Journal of Algebraic Combinatorics **14** (2001), 27–48

Spectral Sequences on Combinatorial Simplicial Complexes

DMITRY N. KOZLOV* kozlov@math.ias.edu
School of Mathematics, Institute for Advanced Study, Olden Lane, Princeton, NJ, USA

Received July 20, 1999; Revised July 26, 2000

Abstract. The goal of this paper is twofold. First, we give an elementary introduction to the usage of spectral sequences in the combinatorial setting. Second we list a number of applications.

In the first group of applications the simplicial complex is the nerve of a poset; we consider general posets and lattices, as well as partition-type posets. Our last application is of a different nature: the $\mathcal{S}_n$-quotient of the complex of directed forests is a simplicial complex whose cell structure is defined combinatorially.

Keywords: spectral sequences, posets, graphs, homology groups, shellability

1. Introduction

In this paper we use spectral sequences to compute homology groups of combinatorially given simplicial complexes, whether they come as nerves of posets or with an explicit combinatorial description of the cell structure.

This idea is not new, in fact spectral sequences have been used for that purpose in a quite general setting, already in, e.g. [1–3, 16]. Recently, these methods have started to take more concrete forms, for example Phil Hanlon used them in [9] to compute the homology groups of the so-called generalized Dowling lattices. In the joint paper [8], Eva Maria Feichtner and the author used spectral sequences to attack an especially difficult case of subspace arrangements, namely the so-called $\mathcal{D}_{n,k}$-arrangements.

In Section 2 we define some basic notions. Then, in Section 3, we give a thorough and elementary, from scratch, description of one possible way to use spectral sequences for poset homology computations.

In Section 4 we derive several corollaries of the properties of the spectral sequences, which can be applied to a wide class of posets. These results include both Möbius function computations and finding the Betti numbers of a poset. We take a look at the Whitney homology of a poset and the intriguing questions coming up in this context. In Theorem 4.1 we prove two inequalities for the Betti numbers of an arbitrary lattice.

In Section 5 we apply these methods to different partition-type posets. In Subsection 5.1 we consider the intersection lattices of orbit arrangements, $\Pi_{\lambda_1,\dots,\lambda_p,2,1^m}$. Furthermore, we

Present address: Department of Mathematics, Royal Institute of Technology, Stockholm, 10044, Sweden; e-mail: kozlov@math.kth.se.
Research at IAS was supported by von Hoffmann, Arcana Foundation.

compute completely the homology groups of the particular lattice $\Pi_{3,2,2,1}$. This example shows that the homology groups of orbit arrangements $\mathcal{A}_\lambda$ can have very irregular structure in general, which was not known before. We remind the reader that it was shown in [11, Theorem 4.1] that when a partition λ has no primitive partition identities then Π_λ is shellable, in particular it is homotopy equivalent to a wedge of spheres. In Subsection 5.2 we take a look at the subspace arrangements associated with certain partitions with restricted block sizes.

In Section 6 we use spectral sequences to study homology groups of the $\mathcal{S}_n$-quotient of the complex of directed forests $\Delta(G_n)/\mathcal{S}_n$. In [12] it was shown that $\Delta(G_n)$ is shellable. Here we derive a formula for the rational Betti numbers of $\Delta(G_n)/\mathcal{S}_n$ and also detect torsion in its integer homology groups.

2. Basic notions and definitions

In this section we introduce the basic notions which we use throughout the text.

Definition 2.1 Let P be a finite poset. The **nerve** of P, $\Delta(P)$, (also known as the **order complex** of P), is the abstract simplicial complex whose vertices are the elements of P and whose faces of dimension k are the chains $x_0 < \cdots < x_k$ of length $k+1$ in P. See [15] for a more general definition.

If P is explicitly given with adjoint elements $\hat{0}$ and $\hat{1}$, then we consider the simplicial complex $\Delta(\bar{P})$, where $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$. Where it causes no confusion we often write $\Delta(P)$ instead of $\Delta(\bar{P})$.

We also use the convention that unless the bar ($\bar{\ }$) is explicitly written, the concerned poset always has adjoint elements $\hat{0}$ and $\hat{1}$.

