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Zassenhaus Lie Idempotents, *q*-Bracketing and a New Exponential/Logarithm Correspondence

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Abstract. We introduce a new q-exponential/logarithm correspondance that allows us to solve a conjecture relating Zassenhauss Lie idempotents with other Lie idempotents related to the q-bracketing operator.

Keywords: Fer-Zassenhauss formula, Lie idempotents, noncommutative symmetric functions, logarithm, exponential

1. Introduction

The algebra of noncommutative symmetric functions **Sym**, introduced in [2], is the free associative algebra (over some field of characteristic zero) generated by an infinite sequence $(S_n)_{n\geq 1}$ of noncommutating inderminates (intended to correspond to the complete noncommutative symmetric functions) and endowed with some extra structure imitated from the usual algebra of commutative symmetric functions. This point of view consists typically in defining other noncommutative symmetric functions, in terms of the complete functions that are initially given, by taking noncommutative analogues of the classical relations that exist between usual commutative symmetric functions.

Noncommutative symmetric functions have already been used in several contexts. They provide a simple and unified approach to several topics such as noncommutative continued fractions, Padé approximants and various generalizations of the characteristic polynomial of noncommutative matrices arising in the study of classical enveloping algebras and their quantum analogues (cf [2, 10]). They also provided a new point of view regarding the classical connections between the free Lie algebra and Solomon's descent algebra (see [2, 4, 5, 11] for more details). One can in particular use the theory of noncommutative symmetric functions in order to characterize the Lie idempotents that belong to Solomon's descent algebra (cf [2]) and obtain new families of Lie idempotents within Solomon's descent algebra that interpolate between all classical Lie idempotents (cf [4]).

More recently quantum interpretations of the algebras of noncommutative symmetric functions and of quasi-symmetric functions (the Hopf dual of **Sym** constructed initially by Gessel in [3]) were obtained (cf [6, 8]). It indeed appears that the algebra of noncommutative symmetric functions (resp. of quasi-symmetric functions) is isomorphic to the Grothendieck ring of finitely generated projective (resp. finitely generated) modules over 0-Hecke algebras (cf [6]). A similar interpretation of these two algebras can also be obtained in terms of the representation theory of Takeuchi's version of $U_q(Gl_n)$ (cf [14]) taken at $q = 0$ (see [7, 8]). Noncommutative ribbon Schur functions and quasi-ribbon functions appear then respectively in these interpretations as

- the cocharacters of the irreducible and the projective comodules over the crystal limit of the Dipper-Donkin version of the quantum linear group (see [1, 6] for more details),
- the characters of the irreducible and the projective polynomial modules over the crystal limit of the Takeuchi version of the quantum enveloping algebra $U_q(Gl_n)$ (see [7, 8, 14] for more details).

In this paper, we are however going back to the beginning of the theory of noncommutative symmetric functions. Indeed, our article solves a conjecture, originally stated in [4], that establishes a strange connection between the family of Zassenhaus Lie idempotents and the Lie idempotents corresponding to the projection onto the Lie algebra associated with the q-bracketing operator. This connection is obtained by introducing a new exponential/logarithm like correspondence which allows us to describe in a very simple way the Lie idempotents associated with the q-bracketing operator as mentioned above.

This paper is therefore organized as follows. In Section 2, we briefly present noncommutative symmetric functions (the reader is referred to [2, 4] for more details on this subject). Section 3 is devoted to the construction of our new analog of the exponential and the obtention of its main properties. Section 4 makes then the connection between this analog of the exponential and noncommutative symmetric functions in order to solve the above mentioned conjecture. In the short concluding Section 5, we finally give some indications for a further possible generalization of our work.

2. Preliminaries

2.1. Noncommutative symmetric functions

The algebra of *formal noncommutative symmetric functions* is the free associative algebra $\mathbf{Sym} = \mathbb{K}\langle S_1, S_2, \ldots\rangle$ (over some field $\mathbb{K}$ of characteristic zero) generated by an infinite sequence of noncommutative indeterminates S_k, called the *complete* symmetric functions (see [2] for more details). We set for convenience $S_0 = 1$. Let t be another variable commuting with all the S_k. Introducing the generating series

$$\sigma(t) := \sum_{k=0}^{\infty} S_k t^k,$$

one can define other noncommutative symmetric functions by the following relations

$$\lambda(t) = \sigma(-t)^{-1}, \quad \frac{d}{dt}\sigma(t) = \sigma(t)\psi(t), \quad \sigma(t) = \exp(\phi(t)),$$

where $\lambda(t)$, $\psi(t)$ and $\phi(t)$ are the generating series

$$\lambda(t) := \sum_{k=0}^{\infty} \Lambda_k t^k, \quad \psi(t) := \sum_{k=1}^{\infty} \Psi_k t^{k-1}, \quad \phi(t) := \sum_{k=1}^{\infty} \frac{\Phi_k}{k} t^k.$$

The noncommutative symmetric functions Λ_k are called *elementary functions*. The elements Ψ_k (resp. Φ_k) are called *power sums* of *first kind* (resp. *second kind*).

The algebra **Sym** is graded by the weight function w defined by $w(S_k) = k$. Its homogeneous component of weight n will be denoted by $\mathbf{Sym}_n$. If (F_n) is a sequence of noncommutative symmetric functions with $F_n \in \mathbf{Sym}_n$ for every $n \geq 1$, we set

$$F^I = F_{i_1} F_{i_2} \ldots F_{i_r}$$

for every composition $I = (i_1, i_2, \ldots, i_r)$. The families (S^I), (Λ^I), (Ψ^I) and (Φ^I) are then homogeneous bases of **Sym**.

The set of all compositions of a given integer n is equipped with the *reverse refinement order*, denoted $\preceq$. For instance, the compositions J of 4 such that $J \preceq (1, 2, 1)$ are $(1, 2, 1)$, $(3, 1)$, $(1, 3)$ and (4). The *ribbon Schur functions* (R_I) can then be defined by one of the two equivalent relations

$$S^I = \sum_{J \preceq I} R_I, \qquad R_I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} S^J,$$

where $\ell(I)$ denotes the *length* of the composition I. One can easily show that the family (R_I) is a homogeneous basis of **Sym**.

The *commutative image* of a noncommutative symmetric function F is the commutative symmetric function f obtained by applying to F the algebra morphism which maps S_n onto h_n, using here the notations of [9]. The commutative image of Λ_n is then e_n. On the other hand, Ψ_n is mapped to p_n. Finally R_I is sent to the ordinary ribbon Schur function r_I.

One can endow **Sym** with a structure of Hopf algebra, its comultiplication Δ being defined by one of the following equivalent formulas

$$\Delta(S_n) = \sum_{i=1}^{n} S_i \otimes S_{n-i}, \qquad \Delta(\Lambda_n) = \sum_{i=1}^{n} \Lambda_i \otimes \Lambda_{n-i},$$
$$\Delta(\Psi_n) = 1 \otimes \Psi_n + \Psi_n \otimes 1, \qquad \Delta(\Phi_n) = 1 \otimes \Phi_n + \Phi_n \otimes 1.$$

It is in fact this Hopf structure which explains in a unified way the properties of Lie idempotents as we will see in the sequel.

2.2. Relations with Solomon's descent algebra

Let $\sigma \in \mathfrak{S}_n$ be a permutation with descent set $E = \{d_1 < \cdots < d_k\} \subseteq [n-1]$. The *descent composition* $I = C(\sigma)$ is the composition $I = (i_1, \ldots, i_{k+1})$ of n defined by $i_s = d_s - d_{s-1}$ where $d_0 = 0$ and $d_{k+1} = n$. The sum in the group algebra of all permutations with descent composition I is denoted by D_I. We also set $I = C(E)$. Conversely the subset of $[1, n-1]$ associated with a composition I of n will be denoted by $E = E(I)$. The D_I with $|I| = n$ form a basis of a subalgebra of $\mathbb{Z}[\mathfrak{S}_n]$, called the *descent algebra* of $\mathfrak{S}_n$ (cf [12]). We denote then by Σ_n the same algebra, with scalars extended to our ground field $\mathbb{K}$.

