i v}, \tilde{T}_i h\rangle = \alpha\langle \tilde{T}_v, h\rangle + \alpha^2\langle \tilde{T}_v, \tilde{T}_i h\rangle$. However, $\langle \tilde{T}_v, \tilde{T}_i h\rangle = 0$ by Lemma 2.10 again, so the corollary follows. □

This translates into a result of Bögeholz; we generalise both of these corollaries below.

Corollary 2.13 [1, Satz 5.1] *Suppose that $n > i_1 > i_2 > \cdots > i_k > 1$ and that w is decreasing. Then w appears in $\tilde{L}_{i_1}\tilde{L}_{i_2}\dots\tilde{L}_{i_k}$ if and only if $w = s_{i_1-1}s_{i_2-1}\dots s_{i_k-1}$. Moreover, in this case, $\langle \tilde{T}_w, \tilde{L}_{i_1}\tilde{L}_{i_2}\dots\tilde{L}_{i_k}\rangle = 1$.*

Proof: By Corollary 2.12, if $1 \leq i < n$ and $h \in \mathscr{H}_i$ then

$$\langle \tilde{T}_w, \tilde{L}_{i+1}h\rangle = \begin{cases} \langle \tilde{T}_v, h\rangle, & \text{if } s_i w = v \in \mathfrak{S}_i, \\ 0, & \text{otherwise}. \end{cases}$$

The result now follows easily by induction on k. □

Remark 2.14 If $w = s_{i_1-1}s_{i_2-1}\dots s_{i_k-1}$, for distinct $i_1, i_2, \dots, i_k$ as above, we see that $\langle T_w, L_{i_1}L_{i_2}\dots L_{i_k}\rangle = q^k$. More generally, if $w \in \mathfrak{S}_n$ is any element of length k and $2 \leq i_1, \dots, i_k \leq n$ then $\langle T_w, L_{i_1}L_{i_2}\dots L_{i_k}\rangle = q^k\langle \tilde{T}_w, \tilde{L}_{i_1}\tilde{L}_{i_2}\dots\tilde{L}_{i_k}\rangle$. We have renormalized the T-basis and the Murphy operators in order to avoid having to keep track of these units.

The corollary tells us which decreasing elements of $\mathfrak{S}_n$ appear in a product of distinct Murphy operators. We now turn to the general case, which will follow by essentially expanding part (ii) of the lemma below.

Lemma 2.15 *Suppose $i \geq 1$ and that $r \geq 1$. Then*
(i) $\tilde{T}_i \mathscr{L}_{i+1}^r = \mathscr{L}_i^r \tilde{T}_i + \alpha \sum_{s=0}^{r-1} \mathscr{L}_{i+1}^{r-s} \mathscr{L}_i^s$.
(ii) $\mathscr{L}_{i+1}^r = \tilde{T}_i \mathscr{L}_i^r \tilde{T}_i + \alpha \sum_{s=1}^{r-1} \mathscr{L}_i^{r-s} \tilde{T}_i \mathscr{L}_i^s + \alpha^2 \sum_{s=1}^{r-1} s\, \mathscr{L}_{i+1}^{r-s} \mathscr{L}_i^s$.

Proof: First consider (i). When $r = 1$ the formula reduces to the observation that $\tilde{T}_i \mathscr{L}_{i+1} = \mathscr{L}_i \tilde{T}_i + \alpha \mathscr{L}_{i+1}$. By Lemma 2.2, $\mathscr{L}_i$ and $\mathscr{L}_{i+1}$ commute so by induction,

$$\tilde{T}_i \mathscr{L}_{i+1}^{r+1} = (\mathscr{L}_i \tilde{T}_i + \alpha \mathscr{L}_{i+1}) \mathscr{L}_{i+1}^r = \mathscr{L}_i \big(\tilde{T}_i \mathscr{L}_{i+1}^r\big) + \alpha \mathscr{L}_{i+1}^{r+1}$$
$$= \mathscr{L}_i^{r+1} \tilde{T}_i + \alpha \sum_{s=0}^{r} \mathscr{L}_{i+1}^{r+1-s} \mathscr{L}_i^s,$$

proving (i). For (ii), note that $\tilde{T}_i^{-1} = \tilde{T}_i - \alpha$, so using (i) twice we find

$$\begin{aligned}
\mathscr{L}_{i+1}^r &= (\tilde{T}_i - \alpha)\left(\mathscr{L}_i^r \tilde{T}_i + \alpha \sum_{s=0}^{r-1} \mathscr{L}_{i+1}^{r-s} \mathscr{L}_i^s \right) \\
&= \tilde{T}_i \mathscr{L}_i^r \tilde{T}_i + \alpha \sum_{s=0}^{r-1} \big(\tilde{T}_i \mathscr{L}_{i+1}^{r-s}\big) \mathscr{L}_i^s - \alpha \mathscr{L}_i^r \tilde{T}_i - \alpha^2 \sum_{s=0}^{r-1} \mathscr{L}_{i+1}^{r-s} \mathscr{L}_i^s \\
&= \tilde{T}_i \mathscr{L}_i^r \tilde{T}_i + \alpha \sum_{s=0}^{r-1} \left(\mathscr{L}_i^{r-s} \tilde{T}_i + \alpha \sum_{m=0}^{r-s-1} \mathscr{L}_{i+1}^{r-s-m} \mathscr{L}_i^m \right) \mathscr{L}_i^s \\
&\quad - \alpha \mathscr{L}_i^r \tilde{T}_i - \alpha^2 \sum_{s=0}^{r-1} \mathscr{L}_{i+1}^{r-s} \mathscr{L}_i^s \\
&= \tilde{T}_i \mathscr{L}_i^r \tilde{T}_i + \alpha \sum_{s=1}^{r-1} \mathscr{L}_i^{r-s} \tilde{T}_i \mathscr{L}_i^s + \alpha^2 \sum_{s=1}^{r-1} s\, \mathscr{L}_{i+1}^{r-s} \mathscr{L}_i^s,
\end{aligned}$$

as required. □

The next few lemmas investigate which decreasing elements can appear in the products of the form $\tilde{T}_i \mathscr{L}_i^r \tilde{T}_i$; these are by far the most difficult terms appearing in (ii) of the lemma. We begin with a technical result which is another application of Proposition 2.11. At first sight the h appearing in the lemma plays no role; however, it is crucial for our applications because the equivalence relation $\overset{\text{dec}}{=}$ does not respect multiplication, as noted in Definition 2.8.

For convenience, given i and j, where $1 \leq i, j < n$, we define

$$\tilde{T}_{i\cdot\cdot j} = \begin{cases} \tilde{T}_i \tilde{T}_{i+1} \dots \tilde{T}_j, & \text{if } i < j, \\ \tilde{T}_i, & \text{if } i = j, \\ \tilde{T}_i \tilde{T}_{i-1} \dots \tilde{T}_j, & \text{if } j < i. \end{cases}$$

In particular, $\mathscr{L}_i = \tilde{T}_{i-1\cdot\cdot 1} \tilde{T}_{1\cdot\cdot i-1}$.