For an arbitrary simplicial complex C, $\tilde{H}_k(C)$ will denote the kth reduced homology group of C (see [17] for a definition). For the sake of brevity we will often write $\tilde{H}_k(P)$ instead of $\tilde{H}_k(\Delta(P))$.

Furthermore we let $\mu_P(x, y)$ denote the value of the Möbius function on the interval (x, y) of the poset P. The definition and many properties of the Möbius function can be found for example in [18]. We use the convention $\mu(\bar{P}) = \mu_P(\hat{0}, \hat{1})$.

Definition 2.2 A poset P is called Cohen-Macaulay (or just CM) if for every interval (x, y) of the poset P we have $\tilde{H}_i((x, y)) = 0$ for $i \neq \text{rk}(y) - \text{rk}(x) - 2$.

Recall that a *chain complex* C of vector spaces (resp. abelian groups) is a sequence $\cdots \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots$ of vector spaces (abelian groups) and maps between them, such that $d^2 = 0$.

A *filtration* on C, $0 = F_{-1} \subseteq F_0 \subseteq \cdots \subseteq F_t = C$ is a collection of filtrations on each $C_i : 0 = F_{-1}C_i \subseteq F_0C_i \subseteq \cdots \subseteq F_tC_i = C_i$ satisfying $d(F_jC_i) \subseteq F_jC_{i-1}$ for all i, j; here we denote $F_i = (\cdots \xrightarrow{d} F_iC_n \xrightarrow{d} F_iC_{n-1} \xrightarrow{d} \cdots)$.

3. Spectral sequences for the nerves of posets

Spectral sequences constitute a convenient tool for computing the homology groups of a simplicial complex. Here we give a short description of one possible way to apply spectral sequences to compute homology groups of the nerve of a poset. A special case of this particular approach has been previously used by Phil Hanlon, in the work cited above.

Of course, the filtrations considered here are very special, but we hope that this may be a good starting point for a combinatorialist. A few good sources for the material on spectral sequences are [13, 14, 17].

3.1. The definition and some properties of spectral sequences

A spectral sequence associated with a chain complex C and a filtration F on C is a sequence of 2-dimensional tableaux $(E^r_{*,*})_{r=0}^{\infty}$, where every component $E^r_{k,i}$ is a vector space (for simplicity we start with considering field coefficients), $E^r_{k,i} = 0$ unless $k \geq -1$ and $i \geq 0$, and a sequence of differential maps $(d^r)_{r=0}^{\infty}$ such that

(0) $E^0_{k,i} = F_i C_k / F_{i-1} C_k$;
(1) $d^r : E^r_{k,i} \longrightarrow E^r_{k-1,i-r}$, $\forall k, i \in \mathbb{Z}$;
(2) $E^{r+1}_{*,*} = H_*(E^r_{*,*}, d^r)$, in other words

$$E^{r+1}_{k,i} = \ker\left(E^r_{k,i} \xrightarrow{d^r} E^r_{k-1,i-r}\right) / \mathrm{im}\left(E^r_{k+1,i+r} \xrightarrow{d^r} E^r_{k,i}\right); \tag{3.1}$$

(3) for all $k \in \mathbb{Z}$,

$$H_k(C) = \bigoplus_{i \in \mathbb{Z}} E^{\infty}_{k,i}. \tag{3.2}$$

Comments.

0. It follows from (0) and (2) that $E^1_{k,i} = H_k(F_i, F_{i-1})$.

1. In the general case $E^{\infty}_{k,i}$ is defined using the notion of convergence of the spectral sequence. We will not explain this notion in general, since for the spectral sequence that we consider only a finite number of components in every tableau $E^r_{*,*}$ are different from zero, so there exists $N \in \mathbb{N}$, such that $d^r = 0$ for $r \geq N$. Then, one sets $E^{\infty}_{*,*} = E^N_{*,*}$, and so $H_k(C) = \bigoplus_{i \in \mathbb{Z}} E^N_{k,i}$.

2. For the case of integer coefficients, (3.2) becomes more involved: rather than just summing the entries of $E^{\infty}_{*,*}$ one needs to solve extension problems to get $H_*(C)$. This difficulty will not arise in our applications, so we refer the interested reader to [14] for the detailed explanation of this phenomena. When considering integer coefficients, $E^r_{*,*}$ are not vector spaces, but just abelian groups.