There is in fact a strong connection between noncommutative symmetric functions and the descent algebras of the symmetric group. One can indeed define an isomorphism of graded vector spaces by setting

$$\begin{aligned} \alpha : \Sigma = \bigoplus_{n=0}^{\infty} \Sigma_n &\longrightarrow \mathbf{Sym} = \bigoplus_{n=0}^{\infty} \mathbf{Sym}_n \\ D_I &\longrightarrow R_I \end{aligned}$$

for any composition I. The existence of this isomorphism shows that an element of $\mathbf{Sym}_n$ is just a certain encoding of an element of the descent algebra Σ_n. Note that the interpretation of S_n and Λ_n in this encoding is simple since one has

$$\begin{cases} \alpha^{-1}(S_n) = D_n = Id_n, \\ \alpha^{-1}(\Lambda_n) = D_{1^n} = \omega_n, \end{cases}$$

where Id_n denotes the identity permutation of order n and where ω_n denotes the maximal permutation $n n-1 \ldots 1$ of $\mathfrak{S}_n$. We will see in the next section that there is also a strong connection between noncommutative symmetric functions and Lie idempotents that passes through Solomon's descent algebra.

2.3. Lie idempotents

Let A be an alphabet. A *Lie projector* is a projection from the free associative algebra $\mathbb{K}\langle A\rangle$ onto the free Lie algebra $L(A)$. In other words, a Lie projector is an endomorphism π of $\mathbb{K}\langle A\rangle$ that satisfies the two following properties:

- $\pi^2 = \pi$ (π is a *projector*);
- $\operatorname{Im} \pi = L(A)$ (π is a *Lie* projector).

The basic property of a Lie projector is that it maps any Lie element on itself.

Recall that the symmetric group $\mathfrak{S}_n$ acts on the homogeneous component of order n of $\mathbb{K}_n\langle A\rangle$ (and hence on $L_n(A)$) by setting

$$a_1 \ldots a_n \cdot \sigma = a_{\sigma(1)} \cdots a_{\sigma(n)}$$

for $a_i \in A$ and $\sigma \in \mathfrak{S}_n$. Recall also that an endomorphism f of $\mathbb{K}\langle A\rangle$ is said to commute with letter substitutions if one has

$$f(s(w)) = s(f(w))$$

for every endomorphism s of $\mathbb{K}\langle A\rangle$ which maps every letter $a \in A$ onto another letter $s(a) \in A$ (such an endomorphism is called a letter substitution). Note that, according to Schur-Weyl duality, an endomorphism commuting with letter substitutions also commutes with the right action of $\mathfrak{S}_n$.

Suppose now that we work with an infinite alphabet $A = \{1, 2, \ldots\}$. We shall only consider in the sequel Lie projectors with the following properties:

1. π is *finely homogeneous* ($\pi(\mathbb{K}_\lambda\langle A\rangle) \subset L_\lambda(A)$ for every multidegree λ),
2. π commutes with letter substitutions.

If π is a Lie projector which satisfies these properties, it is easy to see that one can recover π from the sequence $(\pi_n)_{n\geq 1}$ where π_n is defined by setting

$$\pi_n = \pi(12\ldots n) \in L_{1^n}(1, 2, \ldots, n)$$

for every $n \geq 1$ (these elements belong to the multilinear components $L_{1^n}(1, \ldots, n)$ of the free Lie algebras on the alphabets $[n] = \{1, \ldots, n\}$). Indeed,

$$\pi(a_1 \ldots a_n) = \pi(s(12\ldots n)) = s(\pi_n)$$

where s denotes the letter substitution of $\mathbb{K}\langle 1, 2, \ldots, n\rangle$ mapping i onto a_i. Since π is finely homogeneous, one can consider π_n as an element of the group algebra $\mathbb{K}[\mathfrak{S}_n]$ (permutations being identified with standard words) which is clearly an idempotent (i.e. $\pi_n^2 = \pi_n$) of this algebra. The study of Lie projectors that satisfy to the two conditions above can therefore be reduced to the study of *Lie idempotents* in $\mathbb{K}[\mathfrak{S}_n]$, i.e. of those idempotents of $\mathbb{K}[\mathfrak{S}_n]$ that can be expressed as Lie polynomials over the alphabet $\{1, 2, \ldots, n\}$.

An element π of the group algebra $\mathbb{K}[\mathfrak{S}_n]$ is a *Lie element* if it can be expressed as a Lie polynomial over the alphabet $\{1, 2, \ldots, n\}$. A *Lie quasi-idempotent* is a Lie element of $\mathbb{K}[\mathfrak{S}_n]$ which is a quasi-idempotent (an element x of a $\mathbb{K}$-algebra is said to be quasi-idempotent if and only if $x^2 = kx$ with $k \in \mathbb{K}$).

It appears that one can use noncommutative symmetric functions in order to classify all the Lie quasi-idempotents that belong to the descent algebra of the symmetric group. Indeed, let us denote by $L(\Psi)$ the free Lie algebra generated by the family $(\Psi_n)_{n\geq 1}$ within **Sym**. We can then state the following result that gives an explicit characterization of all Lie quasi-idempotents of the descent algebra of the symmetric group.

Theorem 2.1 (Gelfand, Krob, Lascoux, Leclerc, Retakh, Thibon [2]) *Let F_n be an element of* $\mathbf{Sym}_n$ *and let $f_n = \alpha^{-1}(F_n)$ be the associated element of Σ_n. The following assertions are then equivalent*:

1. *f_n is a Lie quasi-idempotent,*
2. *F_n belongs to the free Lie algebra $L(\Psi)$,*
3. *F_n is a primitive element for the coproduct Δ.*

It is interesting to stress the fact that the non commutative power sums of first and second kinds correspond to some remarkable Lie idempotents. On can indeed check that the inverse image under α of the noncommutative symmetric function Ψ_n is equal to Dynkin's (quasi)-idempotent, i.e.

$$\alpha^{-1}(\Psi_n) = [[\ldots[[1, 2], 3], \ldots], n] = \vartheta_n.$$

On the other hand, the inverse image of Φ_n under α correspond to the so-called Solomon's (quasi)-idempotent, φ_n, that encodes (up to a constant) the projection onto the free Lie algebra with respect to the classical Poincaré-Birkhoff-Witt decomposition of the free associative algebra (see [2], [11] and [13] for more details).

2.4. The transformation $A \longrightarrow (1-q)A$ and the q-bracketing operator

The q-bracketing operator (of order n) is the linear operator $\vartheta_n(q)$ over the free associative algebra $\mathbb{K}\langle A\rangle$ defined by setting

$$\vartheta_n(q)(a_1a_2\ldots a_n) = [[\ldots[[a_1, a_2]_q, a_3]_q, \ldots], a_n]_q$$

for every word $a_1a_2\ldots a_n$ of A^*, where we set

$$[u, v]_q = uv - qvu$$

for every u and v in $\mathbb{K}\langle A\rangle$. It happens that this operator can be interpreted in terms of noncommutative symmetric functions. Let us indeed consider the noncommutative analogs of commutative complete symmetric functions of the alphabet $(1-q)A$ (in the generalized λ-ring style notation introduced in [4]) which are defined as follows (the generating series $\sigma(t)$ and $\lambda(t)$ are here denoted by $\sigma(A;t)$ and $\lambda(A;t)$ in order to put the stress on the different alphabets that we are using (cf [4] for more details)).