Lemma 2.16 *Let $h \in \mathscr{H}_j^+$ and suppose that $n \geq j \geq i \geq 2$ and $s \geq 1$. Then*

$$\tilde{T}_{j\cdot\cdot i}\,\mathscr{L}_{i-1}^s \tilde{T}_{i-1}\tilde{T}_{i\cdot\cdot j}h \overset{\mathrm{dec}}{=} 0.$$

Proof: For $k \geq i$ let $A_k = \tilde{T}_{k\cdot\cdot i}\,\mathscr{L}_{i-1}^s \tilde{T}_{i-1}\tilde{T}_{i\cdot\cdot k}$; then we must show that $A_j h \overset{\mathrm{dec}}{=} 0$ for all $h \in \mathscr{H}_j^+$. If this is not so then there must exist a decreasing element $w \in \mathfrak{S}_{j+1}$ which appears in $A_j h$ for some $h \in \mathscr{H}_j^+$. By Lemma 2.9 we may assume that $w = s_j v_j$, where $v_j \in \mathfrak{S}_j$. Since $A_j h = T_j A_{j-1} T_j h$ we can apply Proposition 2.11 to find an element $v_{j-1} \in \mathfrak{S}_{j-1}$ which appears in A_{j-1}. By Lemma 2.9, $s_{j-1}v_{j-1}$ appears in A_{j-1}, so we can apply Proposition 2.11 once more to find an element $v_{j-2} \in \mathfrak{S}_{j-2}$ which appears in A_{j-2}. Continuing in this way we see that there exists an element $v_{i-1} \in \mathfrak{S}_{i-1}$ which appears in $\mathscr{L}_{i-1}^s \tilde{T}_{i-1}$. However, this is impossible by Lemma 2.10, so $A_j h \overset{\mathrm{dec}}{=} 0$ as claimed. □

To proceed with the expansion of $\tilde{T}_i\,\mathscr{L}_i^r \tilde{T}_i$ we require two closely related families of polynomials.

Definition 2.17 Define $\mathfrak{a}(s)$ and $\mathfrak{b}(s)$ to be the polynomials in $\mathbb{N}[\alpha^2]$ given by $\mathfrak{a}(0) = \mathfrak{b}(0) = 1$,

$$\mathfrak{a}(s) = \sum_{m=1}^{s} \binom{s+m-1}{2m-1}\alpha^{2m}, \quad \text{and} \quad \mathfrak{b}(s) = \sum_{m=0}^{s} \binom{s+m}{2m}\alpha^{2m},$$

for $s \geq 1$.

We remark that even though it would have been more natural to define $\mathfrak{a}(0)$ to be the 0-polynomial it is much more convenient to have it equal to 1. Also note that because $\alpha^2 = q - 2 + q^{-1}$ both $\mathfrak{a}(s)$ and $\mathfrak{b}(s)$ are actually Laurent polynomials in q (rather than $q^{\frac{1}{2}}$).

The closed forms for $\mathfrak{a}(s)$ and $\mathfrak{b}(s)$ were noticed by John Graham; all that we really need however is that they are the polynomials determined by the recurrence formulae below.

Lemma 2.18 *Suppose that $s \geq 1$. Then*

(i) $\mathfrak{a}(s+1) = \mathfrak{a}(s) + \alpha^2\mathfrak{b}(s)$.

(ii) $\mathfrak{b}(s) = \mathfrak{a}(s) + \mathfrak{b}(s-1)$.

(iii) $\mathfrak{a}(s) = \alpha^2 \sum_{m=1}^{s} m\,\mathfrak{a}(s-m)$.

(iv) $\mathfrak{b}(s) = 1 + \alpha^2 \sum_{m=1}^{s} m\,\mathfrak{b}(s-m)$.

Proof: The first two statements follow from the definitions using the well-known identity $\binom{s}{t} = \binom{s-1}{t} + \binom{s-1}{t-1}$; the final two statements then follow from (i) and (ii) and induction. □

We now return to the expansion of $\mathscr{L}_{i+1}^r$. The next lemma reveals the origin of the $\mathfrak{a}$-polynomials.

Lemma 2.19 *Let r, i and j be integers such that $r \geq 1$ and $n \geq j \geq i \geq 2$, and suppose that $h \in \mathscr{H}_j^+$. Then*

$$\tilde{T}_{j\cdot\cdot i}\,\mathscr{L}_i^r\,\tilde{T}_{i\cdot\cdot j}h \overset{\text{dec}}{=} \sum_{s=0}^{r-1}\mathfrak{a}(s)\tilde{T}_{j\cdot\cdot i-1}\,\mathscr{L}_{i-1}^{r-s}\tilde{T}_{i-1\cdot\cdot j}\,\mathscr{L}_{i-1}^s h.$$

Proof: We argue by induction on r. When $r=1$ it is clear from the definitions that $\tilde{T}_{j\cdot\cdot i}\,\mathscr{L}_i\,\tilde{T}_{i\cdot\cdot j} = \tilde{T}_{j\cdot\cdot i-1}\,\mathscr{L}_{i-1}\tilde{T}_{i-1\cdot\cdot j}h$ so there is nothing to prove. Suppose then that $r > 1$. By Lemma 2.2, $\mathscr{L}_{i-1}$ commutes with $\tilde{T}_{j\cdot\cdot i}$ and with $\tilde{T}_{i\cdot\cdot j}$. Therefore, using Lemma 2.15(ii) and Lemma 2.16,

$$\tilde{T}_{j\cdot\cdot i}\,\mathscr{L}_i^r\,\tilde{T}_{i\cdot\cdot j}h \overset{\text{dec}}{=} \tilde{T}_{j\cdot\cdot i-1}\,\mathscr{L}_{i-1}^r\tilde{T}_{i-1\cdot\cdot j}h + \alpha^2\sum_{m=1}^{r-1} m\tilde{T}_{j\cdot\cdot i}\,\mathscr{L}_i^{r-m}\tilde{T}_{i\cdot\cdot j}\,\mathscr{L}_{i-1}^m h,$$

Because $\mathscr{L}_{i-1}^m h \in \mathscr{H}_j^+$ we may apply induction to expand the right-hand sum

$$\begin{aligned}\sum_{m=1}^{r-1} m\tilde{T}_{j\cdot\cdot i}\,\mathscr{L}_i^{r-m}\tilde{T}_{i\cdot\cdot j}\,\mathscr{L}_{i-1}^m h &\overset{\text{dec}}{=} \sum_{m=1}^{r-1}\sum_{t=0}^{r-m-1} m\mathfrak{a}(t)\tilde{T}_{j\cdot\cdot i-1}\,\mathscr{L}_{i-1}^{r-m-t}\tilde{T}_{i-1\cdot\cdot j}\,\mathscr{L}_{i-1}^{t+m}h\\ &= \sum_{m=1}^{r-1}\sum_{s=m}^{r-1} m\mathfrak{a}(s-m)\tilde{T}_{j\cdot\cdot i-1}\,\mathscr{L}_{i-1}^{r-s}\tilde{T}_{i-1\cdot\cdot j}\,\mathscr{L}_{i-1}^s h\\ &= \sum_{s=1}^{r-1}\tilde{T}_{j\cdot\cdot i-1}\,\mathscr{L}_{i-1}^{r-s}\tilde{T}_{i-1\cdot\cdot j}\,\mathscr{L}_{i-1}^s h\cdot\sum_{m=1}^{s} m\mathfrak{a}(s-m).\end{aligned}$$

Now $\mathfrak{a}(s) = \alpha^2\sum_{m=1}^s m\mathfrak{a}(s-m)$ by Lemma 2.18(iii). Since $\mathfrak{a}(0) = 1$, combining the last two equations yields the lemma. □

A composition of r into l-parts is a sequence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_l)$ of non-negative integers such that $\sum_{i\geq 1}^l \sigma_i = r$. A strict composition of r is a composition $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_l)$ of r in which each part σ_i is strictly positive. Let $\mathscr{C}(r, l)$ be the set of all compositions of r into l parts and $\mathscr{C}(r)$ be the set of all strict compositions of r. For example, $\mathscr{C}(3, 2) = \{(3, 0), (2, 1), (1, 2), (0, 3)\}$ and $\mathscr{C}(3) = \{(3), (2, 1), (1, 2), (1, 1, 1)\}$.

Lemma 2.20 *Let i, j and r be integers such that $r \geq 1$ and $n \geq j \geq i \geq 2$ and suppose that $h \in \mathscr{H}_j^+$. Then*

$$\tilde{T}_{j\cdot\cdot i}\,\mathscr{L}_i^r\,\tilde{T}_{i\cdot\cdot j}h \overset{\text{dec}}{=} \sum_{\sigma\in\mathscr{C}(r-1,i)}\mathfrak{a}(\sigma_{i-1})\ldots\mathfrak{a}(\sigma_1)\,\mathscr{L}_{j+1}\,\mathscr{L}_{i-1}^{\sigma_{i-1}}\ldots\mathscr{L}_2^{\sigma_2}h.$$

Proof: This time we argue by induction on i. If $i = 2$ then, using Lemma 2.19 and noting that $\mathscr{L}_1 = 1$, we have

$$\begin{aligned}&\tilde{T}_{j\cdot\cdot i}\,\mathscr{L}_i^r\,\tilde{T}_{i\cdot\cdot j}h\\ &\overset{\text{dec}}{=} \sum_{s=0}^{r-1}\mathfrak{a}(s)\tilde{T}_{j\cdot\cdot 1}\tilde{T}_{1\cdot\cdot j}\,\mathscr{L}_1^s h = \sum_{s=0}^{r-1}\mathfrak{a}(s)\,\mathscr{L}_{j+1}h = \sum_{\sigma\in\mathscr{C}(r-1,2)}\mathfrak{a}(\sigma_1)\,\mathscr{L}_{j+1}\end{aligned}$$

as required. Now suppose that $i > 2$. Then, by Lemma 2.19 and induction,

$$
\begin{aligned}
&\tilde{T}_{j\cdot\cdot i}\,\mathscr{L}_i^r\,\tilde{T}_{i\cdot\cdot j}h\\
&\overset{\text{dec}}{=}\sum_{s=0}^{r-1}\mathfrak{a}(s)\tilde{T}_{j\cdot\cdot i-1}\,\mathscr{L}_{i-1}^{r-s}\tilde{T}_{i-1\cdot\cdot j}\,\mathscr{L}_{i-1}^{s}h\\
&=\sum_{s=0}^{r-1}\sum_{\sigma\in\mathscr{C}(r-s-1,i-1)}\mathfrak{a}(s)\mathfrak{a}(\sigma_{i-2})\dots\mathfrak{a}(\sigma_1)\,\mathscr{L}_{j+1}\,\mathscr{L}_{i-2}^{\sigma_{i-2}}\dots\mathscr{L}_2^{\sigma_1}\,\mathscr{L}_{i-1}^{s}h\\
&=\sum_{\sigma\in\mathscr{C}(r-1,i)}\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\,\mathscr{L}_{j+1}\,\mathscr{L}_{i-1}^{\sigma_{i-1}}\dots\mathscr{L}_2^{\sigma_2}h,
\end{aligned}
$$

as required. □

For convenience, given a composition $\sigma\in\mathscr{C}(r-1,i)$ we let $\mathscr{L}^\sigma=\mathscr{L}_i^{\sigma_i}\,\mathscr{L}_{i-1}^{\sigma_{i-1}}\dots\mathscr{L}_1^{\sigma_1}$. We are now ready to compute the needed inner of products of the form $\langle\tilde{T}_w,h\rangle$ where h is a product of Murphy operators.

Proposition 2.21 *Let r, i and k be integers such that $r\geq 1$, $n>i\geq 2$ and $k\geq r+1$ and suppose that $w=s_iv$ is a decreasing element of length at least k. Finally, let $h=\mathscr{L}_{j_{r+1}}\mathscr{L}_{j_{r+2}}\dots\mathscr{L}_{j_k}$ where $i\geq j_{r+1}\geq\dots\geq j_k\geq 2$. If $\ell(w)=k$ then*

$$\big\langle\tilde{T}_w,\,\mathscr{L}_{i+1}^rh\big\rangle=\sum_{\substack{\sigma\in\mathscr{C}(r-1,i)\\ \sigma_1=0}}\alpha\mathfrak{b}(\sigma_i)\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\langle\tilde{T}_v,\,\mathscr{L}^\sigma h\rangle.$$

If $\ell(w)>k$ then $\langle\tilde{T}_w,\,\mathscr{L}_{i+1}^rh\rangle=0$.

Proof: We proceed by simultaneous induction on k and r. If $r=1$ then $\langle\tilde{T}_w,\tilde{L}_{i+1}h\rangle=\alpha\langle\tilde{T}_v,h\rangle$, by Corollary 2.12, and the theorem follows.

Suppose then that $r>1$ and that the theorem holds for smaller k. Now by Lemma 2.15(ii),

$$\mathscr{L}_{i+1}^rh=\tilde{T}_i\,\mathscr{L}_i^r\tilde{T}_ih+\alpha\sum_{s=1}^{r-1}\mathscr{L}_i^{r-s}\tilde{T}_i\,\mathscr{L}_i^sh+\alpha^2\sum_{s=1}^{r-1}s\,\mathscr{L}_{i+1}^{r-s}\,\mathscr{L}_i^sh.$$

We deal with each of these three terms in turn.