Definition 2.2 The generating series of the family $(S_n((1-q)A))_{n\geq 1}$ of complete symmetric functions of the alphabet $(1-q)A$ is given by

$$\begin{aligned}\sigma((1-q)A);t) &:= \sum_{n\geq 0} S_n((1-q)A)t^n = \sigma(A;qt)^{-1}\sigma(A;t)\\ &= \lambda(A;-qt)\sigma(A;t).\end{aligned} \tag{1}$$

One can then show (see [4] for all the details) that

$$S_n((1-q)A) = \begin{cases}(1-q)\Theta_n(q) & \text{if } n \geq 1\\ 1 & \text{if } n = 0\end{cases}.$$

where we set

$$\Theta_n(q) = \alpha^{-1}(\vartheta_n(q)) = \sum_{i=0}^{n-1} (-q)^i R_{1^i,\ n-i}$$

where $\vartheta_n(q)$ stands for the image of the identity by the q-bracketing operator, i.e. for the element

$$\vartheta_n(q) = [[\ldots[[1,2]_q,3]_q,\ldots],n]_q = \sum_{i=0}^{n-1} (-q)^i D_{1^i,n-i}$$

of Solomon's descent algebra (we use here the same notation for the q-bracketing operator and for the image of the identity by this operator). In other words, the q-bracketing operator is essentially equal to the image through the isomorphism α of the noncommutative complete symmetric function of the alphabet $(1-q)A$.

It appears that the q-bracketing operator is diagonalizable with eigenvalues

$$u_\lambda = \frac{1}{1-q}\psi^\lambda(1-q) = \frac{1}{1-q}\big(1-q^{\lambda_1}\big)\big(1-q^{\lambda_2}\big)\cdots\big(1-q^{\lambda_r}\big)$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ runs through all partitions of the integer n (see [4] for all details). The eigenspace corresponding to the eigenvalue

$$\frac{1}{1-q}(1-q^n)$$

is remarkable since it is exactly the free Lie algebra. In other words, we have the following q-Dynkin criterion for a non commutative polynomial P of $\mathbb{K}\langle A\rangle$ to belong to the free Lie algebra:

$$P \in L_n(A) \Longleftrightarrow \vartheta_n(q)(P) = \frac{1-q^n}{1-q}P.$$

Let us denote by $E_\lambda(A)$ the eigenspace of the q-bracketing operator associated with the eigenvalue u_λ. The diagonalizability of $\vartheta_n(q)$ allows then us to write

$$\mathbb{K}\langle A\rangle_n = L_n(A) \oplus \bigoplus_{\substack{\lambda\vdash n\\ l(\lambda)\geq 2}} E_\lambda(A)$$

since we have $E_n(A) = L_n(A)$ as explained above. There exists therefore a Lie projector associated with this decomposition of the homogeneous component of weight n of the free associative algebra, i.e. a Lie projector with range $L_n(A)$ and kernel

$$\bigoplus_{\substack{\lambda\vdash n\\ l(\lambda)\geq 2}} E_\lambda(A).$$

It happens that the Lie idempotent, denoted $\Pi_n(q)$, corresponding to this Lie projector (in the sense of Section 2.3) belongs to the descent algebra. We can therefore define a noncommutative symmetric function, that we shall denote by $\pi_n(q)$, by setting

$$\pi_n(q) = \alpha^{-1}(\Pi_n(q))$$

for every $n \geq 1$. These new noncommutative symmetric functions $\pi_n(q)$ can be characterized as follows (see [4] for all the details).

Theorem 2.3 (Krob, Leclerc, Thibon [4]) *The noncommutative symmetric function $\pi_n(q)$ (associated with the Lie idempotent $\Pi_n(q)$) is characterized by the property*

$$\pi_n(q)((1-q)A) = (1-q^n)\pi_n(q)(A). \tag{2}$$

The aim of the present paper is to study in deep details these noncommutative symmetric functions. We will in particular give in the sequel a solution for a conjecture, initially stated in [4], that connects the specialization at $q = 0$ of these elements to the so-called Zassenhaus Lie idempotents.

3. A new exponential/logarithm correspondence

3.1. A new analog of the exponential

Let us consider some infinite alphabet $A = \{a_k, k \geq 1\}$. Let now

$$X(A) = \sum_{w \in A^*} x_w w$$

be a formal power series of $\mathbb{K}(q)\langle\langle A\rangle\rangle$. We associate then with this formal power series the two other formal power series $X(qA)$ and $X((1-q)A)$ of $\mathbb{K}(q)\langle\langle A\rangle\rangle$ defined by setting

$$\begin{cases} X(qA) = \sum_{w \in A^*} x_w q^{\|w\|} w, \\ X((1-q)A) = \sum_{w \in A^*} x_w l_q(w) w, \end{cases}$$

where we have

$$\|w\| = \sum_{k=1}^{r} i_k \quad \text{and} \quad l_q(w) = \prod_{k=1}^{r} \left(1 - q^{i_k}\right)$$

for every $w = a_{i_1} a_{i_2} \ldots a_{i_r}$ of A^*. In other words, $X(qA)$ (resp. $X((1-q)A)$) is obtained by applying to $X(A)$ the substitution $a_k \to q^k a_k$ (resp. $a_k \to (1-q^k)a_k$).

We can now state the following result that will allow us to introduce further a new analog of the exponential.

Proposition 3.1 *Let $X(A)$ be a formal power series of $\mathbb{K}(q)\langle\langle A\rangle\rangle$, i.e.*

$$X(A) = \sum_{w\in A^*} x_w w$$

where $x_w \in \mathbb{K}$ denotes the coefficient of X on w. Then the conditions given below are equivalent:

1. *X satisfies the following functional equation*

$$X(qA)X((1-q)A) = X(A), \tag{3}$$

2. *for every word $w \in A^*$, one has*

$$x_w\big(1 - x_1\big(q^{\|w\|} + l_q(w)\big)\big) = \sum_{\substack{uv=w\\u,v\neq 1}} x_u x_v q^{\|u\|} l_q(v). \tag{4}$$

When $x_1 = 1$, all coefficients x_w are in particular uniquely defined by the identity (4) *when the coefficients x_{a_i} are fixed for every $a_i \in A$.*

Proof: By taking the cofficients of a word $w \in A^*$ in both sides of the functional equation, we can obtain the following relation:

$$x_w = \sum_{uv=w} x_u x_v q^{\|u\|} l_q(v). \tag{5}$$

Collecting all the terms containing x_w, we get

$$x_w = x_1 x_w l_q(w) + x_w x_1 q^{\|w\|} + \sum_{\substack{uv=w\\u,v\neq 1}} x_u x_v q^{\|u\|} l_q(v),$$

which immediately leads to the desired identity (4).

Let us now consider a word $w = a_{i_1} a_{i_2} \dots a_{i_r}$ of A^* and let us denote by $D_w(q)$ the polynomial involved in the left hand side of identity (4), i.e.

$$D_w(q) = 1 - x_1\big(l_q(w) + q^{\|w\|}\big).$$

When $x_1 = 1$, we then have

$$D_w(q) = 1 - q^{\|w\|} - \prod_{k=1}^{r}\big(1 - q^{i_k}\big). \tag{6}$$

Let us now define the two polynomials

$$D_w^{(1)}(q) = 1 - q^{\|w\|} \quad \text{and} \quad D_w^{(2)}(q) = \prod_{k=1}^{r}\big(1 - q^{i_k}\big).$$

If $r \geq 2$, the multiplicities of the root $q = 1$ in these two last polynomials are clearly different: the multiplicity of the root $q = 1$ in $D_w^{(1)}(q)$ is one when this multiplicity in $D_w^{(2)}(q)$ equals r. It follows that the difference of these two polynomials, which is exactly $D_w(q)$, can not be zero. It is now immediate to conclude that identity (4) defines in a unique way all the coefficients x_w when the coefficients x_{a_i} are fixed for every $a_i \in A$. □

We can now give the definition of the analog of the exponential that we will study in this paper.

Definition 3.2 The series $E_q(A)$ is by definition the unique formal power series

$$E_q(A) = \sum_{w \in A^*} x_w w$$

of $\mathbb{K}(q)\langle\langle A\rangle\rangle$ which satisfies both to the functional Eq. (3) of Proposition (3.1) and to the conditions $x_1 = 1$ and $x_{a_k} = 1$ for every $k \geq 1$.