Since $h\in\mathscr{H}_i^+$ we may use Lemma 2.20, with $j=i$, and Corollary 2.12 to see that

$$
\begin{aligned}
\big\langle\tilde{T}_w,\tilde{T}_i\,\mathscr{L}_i^r\tilde{T}_ih\big\rangle&=\sum_{\sigma\in\mathscr{C}(r-1,i)}\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\big\langle\tilde{T}_w,\,\mathscr{L}_{i+1}\,\mathscr{L}_{i-1}^{\sigma_{i-1}}\dots\mathscr{L}_1^{\sigma_1}h\big\rangle\\
&=\sum_{\sigma\in\mathscr{C}(r-1,i)}\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\big\langle\tilde{T}_v,\,\mathscr{L}_{i-1}^{\sigma_{i-1}}\dots\mathscr{L}_1^{\sigma_1}h\big\rangle
\end{aligned}
$$

Now v is of length at least $k-1$, so by induction on k it cannot appear in $\mathscr{L}_{i-1}^{\sigma_{i-1}}\dots\mathscr{L}_1^{\sigma_1}h$ unless this is a product of at least $k-1$ non-trivial $\mathscr{L}$-Murphy operators. Consequently, to get a non-zero contribution to this sum we require that $\sigma_1=\sigma_i=0$. Therefore, noting that $\mathfrak{b}(0)=1$,

$$\langle\tilde{T}_w,\,\tilde{T}_i\,\mathscr{L}_i^r\,\tilde{T}_i h\rangle=\sum_{\substack{\sigma\,\in\,\mathscr{C}(r-1,\,i)\\ \sigma_1=0\\ \sigma_i=0}}\mathfrak{b}(\sigma_i)\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\langle\tilde{T}_v,\,\mathscr{L}^\sigma h\rangle.$$

By Lemma 2.20 and Corollary 2.12,

$$\begin{aligned}
&\sum_{s=1}^{r-1}\big\langle\tilde{T}_w,\,\mathscr{L}_i^{r-s}\tilde{T}_i\,\mathscr{L}_i^s h\big\rangle\\
&=\sum_{s=1}^{r-1}\big\langle\tilde{T}_v,\,\tilde{T}_i\,\mathscr{L}_i^{r-s}\tilde{T}_i\,\mathscr{L}_i^s h\big\rangle\\
&=\sum_{s=1}^{r-1}\sum_{\sigma\in\mathscr{C}(r-s-1,i)}\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\big\langle\tilde{T}_v,\,\mathscr{L}_{i+1}\mathscr{L}_i^s\,\mathscr{L}_{i-1}^{\sigma_{i-1}}\dots\mathscr{L}_1^{\sigma_1}h\big\rangle\\
&=\sum_{s=1}^{r-1}\sum_{\substack{\sigma\in\mathscr{C}(r-s-1,\,i)\\ \sigma_1=\sigma_i=0}}\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\big\langle\tilde{T}_v,\,\mathscr{L}_{i+1}\mathscr{L}_i^s\,\mathscr{L}_{i-1}^{\sigma_{i-1}}\dots\mathscr{L}_1^{\sigma_1}h\big\rangle\\
&=\sum_{\substack{\sigma\in\mathscr{C}(r-1,\,i)\\ \sigma_1=0\\ \sigma_i\neq 0}}\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\langle\tilde{T}_v,\,\mathscr{L}^\sigma h\rangle,
\end{aligned}$$

where the terms in the third line corresponding to $\sigma_1\neq 0$ or $\sigma_i\neq 0$ disappear because, as above, v cannot appear in a product of fewer than $k-1$ non-trivial $\mathscr{L}$-Murphy operators.

Finally, by induction on r,

$$\begin{aligned}
&\sum_{s=1}^{r-1}s\big\langle\tilde{T}_w,\,\mathscr{L}_{i+1}^{r-s}\,\mathscr{L}_i^s h\big\rangle\\
&=\sum_{s=1}^{r-1}\sum_{\substack{\sigma\in\mathscr{C}(r-s-1,i)\\ \sigma_1=0}}s\alpha\mathfrak{b}(\sigma_i)\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\big\langle\tilde{T}_v,\,\mathscr{L}_i^s\,\mathscr{L}^\sigma h\big\rangle\\
&=\alpha\sum_{\substack{\sigma\in\mathscr{C}(r-1,\,i)\\ \sigma_1=0\\ \sigma_i\neq 0}}\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_1)\big\langle\tilde{T}_v,\,\mathscr{L}_i^\sigma h\big\rangle\cdot\sum_{m=1}^{\sigma_i}m\mathfrak{b}(\sigma_i-m).
\end{aligned}$$

By Lemma 2.18(iv), $\mathfrak{b}(\sigma_i) = 1 + \alpha^2 \sum_{m=1}^{\sigma_i} m\mathfrak{b}(\sigma_i - m)$. Therefore, adding up the three contributions to $\langle \tilde{T}_w, \mathscr{L}_{i+1}^r h\rangle$ calculated above, we find

$$\langle \tilde{T}_w, \mathscr{L}_{i+1}^r h\rangle = \sum_{\substack{\sigma \in \mathscr{C}(r-1, i)\\ \sigma_1 = 0}} \alpha \mathfrak{b}(\sigma_i)\mathfrak{a}(\sigma_{i-1}) \dots \mathfrak{a}(\sigma_1)\langle \tilde{T}_v, \mathscr{L}^\sigma h\rangle.$$

By induction the inner products on the right-hand side are zero if $\ell(w) > k$ so the result follows. □

As a corollary we obtain Theorem 2.7.

Corollary 2.22 *Suppose that w is decreasing and that $n \geq i_1 \geq i_2 \geq \cdots \geq i_k > 1$ is a sequence of positive integers. Then w appears in $\mathscr{L}_{i_1}\mathscr{L}_{i_2} \dots \mathscr{L}_{i_k}$ only if $\ell(w) \leq k$. Moreover, if $\ell(w) = k$ then w appears in $\mathscr{L}_{i_1}\mathscr{L}_{i_2} \dots \mathscr{L}_{i_k}$ only if $w = s_{i_1+1}v$ for some $v \in \mathfrak{S}_{i_1-1}$.*

Proof: If $k = 1$ there is nothing to prove, so suppose $k > 1$. Then, if w appears in $\mathscr{L}_{i_1}\mathscr{L}_{i_2} \dots \mathscr{L}_{i_k}$, by Lemma 2.9 either $w = s_{i_1-1}v$, for some $v \in \mathfrak{S}_{i_1-1}$, or $\ell(s_{i_1-1}w) > \ell(w)$ and $s_{i_1-1}w$ appears in $\mathscr{L}_{i_1}\mathscr{L}_{i_2} \dots \mathscr{L}_{i_k}$. Now apply Theorem 2.21. □

Next we describe exactly which decreasing elements of maximal length appear in a product of Murphy operators; for this we need some definitions.

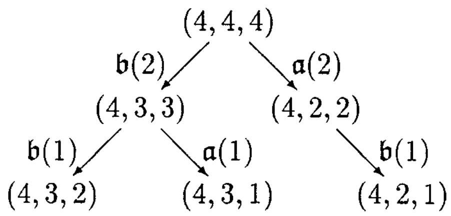
A sequence of integers $\mathbf{i} = (i_1, i_2, \dots, i_k)$ is decreasing if $n > i_1 > i_2 > \cdots > i_k > 0$; $\mathbf{i}$ is weakly decreasing if $i_1 \geq i_2 \geq \cdots \geq i_k$. If $\mathbf{i}$ is decreasing then we write $w_{\mathbf{i}} = s_{i_1}s_{i_2} \dots s_{i_k}$ (cf. Definition 2.4). Note that $w_{\mathbf{i}}$ is a decreasing element of $\mathfrak{S}_{i_1+1}$ of length k.

Definition 2.23 Let $\mathbf{i} = (i_1, i_2, \dots, i_k)$ and $\mathbf{j} = (j_1, j_2, \dots, j_k)$ be two weakly decreasing sequences of length k. Then $\mathbf{i} \to \mathbf{j}$ if

(i) $i_m \geq j_m$ for $m = 1, 2, \dots, k$, and
(ii) $\{i_1, i_2, \dots, i_k\} \subseteq \{j_1, j_2, \dots, j_k\}$.

In particular, note that $\to$ is transitive and that $\mathbf{i} \to \mathbf{j}$ only if $i_1 = j_1$.

Example 2.24 Starting from $\mathbf{i} = (4, 4, 4)$ the paths leading to decreasing 3-tuples (omitting (4, 4, 3) and (4, 4, 2) and the edges from (4, 4, 4) to the three decreasing sequences) are



The next result says, for example, that the only decreasing elements of length 3 which appear in $\mathscr{L}_5^3$ are $s_4s_3s_2$, $s_4s_3s_1$ and $s_4s_2s_1$.

By Theorem 2.21 the labels on the graph give the inner products $\langle\tilde{T}_w,\ \mathscr{L}_5^3\rangle$ when w is one of these elements. For example, $\langle\tilde{T}_4\tilde{T}_3\tilde{T}_1,\ \mathscr{L}_5^3\rangle = \alpha^2\mathfrak{b}(2)\mathfrak{a}(1) + \alpha^2\mathfrak{b}(1)\mathfrak{a}(1)$. The second term comes from the omitted edge $(4, 4, 4) \overset{\mathfrak{b}(1)\mathfrak{a}(1)}{\longrightarrow} (4, 3, 1)$.

Theorem 2.25 *Suppose that $\mathfrak{j} = (j_1, j_2, \dots, j_k)$ is a decreasing sequence and suppose that $\mathfrak{i} = (i_1, i_2, \dots, i_k)$ is weakly decreasing. Then $w = w_\mathfrak{j}$ appears in $\mathscr{L}_{i_1}\mathscr{L}_{i_2}\dots\mathscr{L}_{i_k}$ if and only if $(\mathfrak{i} - (1^k)) \to \mathfrak{j}$ where $\mathfrak{i} - (1^k) = (i_1 - 1, i_2 - 1, \dots, i_k - 1)$.*

Proof: Let $i = i_1 - 1$ and suppose that $i_1 = \cdots = i_r > i_{r+1}$ (if necessary, let $i_{k+1} = 0$). We argue by induction on k. When $k = 1$ there is nothing to prove so suppose that $k > 1$. Since w is of length k, by Proposition 2.21(i) and Corollary 2.12, w appears in $\mathscr{L}_{i_1}\mathscr{L}_{i_2}\dots\mathscr{L}_{i_k}$ only if $w = s_i v$ for some $v \in \mathfrak{S}_i$. As $(\mathfrak{i} - (1^k)) \to \mathfrak{j}$ only if $i_1 - 1 = j_1$, we may assume that $w = s_i v$ for some $v \in \mathfrak{S}_i$.

If $r = 1$ then $\langle\tilde{T}_w,\ \mathscr{L}_{i+1}^r\mathscr{L}_{i_{r+1}}\dots\mathscr{L}_{i_k}\rangle = \alpha\langle\tilde{T}_v,\ \mathscr{L}_{i_{r+1}}\dots\mathscr{L}_{i_k}\rangle$, by Corollary 2.12, and the result follows by induction on k. If $r > 1$ then, by Proposition 2.21,

$$(\dagger)\qquad \big\langle\tilde{T}_w,\ \mathscr{L}_{i+1}^r\mathscr{L}_{i_{r+1}}\dots\mathscr{L}_{i_k}\big\rangle = \alpha \sum_{\substack{\sigma\in\mathscr{C}(r-1,\,i)\\ \sigma_1 = 0}} \mathfrak{a}(\sigma_1)\dots\mathfrak{a}(\sigma_{i-1})\mathfrak{b}(\sigma_i)\big\langle\tilde{T}_v,\ \mathscr{L}^\sigma\mathscr{L}_{i_{r+1}}\dots\mathscr{L}_{i_k}\big\rangle.$$

Now if $(\mathfrak{i} - (1^k)) \to \mathfrak{j}$ then by induction there exists a composition σ such that v appears in $\mathscr{L}^\sigma\mathscr{L}_{i_{r+1}}\dots\mathscr{L}_{i_k}$, which is a summand on the right-hand side of (†). Now all of the terms in this sum belong to $\mathscr{H}^+$, so no cancellation can occur. Therefore, w appears in $\mathscr{L}_{i+1}^r\mathscr{L}_{i_{r+1}}\dots\mathscr{L}_{i_k}$ as required.

Conversely, if w appears in $\mathscr{L}_{i_1}\mathscr{L}_{i_2}\dots\mathscr{L}_{i_k}$ then v appears in $\mathscr{L}^\sigma\mathscr{L}_{i_{r+1}}\dots\mathscr{L}_{i_k}$ for some $\sigma \in \mathscr{C}(r-1, i)$ with $\sigma_1 = 0$ by (†). Let $\mathscr{L}^\sigma\mathscr{L}_{i_{r+1}}\dots\mathscr{L}_{i_k} = \mathscr{L}_{i'_2}\dots\mathscr{L}_{i'_k}$ for some weakly decreasing sequence $(i'_2, \dots, i'_k)$. By induction $(i'_2 - 1, \dots, i'_k - 1) \to (j_2, \dots, j_k)$. On the other hand, $(i_1 - 1, \dots, i_k - 1) \to (i_1 - 1, i'_2 - 1, \dots, i'_k - 1)$. By transitivity, $(\mathfrak{i} - (1^k)) \to \mathfrak{j}$. □

Finally, we deduce the corresponding results for products of the Murphy operators $\tilde{L}_i$.