Note 3.3 The coefficients x_w of the series E_q defined above satisfy therefore to the following induction relation

$$x_w = \frac{1}{1 - q^{\|w\|} - l_q(w)} \left(\sum_{\substack{uv=w \\ u,v \neq 1}} x_u x_v q^{\|u\|} l_q(v) \right) \tag{7}$$

that holds for every word $w \in A^*$ of length at least 2.

The series E_q defined by the previous definition has several connections with the ordinary exponential (see for instance Section 3.3 and Proposition 3.11). In a first approach, we can however immediately state the following result that gives a very first relation between E_q and the ordinary exponential.

Proposition 3.4 *Let $k \geq 1$ be an integer and let $E_q(0, \ldots, 0, a_k, 0, \ldots)$ denote the formal power series of $\mathbb{K}\langle\langle A\rangle\rangle$ obtained by specializing a_i to 0 for every $i \neq k$. Then one has*

$$E_q(0, \ldots, 0, a_k, 0, \ldots) = \exp(a_k) = \sum_{i=0}^{\infty} \frac{a_k^i}{i!}.$$

Proof: The announced result is equivalent to the fact that one has

$$x_{a_k^i} = \frac{1}{i!}$$

for every $i \geq 0$. This property beeing clearly true at $i = 0$ and $i = 1$, we can prove it by induction on i. Note now that formula (7) shows that proving the corresponding induction step at order i is equivalent to proving the following identity

$$\frac{1}{i!}(1 - q^{ik} - (1 - q^k)^i) = \sum_{j=1}^{i-1} \frac{1}{j!}\frac{1}{(i-j)!} q^{jk}(1 - q^k)^{i-j},$$

which is itself clearly equivalent to

$$\sum_{j=0}^{i} \frac{(q^k)^j}{j!}\frac{(1 - q^k)^{i-j}}{(i-j)!} = \frac{1}{i!}.$$

Note now that this last relation is obvious since it just expresses that the coefficient of order i of the (commutative) series $\exp(t)$ is also the coefficient of order i of the Cauchy product

$$\exp(q^k t)\ \exp((1 - q^k)t)$$

(which is clearly equal to the series $\exp(t)$). This ends our proof. □

3.2. Existence of an analog of the logarithm

Let us consider again some infinite alphabet $A = \{a_k, k \geq 1\}$ and let

$$X(A) = \sum_{w \in A^*} x_w w$$

be a formal power series of $\mathbb{K}(q)\langle\langle A\rangle\rangle$. The coefficient x_1 (over the empty word of A^*) of X is called the constant coefficient of X. When a series has a constant coefficient equal to 0 (i.e. when x_1 is equal to 0), it is called a zero constant coefficient series.

Let now $Y(A)$ be a zero constant coefficient series of $\mathbb{K}(q)\langle\langle A\rangle\rangle$, i.e.

$$Y(A) = \sum_{w \in A^+} y_w w$$

(where A^+ stands for the set of all non empty words over A). Then one can use the grading δ defined by setting $\delta(a_i) = i$ in order to separate $Y(A)$ into homogeneous components, i.e.

$$Y(A) = \sum_{i=1}^{+\infty} Y_i(A)$$

where $Y_i(A)$ stands for the polynomial

$$Y_i(A) = \sum_{\substack{w \in A^+ \\ \delta(w) = i}} y_w w.$$

We can now define the *composition* $X(A) \circ Y(A)$ of $Y(A)$ with $X(A)$ by setting

$$X(A) \circ Y(A) = \sigma_Y(X(A))$$

where σ_Y stands for the algebra morphism from $\mathbb{K}(q)\langle\langle A\rangle\rangle$ into $\mathbb{K}(q)\langle\langle A\rangle\rangle$ which maps every letter a_i of A onto $Y_i(A)$. We are now in a position to state the following result that shows the existence of an analog of the logarithm (more exactly of the series $\log(1+X)$) as the reciprocal (in the sense of our series composition) of our analog of the exponential.

Theorem 3.5 *There exists a unique series $L_q(A)$ of $\mathbb{K}(q)\langle\langle A\rangle\rangle$ with zero constant coefficient such that the two following properties hold*:

$$\begin{cases} E_q(A) \circ L_q(A) = 1 + \sum_{i=1}^{+\infty} a_i\,, \\ L_q(A) \circ (E_q(A) - 1) = \sum_{i=1}^{+\infty} a_i\,. \end{cases}$$

Proof: Let us first prove that there exists a unique series $L_q(A)$ of $\mathbb{K}(q)\langle\langle A\rangle\rangle$ that satisfies to the first property above. By definition, there exists a series

$$L_q(A) = \sum_{w\in A^+} y_w\, w = \sum_{i=1}^{+\infty}\left(y_{a_i} a_i + \sum_{\substack{|w|\geq 2\\ \delta(w)=i}} y_w\, w\right)$$

of $\mathbb{K}(q)\langle\langle A\rangle\rangle$ with zero constant coefficient such that

$$E_q(A) \circ L_q(A) = 1 + \sum_{i=1}^{+\infty} a_i \tag{8}$$

if and only if one has

$$1 + \sum_{\substack{i_1,\ldots,i_r\geq 1\\ r\geq 1}} x_{a_{i_1}\ldots a_{i_r}} \prod_{j=1}^{r}\left(y_{a_{i_j}} a_{i_j} + \sum_{\substack{|w|\geq 2\\ \delta(w)=i_j}} y_w\, w\right) = 1 + \sum_{i=1}^{+\infty} a_i$$

where the x_w's stand for the coefficients of our analog of the exponential. It is now easy to see that the above identity is equivalent to the fact that one has first $y_{a_i} = 1$ for every $i \geq 1$ and next

$$\sum_{r=1}^{|w|}\left(\sum_{\substack{u_1,\ldots,u_r\in A^+\\ u_1\ldots u_r = w}} x_{a_{\delta(u_1)}\ldots a_{\delta(u_r)}} y_{u_1}\ldots y_{u_r}\right) = 0$$

for every word w of length at least 2. Note now that this condition is equivalent to

$$y_w = -x_w - \sum_{r=2}^{|w|-1} \left(\sum_{\substack{u_1,\ldots,u_r \in A^+ \\ u_1 \ldots u_r = w}} x_{a_{\delta(u_1)} \ldots a_{\delta(u_r)}} y_{u_1} \ldots y_{u_r} \right)$$

for every word w of length at least 2. Since these last relations together with the requirement that $y_{a_i} = 1$ for every $i \geq 1$, define in a unique way the family $(y_w)_{w \in A^+}$, it is now immediate to conclude to the existence of a unique series $L_q(A)$ of $\mathbb{K}(q)\langle\langle A \rangle\rangle$ that satisfies Eq. (8).

It follows now immediately that one has

$$\begin{aligned} L_q(A) \circ (E_q(A) - 1) \circ L_q(A) &= L_q(A) \circ (E_q(A) \circ L_q(A) - 1) \\ &= L_q(A) \circ \left(\sum_{i=1}^{+\infty} a_i \right) = L_q(A). \end{aligned}$$

However, using the same method as above, it is easy to prove that there exists a series T with zero constant coefficient in $\mathbb{K}(q)\langle\langle A \rangle\rangle$ such that

$$L_q(A) \circ T = \sum_{i=1}^{+\infty} a_i .$$

Composing at the right this identity by T the last identity, we now get

$$L_q(A) \circ (E_q(A) - 1) \circ L_q(A) \circ T = L_q(A) \circ T.$$

This last identity is therefore equivalent to the relation

$$L_q(A) \circ (E_q(A) - 1) \circ \left(\sum_{i=1}^{+\infty} a_i \right) = \sum_{i=1}^{+\infty} a_i$$

which is itself clearly equivalent to

$$L_q(A) \circ (E_q(A) - 1) = \sum_{i=1}^{+\infty} a_i ,$$

i.e. to the second required identity. □

Note 3.6 Note that the above theorem shows essentially that the pair of series $(E_q(A), L_q(A))$ have the same formal properties than the pair of formal power series $(\exp(X), \log(1+X))$. In other words, the series L_q plays exactly the role of a q-logarithm naturally associated with the series E_q.