Theorem 2.26 *Suppose that $\mathfrak{i} = (i_1, i_2, \dots, i_k)$ is weakly decreasing and $\mathfrak{j} = (j_1, j_2, \dots, j_l)$ is decreasing and let $w = w_\mathfrak{j}$. Then*

(i) *$w_\mathfrak{j}$ appears in $\tilde{L}_{i_1}\tilde{L}_{i_2}\dots\tilde{L}_{i_k}$ only if $\ell(w_\mathfrak{j}) \le k$.*

(ii) *If $l = \ell(w_\mathfrak{j}) = k$ then $w_\mathfrak{j}$ appears in $\tilde{L}_{i_1}\tilde{L}_{i_2}\dots\tilde{L}_{i_k}$ if and only if $(\mathfrak{i} - (1^k)) \to \mathfrak{j}$.*

(iii) *Suppose that $\ell(w) = k$, $i_1 = j_1$, and let $i = i_1 - 1$ and let r be the integer such that $i_1 = \cdots = i_r$ and $i_r > i_{r+1}$. Then*

$$\big\langle\tilde{T}_{w_\mathfrak{j}},\ \tilde{L}_{i+1}^r\tilde{L}_{i_{r+1}}\dots\tilde{L}_{i_k}\big\rangle = \sum_{\substack{\sigma\in\mathscr{C}(r-1,\,i)\\ \sigma_1 = 0}} \mathfrak{b}(\sigma_i)\mathfrak{a}(\sigma_{i-1})\dots\mathfrak{a}(\sigma_2)\big\langle\tilde{T}_v,\ \tilde{L}^\sigma\tilde{L}_{i_{r+1}}\dots\tilde{L}_{i_k}\big\rangle,$$

where $v = s_{j_2}\dots s_{j_k}$ and $\tilde{L}^\sigma = \tilde{L}_i^{\sigma_i}\tilde{L}_{i-1}^{\sigma_{i-1}}\dots\tilde{L}_2^{\sigma_2}$.

Proof: Since $\mathscr{L}_i = \alpha\tilde{L}_i + 1$ by Lemma 2.3, (i) follows from Proposition 2.21. Consequently, when $\ell(w) = k$ we may expand the $\mathscr{L}_{i_j}$'s using (i) to see that $\langle \tilde{T}_w, \mathscr{L}_{i_1}\mathscr{L}_{i_2}\dots\mathscr{L}_{i_k}\rangle = \alpha^k\langle \tilde{T}_w, \tilde{L}_{i_1}\tilde{L}_{i_2}\dots\tilde{L}_{i_k}\rangle$; (ii) and (iii) follow from this observation combined with Theorem 2.25 and Proposition 2.21 respectively. □

Applying Theorem 2.26(iii) recursively gives a purely combinatorial algorithm for computing the inner products $\langle \tilde{T}_w, \tilde{L}_{i_1}\dots\tilde{L}_{i_k}\rangle$ for any decreasing element w of length k and any weakly decreasing sequence $(i_1, \dots, i_k)$.

3. Symmetric polynomials

In this section we return to Conjecture 1.1 and reduce it to the conjecture that $\sum_\nu \Gamma_{\lambda\nu}\Gamma_{\nu\mu} = \delta_{\lambda\mu}$ (Kronecker delta), for certain polynomials $\Gamma_{\lambda\nu} \in \mathbb{N}[\alpha^2]$ (λ, μ, and ν partitions of some integer $k < n$).

If $\sigma = (\sigma_1, \sigma_2, \dots)$ is a composition, let $\bar{\sigma}$ be the partition obtained from σ by reordering its parts in weakly decreasing order. For example, if $\sigma = (3, 0, 2, 3, 1)$ then $\bar{\sigma} = (3, 3, 2, 1)$.

Definition 3.1 Let k and n be positive integers and let μ a partition of k. Then the μth monomial symmetric function in $\tilde{L}_2, \dots, \tilde{L}_n$ is

$$\tilde{M}_\mu(n) = \sum_{\substack{\sigma\in\mathscr{C}(k,n-1)\\ \bar{\sigma}=\mu}} \tilde{L}_2^{\sigma_1}\tilde{L}_3^{\sigma_2}\dots\tilde{L}_n^{\sigma_{n-1}}.$$

Notice that $\tilde{M}_\mu(n)$ belongs to the centre of $\mathscr{H}_n$ by Lemma 2.2 since it is a symmetric polynomial in the Murphy operators. Also, $\tilde{M}_\mu(n)$ is an element of $\mathscr{H}_n^+$ and $\tilde{M}_\mu(n) = \tilde{M}_\mu(m)$ if and only if $n = m$. For example,

$$\tilde{M}_{(2,1)}(3) = \tilde{L}_2^2\tilde{L}_3 + \tilde{L}_2\tilde{L}_3^2.$$
$$\tilde{M}_{(2,1)}(4) = \tilde{L}_2^2\tilde{L}_3 + L_2\tilde{L}_3^2 + \tilde{L}_2^2\tilde{L}_4 + \tilde{L}_3^2\tilde{L}_4 + \tilde{L}_2\tilde{L}_4^2 + \tilde{L}_3\tilde{L}_4^2.$$

Even though $\tilde{M}_\mu(n) \neq \tilde{M}_\mu(m)$ for $n \neq m$, these elements look the same as far as decreasing elements of length k are concerned. Recall the permutation u_λ from Definition 2.4.

Theorem 3.2 *Suppose that λ is a partition of n and that μ is a partition of k where $k < n$. Then*

(i) $\langle \tilde{T}_{u_\lambda}, \tilde{M}_\mu(n)\rangle \neq 0$ *if and only if* $\ell(u_\lambda) \leq k$.

(ii) *If* $\ell(u_\lambda) \leq k$ *and* $m > n$ *then* $\langle \tilde{T}_{u_\lambda}, \tilde{M}_\mu(m)\rangle = \langle \tilde{T}_{u_\lambda}, \tilde{M}_\mu(n)\rangle$.

Proof: Because all of the summands of $\tilde{M}_\mu(n)$ belong to $\mathscr{H}^+$ no cancellation can occur in $\langle \tilde{T}_{u_\lambda}, \tilde{M}_\mu(n)\rangle$ and the first statement is immediate from Theorem 2.26(ii). For the second statement, suppose that $u_\lambda = s_i v$ for some decreasing element $v \in \mathfrak{S}_i$. Then

$u_\lambda \in \mathfrak{S}_{i+1}$ so without loss of generality we may take $n = i + 1$ and $m > n$. Therefore, by Theorem 2.26(ii),

$$\begin{aligned}\langle \tilde{T}_{u_\lambda}, \tilde{M}_\mu(m)\rangle &= \sum_{\substack{\sigma \in \mathscr{C}(k, m-1)\\ \bar{\sigma} = \mu}} \langle \tilde{T}_{n-1}\tilde{T}_v, \tilde{L}_m^{\sigma_{m-1}} \dots \tilde{L}_2^{\sigma_1}\rangle \\ &= \sum_{\substack{\sigma \in \mathscr{C}(k, n-1)\\ \bar{\sigma} = \mu}} \langle \tilde{T}_{n-1}\tilde{T}_v, \tilde{L}_n^{\sigma_{n-1}} \dots \tilde{L}_2^{\sigma_1}\rangle \\ &= \langle \tilde{T}_{u_\lambda}, \tilde{M}_\mu(n)\rangle,\end{aligned}$$

proving (ii). □

Let λ be a partition of n. As illustrated in Example 2.5, if $\lambda = (\lambda_1, \dots, \lambda_k)$ then u_λ is the permutation

$$(1, \dots, \lambda_1)(\lambda_1 + 1, \dots, \lambda_1 + \lambda_2)(\lambda_1 + \dots + \lambda_{k-1} + 1, \dots, n)$$

and $\ell(u_\lambda) = (\lambda_1 - 1) + (\lambda_2 - 1) + \dots + (\lambda_k - 1)$. Using this observation the next Lemma follows easily.

Lemma 3.3 *Suppose that $n \geq 2k$. Then the map $\mu \mapsto \lambda = \mu + (1^{n-k})$ gives a one-to-one correspondence between the partitions μ of k and the partitions λ of n such that $\ell(u_\lambda) = k$ (where we write $\mu + (1^{n-k})$ for the partition $(\mu_1 + 1, \mu_2 + 1, \dots, \mu_{n-k} + 1)$).*

Consequently, if μ is a partition of k of length l then

$$u_{\mu+(1^l)} = u_{\mu+(1^{l+1})} = u_{\mu+(1^{l+2})} = \cdots$$

where we identify these permutations under the natural embeddings $\mathfrak{S}_1 \hookrightarrow \mathfrak{S}_2 \hookrightarrow \cdots$. Accordingly, we relabel the permutations u_λ using partitions of k, where k is the length of u_λ.

Definition 3.4 Suppose that λ is a partition of k of length l. Let $v_\lambda = u_{\lambda+(1^l)}$.

Thus, $v_\lambda \in \mathfrak{S}_{k+l}$ and we can consider v_λ as an element of $\mathfrak{S}_n$ whenever $n \geq k + l$. In particular, $v_\lambda \in \mathfrak{S}_{2k}$ for any partition λ of k.

Consequently, if λ and μ are partitions of $k < n$ then by Theorem 3.2,

$$m_{\lambda\mu} = \langle \tilde{T}_{v_\lambda}, \tilde{M}_\mu(n)\rangle = \langle \tilde{T}_{v_\lambda}, \tilde{M}_\mu(2k)\rangle \neq 0$$

depends only upon k and not upon n (and $\langle \tilde{T}_{v_\lambda}, \tilde{M}_\mu(n)\rangle = 0$ when $v_\lambda \notin \mathfrak{S}_n$). By definition, $m_{\lambda\mu} \in \mathbb{N}[\alpha^2]$. In principle, Theorem 2.26 can be used to calculate the polynomials $m_{\lambda\mu}$; for example, $m_{(2,1)(3)} = \mathfrak{a}(2)\mathfrak{b}(1) + \mathfrak{a}(1)^2$ by Example 2.24.

We let $\tilde{M}_k = (m_{\lambda\mu})$ be the matrix of these inner products as λ and μ run over the partitions of k. Letting $v = \alpha^2$, and indexing the rows and columns lexicographically (beginning with (1^3)), the full matrix when $k = 3$ is

$$\tilde{M}_3 = \begin{pmatrix} 1 & v^2+3v & v^3+3v^2 \\ 1 & v^2+4v+1 & v^3+4v^2+2v \\ 1 & v^2+5v+3 & v^3+5v^2+5v+1 \end{pmatrix}.$$

Notice that when we set $v = 0$, or equivalently $q = 1$, $\tilde{M}_3$ is unitriangular. Using Theorem 2.26 one can show that $m_{\lambda\lambda}(0) = 1$ and that $m_{\lambda\mu}$ has non-zero constant term if and only if $\lambda \trianglerighteq \mu$, where $\triangleright$ is the usual dominance ordering on partitions. The point is that in the expansion of Theorem 2.26(iii) only the $\mathfrak{b}$ polynomials can occur if we are to get a non-zero constant term in the coefficients. So in general the matrix $\tilde{M}_k$ is lower unitriangular when $v = 0$.

More surprising is the following:

Conjecture 3.5 *Suppose $k \geq 0$ is any positive integer. Then* $\det \tilde{M}_k = 1$. *In particular, $\tilde{M}_k$ is invertible over the Laurent polynomial ring $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$.*

This conjecture was made by Gordon James during discussions with Richard Dipper and the author. James also conjectured that Theorems 2.7 and 3.2 were true. Using these results we next show that Conjecture 3.5 implies Conjecture 1.1. Below we give an explicit conjecture for the inverse of $\tilde{M}_k$.

Let R be a commutative ring, $\bar{q}$ an invertible element in R. As in Definition 3.1, given a partition μ of $k < n$ we define $M_\mu(n) \in \mathscr{H}_{R,\bar{q}}$ to be the monomial symmetric function in the q-Murphy operators $L_2, \ldots, L_n$ of $\mathscr{H}_{R,\bar{q}}$. By Lemma 2.2, $M_\mu(n)$ belongs to the centre of $\mathscr{H}_{R,\bar{q}}$.

Theorem 3.6 *Let R be a commutative ring with* 1, *$\bar{q}$ an invertible element in R, and suppose that the matrices $\tilde{M}_k$ are invertible for all $0 \leq k < n$. Then*

$$\{M_\mu(n) \mid \mu \text{ a partition of } 0 \leq k < n \text{ such that } v_\mu \in \mathfrak{S}_n\}$$

is a basis for the centre of $\mathscr{H}_{R,\bar{q}}$. Consequently, the centre of $\mathscr{H}_{R,\bar{q}}$ is precisely that set of symmetric polynomials in the Murphy operators; thus, Conjecture 3.5 *implies Conjecture* 1.1.

Proof: By Remark 2.14, for any partitions λ and μ of k, the inner product $\langle T_{v_\lambda}, M_\mu(n)\rangle$ in $\mathscr{H}_{R,\bar{q}}$ is equal to the polynomial $q^{\ell(v_\lambda)}\langle \tilde{T}_{v_\lambda}, \tilde{M}_\mu(n)\rangle$ evaluated at $q = \bar{q}$ (note that $m_{\lambda\mu}$

is a polynomial in $\alpha^2 = q - 2 + q^{-1}$). Thus we may work in the generic Hecke algebra and then specialize q to $\bar{q}$.

First consider the case where R is the rational function field $\mathbb{Q}(q)$ and let $\tilde{M}(n)$ be the matrix of inner products $\langle \tilde{T}_{v_\lambda}, \tilde{M}_\mu(n)\rangle$ where λ and μ vary over all partitions of all positive integers k, where $0 \le k < n$. By Theorem 2.26(i), $\tilde{M}(n)$ has the form

$$\tilde{M}(n) = \begin{pmatrix} \tilde{M}_1(n) & & & 0 \\ & \tilde{M}_2(n) & & \\ & & \tilde{M}_3(n) & \\ * & & & \ddots \end{pmatrix}$$

where $\tilde{M}_k(n) = (\langle \tilde{T}_{v_\lambda}, \tilde{M}_\mu(n)\rangle)$ is the submatrix of inner products in $\tilde{M}(n)$ where λ and μ run over all of the partitions of k. By Theorem 3.2, when $v_\lambda \in \mathfrak{S}_n$ the entries in the λth row of $\tilde{M}_k(n)$ are equal to the corresponding entries of the matrix $\tilde{M}_k$; in particular they are independent of n. When $v_\lambda \notin \mathfrak{S}_n$, all of the entries in this row of $\tilde{M}_k(n)$ are zero.