3.3. The exponential/logarithm correspondence

Before giving the main result of this subsection, let us first introduce some new notations. Let $A = \{a_i, i \geq 1\}$ and $B = \{b_i, i \geq 1\}$ be two noncommutative alphabets. Then $E_q(A+B)$ stands for the series of $\mathbb{K}(q)\langle\langle A \cup B\rangle\rangle$ defined by setting

$$E_q(A + B) = \sigma_{A,B}(E_q(A))$$

where $\sigma_{A,B}$ stands for the algebra morphism from $\mathbb{K}(q)\langle\langle A\rangle\rangle$ into $\mathbb{K}(q)\langle\langle A \cup B\rangle\rangle$ which maps every letter a_i of A onto $a_i + b_i$. For every composition $I = (i_1, \ldots, i_n)$, we shall also denote by a_I the monomial defined by

$$a_I = a_{i_1} \ldots a_{i_r}.$$

Let us finally also recall that the shuffle product $\sqcup\!\sqcup$ is the bilinear product of $\mathbb{K}\langle A\rangle$ which is defined on words of A^* by requiring that one has

$$\begin{cases} 1 \sqcup\!\sqcup w = w \sqcup\!\sqcup 1 = w, \\ (au) \sqcup\!\sqcup (bv) = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) \end{cases}$$

for every words u, v, w in A^* and every letters a, b in A. Let us now recall that one can define for every words u, v, w of A^* the coefficient

$$\binom{w}{u, v}$$

(which is a generalization of the classical binomial coefficient) by setting

$$u \sqcup\!\sqcup v = \sum_{w \in A^*} \binom{w}{u, v} w.$$

In other words, this last coefficient is just the number of times that the word w can be obtained in the shuffle product of u with v. We are now in a position to state the following theorem.

Theorem 3.7 *Let $A = \{a_i, i \geq 1\}$ and $B = \{b_i, i \geq 1\}$ be two noncommutative alphabets such that $a_i b_j = b_j a_i$ for every $i, j \geq 1$. Then one has*

$$E_q(A + B) = E_q(A)E_q(B) = E_q(B)E_q(A).$$

Proof: Let $A = \{a_i, i \geq 1\}$ and $B = \{b_i, i \geq 1\}$ be two alphabets such that $a_i b_j = b_j a_i$ for every $i, j \geq 1$. Then we can write

$$E_q(A+B) = \sum_{\substack{i_1,\dots,i_r \geq 1 \\ r \geq 1}} x_{a_{i_1}\dots a_{i_r}} \left(a_{i_1} + b_{i_1}\right) \dots \left(a_{i_r} + b_{i_r}\right)$$

$$= \sum_{\substack{i_1,\dots,i_r \geq 1 \\ r \geq 1}} x_{a_{i_1}\dots a_{i_r}} \left(\sum_{\substack{I,J \\ (i_1,\dots,i_r) \in I \sqcup\!\sqcup J}} \binom{(i_1,\dots,i_r)}{I,\ J} a_I b_J \right)$$

where the x_w's stand for the coefficients of our analog of the exponential. This leads us to the relation

$$E_q(A+B) = \sum_{I,J} a_I b_J \left(\sum_{(i_1,\dots,i_r) \in I \sqcup\!\sqcup J} \binom{(i_1,\dots,i_r)}{I,\ J} x_{a_{i_1}\dots a_{i_r}} \right). \tag{9}$$

Let us now give the following lemma that shows an important (and rather surprising) property of the coefficients x_w of our analog of the exponential which means exactly that the functional $x(w) = x_w$ is a character of the shuffle algebra.

Lemma 3.8 *For every words u and v of A^*, one has*

$$x_u x_v = \sum_{w \in u \sqcup\!\sqcup v} \binom{w}{u,\ v} x_w. \tag{10}$$

Proof of the lemma: The proof goes by induction on $L(u,v) = |u| + |v|$. Note first that there is nothing to prove when $L(u,v) = 0$, i.e. when u and v are both equal to the empty word.

Let now $u = u_1 \dots u_r$ and $v = v_1 \dots v_s$ be two words of A^* (where u_i and v_i stand for letters of A) such that identity (10) holds for every pair (x, y) of words of A^* such that $L(x, y) < L(u, v)$. Using the defining relation (7) of the coefficients of the series E_q, we can then write

$$\sum_{w \in u \sqcup\!\sqcup v} \binom{w}{u,\ v} x_w = \frac{1}{D_{u,v}(q)} \left(\sum_{w \in u \sqcup\!\sqcup v} \binom{w}{u,\ v} \left(\sum_{\substack{\alpha\beta = w \\ \alpha\beta \neq 1}} x_\alpha x_\beta q^{\|\alpha\|} l_q(\beta) \right) \right) \tag{11}$$

where $D_{u,v}(q)$ stands for the polynomial defined by

$$D_{u,v}(q) = 1 - q^{\|u\|+\|v\|} - l_q(u) l_q(v).$$

Note now that identity (11) can be clearly rewritten as follows

$$\sum_{w\in u\sqcup\!\sqcup v}\binom{w}{u,\ v}x_w=\frac{1}{D_{u,v}(q)}\left(\sum_{w\in u\sqcup\!\sqcup v}\ \sum_{\substack{\alpha\beta=w\\ \alpha\beta\neq 1}}\binom{\alpha\beta}{u,\ v}x_\alpha x_\beta q^{\|\alpha\|}l_q(\beta)\right). \tag{12}$$

However a pair (α, β) of non empty words satisfies to the relation $\alpha\beta = w$ with $w \in u \sqcup\!\sqcup v$ if and only if there exists a pair $(i,\ j) \in [0, r] \times [0, s]$ with $(i,\ j) \notin \{(0, 0), (r, s)\}$ such that

$$\alpha \in u_1 \ldots u_i \sqcup\!\sqcup v_1 \ldots v_j \quad \text{and} \quad \beta \in u_{i+1} \ldots u_r \sqcup\!\sqcup v_{j+1} \ldots v_s$$

(with the convention that a sequence of letters is empty when its indexation is decreasing). The right hand-side of identity (12) is therefore equal to

$$\frac{1}{D_{u,v}(q)}\left\{\sum_{\substack{(i,j)\in[0,r]\times[0,s]\\ (i,j)\neq(0,0),(r,s)}}\left[\left(\sum_{\alpha\in u_1\ldots u_i\sqcup\!\sqcup v_1\ldots v_j}\binom{\alpha}{u_1\ldots u_i,\ v_1\ldots v_j}x_\alpha\right)q^{\|\alpha\|}\right]\right.$$
$$\left.\times\left[\left(\sum_{\beta\in u_{i+1}\ldots u_r\sqcup\!\sqcup v_{j+1}\ldots v_s}\binom{\beta}{u_{i+1}\ldots u_r,\ v_{j+1}\ldots v_s}x_\beta\right)l_q(\beta)\right]\right\}$$

since it is quite immediate to see that one has

$$\binom{\alpha\beta}{u,\, v}=\sum_{\substack{(i,j)\in[0,r]\times[0,s]\\ (i,j)\neq(0,0),(r,s)}}\binom{\alpha}{u_1\ldots u_i,\, v_1\ldots v_j}\binom{\beta}{u_{i+1}\ldots u_r,\, v_{j+1}\ldots v_s}$$

for every non empty words α and β of A^*. By using our induction hypothesis, we can now see that our last expression is equal to

$$\frac{1}{D_{u,v}(q)}\left\{\sum_{\substack{(i,j)\in[0,r]\times[0,s]\\ (i,j)\neq(0,0),(r,s)}}\left(x_{u_1\ldots u_i}x_{v_1\ldots v_j}q^{\|u_1\ldots u_i\|}q^{\|v_1\ldots v_j\|}\right)\right.$$
$$\left.\times\left(x_{u_{i+1}\ldots u_r}x_{v_{j+1}\ldots v_s}l_q(u_{i+1}\ldots u_r)l_q(v_{j+1}\ldots v_s)\right)\right\}$$

which can be itself rewritten in the following way

$$\frac{1}{D_{u,v}(q)}\left\{\left(\sum_{i=0}^{r}x_{u_1\ldots u_i}x_{u_{i+1}\ldots u_r}q^{\|u_1\ldots u_i\|}l_q(u_{i+1}\ldots u_r)\right)\right.$$
$$\left.\times\left(\sum_{j=0}^{s}x_{v_1\ldots v_j}x_{v_{j+1}\ldots v_s}q^{\|v_1\ldots v_j\|}l_q(v_{j+1}\ldots v_s)\right)-x_u x_v\left(q^{\|u\|+\|v\|}+l_q(u)l_q(v)\right)\right\}.$$

Using now the defining relation (5) of the coefficients of the series E_q, we can immediately simplify the previous expression and rewrite it as follows

$$\frac{1}{D_{u,v}(q)}\left\{x_u x_v - x_u x_v\left(q^{\|u\|+\|v\|} + l_q(u)l_q(v)\right)\right\} = x_u x_v.$$

This ends therefore our induction and the proof of our lemma. □

Using the previous lemma in connexion with Eq. (9) leads us now immediately to the following identity

$$E_q(A+B) = \sum_{I,J} x_I x_J a_I b_J = \left(\sum_I x_I a_I\right)\left(\sum_J x_J b_J\right) = E_q(A)E_q(B)$$

which was one of the relation to prove. The other identity can be immediately obtained from this last one. □

The following corollary is now immediate to obtain. It is important to note that this corollary essentially shows that our analog of the exponential transforms Lie elements into group like elements for the natural comultiplication on $\mathbb{K}(q)\langle\langle A\rangle\rangle$ (which is clearly a basic property of any exponential/logarithm correspondence).

Corollary 3.9 *Let Δ be the comultiplication of $\mathbb{K}(q)\langle\langle A\rangle\rangle$ defined by setting*

$$\Delta(a_i) = 1 \otimes a_i + a_i \otimes 1.$$

Then the series $E_q(A)$ is a group-like element for Δ, i.e.

$$\Delta(E_q(A)) = E_q(A) \otimes E_q(A).$$

3.4. Specialization properties

The first result stated below gives the specialization of the series E_q at $q = 0$. We begin by proving the following lemma.

Lemma 3.10 *Let $I = (i_1, \ldots, i_r)$ be a composition, let $w = a_{i_1} \ldots a_{i_r}$ be the word indexed by I and let m_w be the minimal part of I, i.e.*

$$m_w = \min_{1\leq k\leq r} \{i_k\}.$$

Then the term of the polynomial $D_w(q)$ (defined by relation (6)) including the lowest power of q is exactly $\alpha_w q^{m_w}$, where α_w is the number of parts of I equal to m_w, i.e.

$$\alpha_w = \sharp\{j, i_j = m_w\}.$$

In other words, we can write:

$$D_w(q) = \alpha_w q^{m_w} + \sum_{i_j > m_w} \alpha_{i_j} q^{i_j}.$$

Proof: It suffices to notice that one has

$$D_w(q) = 1 - q^{i_1 + \cdots + i_r} - \left(1 - \sum_{k=1}^{r} q^{i_k} + \sum_{k,l=1}^{r} q^{i_k + i_l} + \cdots + (-1)^r q^{i_1 + \cdots + i_r}\right).$$

This last expression can be clearly rewritten as follows:

$$D_w(q) = \sum_{k=1}^{r} q^{i_k} - \sum_{k,l=1}^{r} q^{i_k + i_l} + \cdots + (-1 - (-1)^r) q^{i_1 + \cdots + i_r}.$$

Note now that only the first summand in the right-hand side of the above identity can give a contribution to the coefficient of the lowest power of q within $D_w(q)$. The lemma follows now immediately from this remark. □

Proposition 3.11 *Let $A = \{a_i, i \geq 1\}$ be a noncommutative alphabet. Then the specialization E_0 of the series E_q at $q = 0$ is given by the following formula*:

$$E_0(A) = \exp(a_1)\ \exp(a_2) \ldots \exp(a_n) \ldots.$$

Proof: According to Lemma 3.10, the defining Eq. (7) of the coefficients (that we will denote here by $x_w(q)$) of the series E_q reduces to

$$x_w(q) = \frac{1}{q^{m_w}(\alpha_w + q P_w(q))} \left(\sum_{\substack{uv=w \\ u,\, v \neq 1}} x_u(q) x_v(q) q^{\|u\|} l_q(v) \right),$$

where $P_w(q)$ denotes some polynomial. This last identity can now be rewritten as follows

$$x_w(q) = \frac{1}{\alpha_w + q P_w(q)} \left(\sum_{k=1}^{r-1} x_{a_{i_1} \cdots a_{i_k}}(q) x_{a_{i_{k+1}} \cdots a_{i_r}}(q) q^{i_1 + \cdots + i_k - m_w} \prod_{j=k+1}^{r} (1 - q^{i_j}) \right).$$

Due to the fact that one has

$$i_1 + \cdots + i_k - m_w = i_1 + \cdots + i_k - \min_{1 \leq j \leq r} \{i_j\} \geq 0,$$

one can immediately deduce (by induction) from the last relation that $x_w(0)$ is well defined. Putting $q = 0$ in this relation, we now get the following recurrence relation for the specialization at $q = 0$ of the generic coefficient $x_w(q)$ of the series E_q:

$$x_w(0) = \frac{1}{\alpha_w}\left(\sum_{k=1}^{r-1} x_{a_{i_1}\cdots a_{i_k}}(0)x_{a_{i_{k+1}}\cdots a_{i_r}}(0)q^{i_1+\cdots+i_k-m_w}\Big|_{q=0}\right). \tag{13}$$

We can now proceed with the calculation of $x_w(0)$. We consider separately the following two cases: when w is an increasing word (i.e. when one has $i_1 \le i_2 \le \cdots \le i_r$) and when w is not increasing.

Let us begin with the case when w is not an increasing word. We shall now prove by induction on the length $|w|$ of w that $x_w(0) = 0$ for every non increasing word w. The first non increasing words are of length 2. Let us take such a word, i.e. $w = a_{i_j}a_{i_{j+1}}$ with $i_j > i_{j+1}$. Using the last equation we obtain immediately

$$x_{a_{i_j}a_{i_{j+1}}}(0) = x_{a_{i_j}}(0)x_{a_{i_{j+1}}}(0)q^{i_j-i_{j+1}}|_{q=0} = 0$$

as required. Assume now the induction hypothesis, i.e. $x_w(0) = 0$ for every non decreasing word w such that $|w| \le m-1$. Note that if $w = a_{i_1}\cdots a_{i_r}$ is not increasing, then either $a_{i_1}\cdots a_{i_j}$ or $a_{i_{j+1}}\cdots a_{i_m}$ is not increasing. From this, it is now immediate to obtain from the recurrence relation (13) that $x_w(0) = 0$. This ends therefore the first part of our reasoning.