Now, by assumption the matrix $\tilde{M}_k$ has full rank. Consequently the rank of $\tilde{M}_k(n)$ is greater than or equal to the number of partitions λ of k such that $v_\lambda \in \mathfrak{S}_n$ $(0 \le k < n)$. Therefore, the $\mathscr{H}_{\mathbb{Q}(q),q}$-submodule spanned by the monomial symmetric functions $M_\mu(q)$ has dimension greater than or equal to the number of partitions of n. However, by [2, Theorem 2.26] (see also Remark 2.6), the centre of $\mathscr{H}_{\mathbb{Q}(q),q}$ has dimension less than or equal to the number of partitions of n. This completes the proof of the proposition when $R = \mathbb{Q}(q)$. (In fact this argument holds whenever R is a field.)

Now because the matrices $\tilde{M}_k$ are invertible over $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{1}{2}}]$, the monomial symmetric functions $\{M_\mu(n)\}$ span an $\mathcal{A}$-lattice $\mathscr{M}$ in the centre of $\mathscr{H}_{\mathbb{Q}(q),q}$ and $\mathscr{M} \otimes_{\mathcal{A}} R$ is the centre of $\mathscr{H}_{R,\bar{q}}$ (here we consider R as an $\mathcal{A}$-module by specifying that q acts as $\bar{q}$ on R). Therefore, the $\{M_\mu(n) = \bar{q}^{\ell(v_\mu)}\tilde{M}_\mu(n) \otimes 1\}$ are linearly independent and span the centre of $\mathscr{H}_{R,\bar{q}}$. □

Although we have not been able to prove Conjecture 3.5, we do have an explicit conjecture for the inverse matrix of $\tilde{M}_k$. Recall that $\mathscr{C}(k)$ is the set of strict compositions of k. There is a well-known one-to-one correspondence between the compositions σ in $\mathscr{C}(k)$ and the subsets of $\{1, 2, \ldots, k-1\}$ where

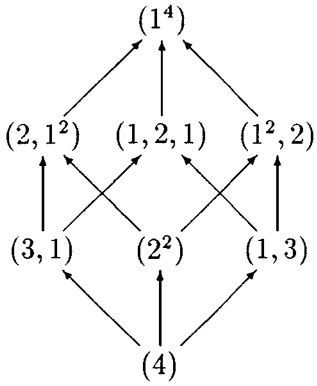
$$\sigma = (\sigma_1, \ldots, \sigma_l) \leftrightarrow J_\sigma = \{\sigma_1, \sigma_1 + \sigma_2, \ldots, \sigma_1 + \cdots + \sigma_{l-1}\}.$$

For example, $J_{(1^k)} = \{1, 2, \ldots, k-1\}$ and $J_{(k)} = \emptyset$ for all r. Given two partitions λ and μ define

$$c_{\lambda\mu} = (-1)^{\ell(\mu)}|\{\sigma \in \mathscr{C}(k) \mid J_\lambda \subseteq J_\sigma \text{ and } \bar{\sigma} = \mu\}|,$$

and let $C_k = (c_{\lambda\mu})$. For example when $k = 4$, and

$\mathscr{C}(4) =$



and

$$C_4 = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot & \cdot \\ 1 & -2 & 1 & \cdot & \cdot \\ 1 & -2 & 0 & 1 & \cdot \\ 1 & -3 & 1 & 2 & -1 \end{pmatrix},$$

where the rows and columns of C_4 are ordered lexicographically and all omitted entries are zero.

So, $|c_{(k)\mu}|$ is equal to the number of compositions σ in $\mathscr{C}(k)$ such that $\bar{\sigma} = \mu$; more generally, $|c_{\lambda\mu}|$ is the number of *refinements* of λ into compositions σ such that $\bar{\sigma} = \mu$.

Conjecture 3.7 *The inverse matrix of $\tilde{M}_k$ is $C_k \tilde{M}_k C_k$.*

In particular this implies Conjecture 3.5 and hence, by Theorem 3.6, Conjecture 1.1.

It is not hard to show that $C_k^2 = I$ for all k; in fact this reduces to the well-known identity

$$\sum_{J \subseteq S} (-1)^{|J|} = \begin{cases} 1, & \text{if } S = \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

So, in fact Conjecture 3.7 claims that $\tilde{M}_k^{-1} = C_k \tilde{M}_k C_k^{-1}$. Further, if $\Gamma_k = C_k \tilde{M}_k$ then Conjecture 3.7 says that $\Gamma_k^2 = I$. Equivalently, if $\Gamma_k = (\Gamma_{\lambda\mu})$ then Conjecture 3.7 holds if and only if $\sum_\nu \Gamma_{\lambda\nu}\Gamma_{\nu\mu} = \delta_{\lambda\mu}$.

By the above remarks Γ_k is lower unitriangular when $v = 0$ and, assuming Conjecture 3.5, $\det \Gamma_k = \det C_k = \pm 1$. It would also appear to be true that $(-1)^{\ell(\lambda)}\Gamma_{\lambda\mu}$ is a polynomial in v with non-negative coefficients. The matrices Γ_k for $k = 1, 2, 3,$ and 4 are as follows.

$$\Gamma_1 = (1), \quad \Gamma_2 = \begin{pmatrix} 1 & v \\ 0 & -1 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} -1 & -v^2 - 3v & -v^3 - 3v^2 \\ 0 & v + 1 & v^2 + 2v \\ 0 & -1 & -v - 1 \end{pmatrix},$$

and Γ_4 is the matrix

$$\begin{pmatrix} 1 & v^3+4v^2+6v & v^4+4v^3+3v^2 & v^5+7v^4+15v^3+12v^2 & v^6+8v^5+20v^4+16v^3 \\ 0 & -v^2-2v-1 & -v^3-3v^2-v & -v^4-6v^3-9v^2-4v & -v^5-7v^4-14v^3-8v^2 \\ 0 & v & v^2+2v+1 & v^3+5v^2+4v & v^4+6v^3+9v^2+4v \\ 0 & v+1 & v^2+2v & v^3+5v^2+5v+1 & v^4+6v^3+9v^2+3v \\ 0 & -1 & -v-1 & -v^2-4v-2 & -v^3-5v^2-5v-1 \end{pmatrix}.$$

The reader may check that $\det \Gamma_k = \pm 1$ and that $\Gamma_k^2 = I$ as we have claimed. We have checked Conjecture 3.7 for $k \leq 7$.

Acknowledgments

I would like to thank Gordon James and Richard Dipper for many invaluable discussions on this problem, without which this work could not have started. I also wish to thank James for his comments on earlier versions of this paper, Meinolf Geck for writing his GAP programs for Iwahori-Hecke algebras [5] which helped me guess the form of Theorem 2.21 and the referee for several useful comments. Supported in part by SERC grant GR/J37690.

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