It remains to consider the case when w is increasing, i.e. $w = a_{i_1}^{\gamma_1}\cdots a_{i_r}^{\gamma_r}$ with $i_j < i_{j+1}$ for every j. In this case, the recurrence relation can be written as

$$x_w(0) = \frac{1}{\gamma_1}\left| \begin{array}{c} x_{a_{i_1}}(0)x_{a_{i_1}^{\gamma_1-1}\cdots a_{i_r}^{\gamma_r}}(0)q^{i_1-i_1}\big|_{q=0} \\ + x_{a_{i_1}a_{i_1}}(0)x_{a_{i_1}^{\gamma_1-2}a_{i_2}^{\gamma_2}\cdots a_{i_r}^{\gamma_r}}(0)q^{i_1+i_1-i_1}\big|_{q=0} \\ +\cdots \end{array}\right.$$

Note that the only nonzero summand in the above relation is the first one. Thus this relation reduces to

$$x_{a_{i_1}^{\gamma_1}\cdots a_{i_r}^{\gamma_r}}(0) = \frac{1}{\gamma_1}x_{a_{i_1}^{\gamma_1-1}\cdots a_{i_r}^{\gamma_r}}(0).$$

By an easy induction argument, it is now immediate to get that

$$x_w(0) = x_{a_{i_1}^{\gamma_1}\cdots a_{i_r}^{\gamma_r}}(0) = \frac{1}{\gamma_1!\ldots\gamma_r!}$$

for every increasing word $w = a_{i_1}^{\gamma_1}\ldots a_{i_r}^{\gamma_r}$.

We can now summarize our calculation with the following table:

$$x_w(0) = \begin{cases} \dfrac{1}{\gamma_1!\ldots\gamma_r!} & \text{if } w = a_{i_1}^{\gamma_1}\ldots a_{i_r}^{\gamma_r} \text{ is increasing,} \\ 0 & \text{if } w \text{ is not increasing.} \end{cases}$$

The reader will easily see that this last result is exactly equivalent to the identity claimed by our proposition. □

The second result given below gives the specialization of the series E_q at $q = 1$.

Proposition 3.12 *Let $A = \{a_i, i \geq 1\}$ be a noncommutative alphabet. Then the specialization E_1 of the series E_q at $q = 1$ is given by the following formula*:

$$E_1(A) = 1 + \sum_{\substack{i_1,\ldots,i_r \geq 1 \\ r \geq 1}} \left(\prod_{k=1}^{r} \frac{i_k}{i_1 + \cdots + i_k} \right) a_{i_1} \ldots a_{i_r}.$$

Proof: Let us denote by $Q_w(q)$ the right-hand side of Eq. (4) (considered in the case $x_1 = 1$ which corresponds to the series E_q), i.e. the element of $\mathbb{K}(q)$ defined by $Q_w(q) = x_w(q)D_w(q)$, which is equal to

$$Q_w(q) = (1-q)[i_r]_q \left(\sum_{k=1}^{r-1} x_{a_{i_1} \cdots a_{i_k}}(q) x_{a_{i_{k+1}} \cdots a_{i_r}}(q) \right.$$
$$\left. \times\, q^{i_1 + \cdots + i_k} (1-q)^{r-k-1} \prod_{j=k+1}^{r-1} [i_j]_q \right).$$

On the other hand, we have

$$D_w(q) = (1-q) \left([i_1 + \cdots + i_r]_q - (1-q)^{r-1} \prod_{k=1}^{r} [i_k]_q \right).$$

So we immediately obtain

$$\begin{aligned} x_w(q) &= \frac{Q_w(q)}{D_w(q)} \\ &= \frac{[i_r]_q \left(\sum_{k=1}^{r-1} x_{a_{i_1} \cdots a_{i_k}}(q) x_{a_{i_{k+1}} \cdots a_{i_r}}(q) q^{i_1 + \cdots + i_k} (1-q)^{r-k-1} \prod_{j=k+1}^{r-1} [i_j]_q \right)}{[i_1 + \cdots + i_r]_q - (1-q)^{r-1} \left(\prod_{k=1}^{r} [i_k]_q \right)} \end{aligned}$$

from which it is immediate to deduce that

$$x_w(1) = \frac{i_r}{i_1 + \cdots + i_r}\, x_{a_{i_1} \ldots a_{i_{r-1}}}(1).$$

By induction on the length of the word w, we then easily get

$$x_w(1) = \prod_{k=1}^{r} \frac{i_k}{i_1 + \cdots + i_k}$$

for every word $w = a_{i_1} \cdots a_{i_r}$ as required. □

4. Relations with q-bracketing

In this section, we will show the relation between our analog of the exponential and the family $(\pi_n(q))_{n\geq 1}$ of q-idempotents associated with the q-bracketing (cf Section 2.4).

4.1. The main result

Before going further, we must first define some new notations. Let us consider again some infinite alphabet $A = \{a_k, k \geq 1\}$ and let

$$X(A) = \sum_{w\in A^*} x_w w$$

be a formal power series of $\mathbb{K}(q)\langle\langle A\rangle\rangle$. Let now

$$Y = \sum_{i=1}^{+\infty} y_i t^i$$

be a series of **Sym**$[[t]]$ with zero constant coefficient such that y_i is an homogeneous noncommutative symmetric function of weight i. We can now define the *composition* $X(A)\circ Y$ (or more simply $X(Y)$) of Y with X by setting

$$X(Y) = X(A)\circ Y = \sigma_Y(X(A))$$

where σ_Y stands for the algebra morphism from $\mathbb{K}(q)\langle\langle A\rangle\rangle$ into **Sym**$[[t]]$ which maps every letter a_i of A onto y_i.

We can now state the main result of this section (and of the paper) which gives an expression (involving our analog of the exponential) for the generating series of the Lie idempotents $(\pi_n(q))_{n\geq 1}$ associated with the q-bracketing.

Theorem 4.1 *Let $(\pi_n(q))_{n\geq 1}$ be the family of Lie idempotents associated with the q-bracketing operator introduced at the end of Section* 2.4. *Then one has*

$$E_q\left(\sum_{n=1}^{+\infty} \pi_n(q)t^n\right) = \sigma(t).$$

Proof: Let us define a family $(P_n(A))_{n\geq 1}$ of homogeneous noncommutative symmetric functions defined by setting

$$\sum_{n=1}^{+\infty} P_n(A)t^n = L_q(\sigma(t)-1). \tag{14}$$

Let us first prove the following lemma.

Lemma 4.2 *The elements $(P_n)_{n\geq 1}$ defined by Eq.* (14) *are primitive for the natural comultiplication Δ on* **Sym**.

Proof of the Lemma: Note that according to Theorem 3.5, we have

$$E_q\left(\sum_{n=1}^{+\infty} P_n(A)t^n\right) = \sigma(t) \tag{15}$$

from which we immediately deduce that

$$\begin{cases} E_q\left(\sum_{n=1}^{+\infty}(1 \otimes P_n(A))t^n\right) = 1 \otimes \sigma(t), \\ E_q\left(\sum_{n=1}^{+\infty}(P_n(A) \otimes 1)t^n\right) = \sigma(t) \otimes 1. \end{cases}$$

Hence we have

$$E_q\left(\sum_{n=1}^{+\infty}(1 \otimes P_n(A))t^n\right) E_q\left(\sum_{n=1}^{+\infty}(P_n(A) \otimes 1)t^n\right) = \sigma(t) \otimes \sigma(t) = \Delta(\sigma(t)).$$

Since we always have

$$(P_n \otimes 1)(1 \otimes P_m) = (1 \otimes P_m)(P_n \otimes 1) = P_n \otimes P_m$$

for every $n, m \geq 1$, we are clearly again in a position to apply Theorem 3.7 which gives us here

$$E_q\left(\sum_{n=1}^{+\infty}(1 \otimes P_n + P_n \otimes 1)t^n\right) = \Delta(\sigma(t)).$$

On the other hand, it is easy to deduce from relation (15) that one has

$$\Delta(\sigma(t)) = E_q\left(\sum_{n=1}^{+\infty} \Delta(P_n)t^n\right).$$

The last two relations now lead to the identity

$$E_q\left(\sum_{n=1}^{+\infty} \Delta(P_n)t^n\right) = E_q\left(\sum_{n=1}^{+\infty}(1 \otimes P_n + P_n \otimes 1)t^n\right).$$

According to Theorem 3.5, the composition on the left by L_q of both sides of this last relation leads finally to the identity

$$\sum_{n=1}^{+\infty} \Delta(P_n)t^n = \sum_{n=1}^{+\infty}(1 \otimes P_n + P_n \otimes 1)t^n,$$

which shows that every element P_n is primitive for Δ as required. □

We are now in a position to prove that the elements P_n are in fact encoding Lie idempotents.

Lemma 4.3 *For every $n \geq 1$, $\alpha^{-1}(P_n)$ is a Lie idempotent.*

Proof of the Lemma: According to the previous lemma and to Theorem 2.1, $p_n = \alpha^{-1}(P_n)$ is a Lie quasi-idempotent. Showing that p_n is a Lie idempotent amounts to show that

$$P_n = \frac{\Psi_n}{n} + L(\Psi)$$

(see [4]), i.e. to show that the coefficient of P_n on Ψ_n in the basis of **Sym** associated with the power sums of first kind is exactly equal to $1/n$. Note now that it is easy to deduce from Eq. (15) that this coefficient is the coefficient of S_n on Ψ_n with respect to the same basis, i.e. to $1/n$ (see [2]) as required. This ends therefore the proof of our lemma. □

We can now prove our last lemma.

Lemma 4.4 *For every $n \geq 1$, one has $P_n = \pi_n(q)$.*

Proof of the Lemma: According to the previous lemma, the noncommutative symmetric function P_n encodes a Lie idempotent. Hence it suffices to show that

$$P_n((1-q)A) = (1-q^n)P_n$$

for every $n \geq 1$ in order to prove that P_n is equal to $\pi_n(q)$ since this last property is a characterization of this Lie idempotent according to Theorem 2.3.

According to Eq. (15), we can write

$$\sigma((1-q)A; t) = E_q\left(\sum_{n=1}^{+\infty} P_n((1-q)A)t^n\right)$$

using here the notations of Definition 2.2. But we also have

$$\sigma((1-q)A; t) = \sigma(A; qt)^{-1}\sigma(A; t)$$

by definition. Using again Eq. (15), we can easily see that we have

$$\sigma(A; qt) = E_q\left(\sum_{n=1}^{+\infty} P_n q^n t^n\right).$$

Hence we immediately get that

$$\sigma((1-q)A; t) = E_q\left(\sum_{n=1}^{+\infty} P_n q^n t^n\right)^{-1} E_q\left(\sum_{n=1}^{+\infty} P_n t^n\right)$$

using again Eq. (15). By using now the characteristic property of the series E_q, i.e. Eq. (3), we now get

$$\sigma((1-q)A; t) = E_q\left(\sum_{n=1}^{+\infty} (1-q^n) P_n t^n\right).$$

Comparing now this equality with the very first one obtained at the beginning of our proof, we can now write

$$E_q\left(\sum_{n=1}^{+\infty} P_n((1-q)A) t^n\right) = E_q\left(\sum_{n=1}^{+\infty} (1-q^n) P_n t^n\right).$$

Applying L_q at the left of the two sides of this equality brings us now immediately to the following relation

$$\sum_{n=1}^{+\infty} P_n((1-q)A) t^n = \sum_{n=1}^{+\infty} (1-q^n) P_n t^n$$

according to Theorem 3.5. Hence we have proved that $P_n((1-q)A) = (1-q^n)P_n$ for every $n \geq 1$ as required. □

It is now immediate to conclude our proof. □

4.2. Zassenhaus Lie idempotents

Before going further, let us recall the definition of Zassenhaus noncommutative symmetric functions (also called power sums of the third type) that were introduced in [4]. This last family is the family $(Z_n)_{n\geq 1}$ of homogeneous noncommutative symmetric functions (with $Z_n \in \mathbf{Sym}_n$ for every $n \geq 1$) defined by setting

$$\sigma(t) := \exp(Z_1 t)\ \exp\left(\frac{Z_2}{2} t^2\right) \dots \exp\left(\frac{Z_n}{n} t^n\right) \dots \tag{16}$$

The Fer-Zassenhaus formula (cf [15]) shows that these new noncommutative symmetric functions are primitive elements for the natural comultiplication on **Sym** and thus encode Lie quasi-idempotents.

Example 4.5 The first values of Z_n are listed below

$$
\begin{aligned}
&Z_1 = \Psi_1, \quad Z_2 = \Psi_2, \quad Z_3 = \Psi_3 + [\Psi_2, \Psi_1],\\
&Z_4 = \Psi_4 + \frac{1}{3}[\Psi_3, \Psi_1] + \frac{1}{6}[[\Psi_2, \Psi_1], \Psi_1],\\
&Z_5 = \Psi_5 + \frac{1}{4}[\Psi_4, \Psi_1] + \frac{1}{3}[\Psi_3, \Psi_2] + \frac{1}{12}[[\Psi_3, \Psi_1], \Psi_1] - \frac{7}{24}[\Psi_2, [\Psi_2, \Psi_1]]\\
&\qquad + \frac{1}{24}[[[\Psi_2, \Psi_1], \Psi_1], \Psi_1].
\end{aligned}
$$

Note 4.6 It is also interesting to note that Goldberg's explicit formula for the Hausdorff series (see [11]) gives the decomposition of Φ_n over the basis (Z^I).

We can now give the following immediate consequence of the last theorem, which solves a conjecture of Krob, Leclerc and Thibon (see [4]).

Corollary 4.7 *The specialization at $q = 0$ of the Lie idempotent $\pi_n(q)$ is the Zassenhaus idempotent Z_n/n, i.e. one has*

$$\pi_n(0) = \frac{Z_n}{n}$$

for every integer $n \geq 0$.

Proof: The result is an immediate consequence of Theorem 4.1 and Proposition 3.11. □

5. Conclusion

The main result of this paper is clearly Theorem 4.1 in which we obtained the identity

$$E_q\left(\sum_{n=1}^{+\infty} \pi_n(q)t^n\right) = \sigma(t)$$

which related the generating series of the family $(\pi_n(q))_{n\geq 1}$ with the generating series of the complete functions. Taking now into account the fact that $\pi_n(1) = \Psi_n/n$ for every

$n \geq 1$ (as shown in [4]), it is interesting to write down the following relations

$$\sigma(t) = \begin{cases} \exp\left(\sum_{n=1}^{+\infty} \frac{\Phi_n}{n} t^n\right), \\ E_1\left(\sum_{n=1}^{+\infty} \frac{\Psi_n}{n} t^n\right), \\ E_0\left(\sum_{n=1}^{+\infty} \frac{Z_n}{n} t^n\right), \end{cases}$$

that hold between the generating series of all kinds of noncommutative power sums that we introduced up to now. All these relations have in common that the generating series of noncommutative complete symmetric functions is equal to some exponential-like operator applied to the generating series of the corresponding noncommutative power sums.

On the other hand, let us recall that it was shown in [5] that an homogeneous noncommutative symmetric function P_n of weight n is the encoding of a Lie idempotent if and only if there exists a totally ordered commutative alphabet X such that

$$P_n = \Pi_n(XA)$$

where Π_n is an arbitrary fixed Lie idempotent of weight n. It seems therefore likely to conjecture that there exists some general quasi-symmetric analog E_X of the exponential such that the identity

$$E_X\left(\sum_{n=1}^{+\infty} \frac{\pi_n}{n} t^n\right) = \sigma(t)$$

holds if and only if and only if the sequence $(\pi_n/n)_{n\geq 1}$ of homogeneous noncommutative symmetric functions consists only in Lie idempotents. Note also that this conjecture is clearly supported by the fact that one can associate a Lie projector with the operator $S_n(XA)$ (see [4]) in the same line than the Lie idempotent $\pi_n(q)$ was associated with the q-bracketing operator (that corresponds to the situation $X = 1 - q$ in the sense of [4]).

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