3. We would like to warn the reader that our indexing is different from the standard (but more convenient for our purposes). The standard indexing is more convenient for the spectral sequences associated to fibrations, an instance we do not discuss in this paper.

For a finitely generated abelian group G, let rk$G$ denote the dimension of the free part of G. When specializing to a spectral sequence for the homology of the nerve of a finite bounded poset, we immediately observe that its Möbius function can be read off from the $E^r_{*,*}$-tableau, for any non-negative integer r.

Proposition 3.1 *Let P be a finite poset and $(E^r_{*,*})_{r=0}^{\infty}$ an associated spectral sequence, then*

$$\mu_P(\hat{0},\hat{1}) = \sum_{k,i\in\mathbb{Z}} (-1)^k \mathrm{rk} E^r_{k,i} - 1. \tag{3.3}$$

Proof: It is a well known fact that

$$\chi(E^r_{*,*}) = \chi(\Delta(P)), \quad \text{for all } r \geq 1, \tag{3.4}$$

where $\chi(E^r_{*,*}) = \sum_{k,i\in\mathbb{Z}} (-1)^k \text{ rk } E^r_{k,i}$, see for example [14, Example 6, pp. 15–16].

Furthermore the theorem of Ph. Hall says that

$$\mu_P(\hat{0},\hat{1}) = \tilde{\chi}(\Delta(P)). \tag{3.5}$$

Formula (3.3) follows from (3.4) and (3.5). □

As we will see later, formula (3.3) specializes to several well-known formulae for Möbius function computations, once the spectral sequence is specified.

Proposition 3.2 *Let P be any poset and $(E^r_{*,*})_{r=0}^{\infty}$ a spectral sequence for $H_*(\Delta(P))$. Then we have*

$$\big(\textit{for some } r : E^r_{k,i} = 0, \forall i \in \mathbb{Z}\big) \Rightarrow H_k(P) = 0, \tag{3.6}$$

and, for any k,

$$\beta_k(P) \leq \sum_{i\in\mathbb{Z}} \text{rk } E^1_{k,i}, \tag{3.7}$$

$$\beta_k(P) - \beta_{k-1}(P) - \beta_{k+1}(P) \geq \sum_{i\in\mathbb{Z}} \text{rk } E^1_{n,i} - \sum_{i\in\mathbb{Z}} \text{rk } E^1_{n-1,i} - \sum_{i\in\mathbb{Z}} \text{rk } E^1_{n+1,i}. \tag{3.8}$$

Proof: From (3.1) we have rk $E^{r+1}_{k,i} \leq$ rk $E^r_{k,i}$, hence ($E^r_{k,i} = 0 \Rightarrow E^{r+1}_{k,i} = 0$), and (3.6) follows. It also follows that

$$\beta_k(P) = \text{rk } H_k(P) = \sum_{i\in\mathbb{Z}} \text{rk } E^{\infty}_{k,i} \leq \sum_{i\in\mathbb{Z}} \text{rk } E^1_{k,i}.$$

We shall now prove (3.8). Let us denote $d_0 = d^r|_{E^r_{n-1,i-r}}$, $d_1 = d^r|_{E^r_{n,i}}$, $d_2 = d^r|_{E^r_{n+1,i+r}}$, $d_3 = d^r|_{E^r_{n+2,i+2r}}$, then we have the following diagram

$$\cdots \leftarrow E^r_{n-2,i-2r} \stackrel{d_0}{\leftarrow} E^r_{n-1,i-r} \stackrel{d_1}{\leftarrow} E^r_{n,i} \stackrel{d_2}{\leftarrow} E^{n+1,i+r} \stackrel{d_3}{\leftarrow} E^r_{n+2,i+2r} \leftarrow \cdots$$

From the definition of the spectral sequence we know that

$$E^{r+1}_{n,i} = \ker d_1/\mathrm{Im}\, d_2, \quad E^{r+1}_{n+1,i+r} = \ker d_2/\mathrm{Im}\, d_3, \quad E^{r+1}_{n-1,i-r} = \ker d_0/\mathrm{Im}\, d_1,$$

hence

$$\begin{aligned} &\mathrm{rk} E^{r+1}_{n,i} - \mathrm{rk} E^{r+1}_{n-1,i-r} - \mathrm{rk} E^{r+1}_{n+1,i+r} \\ &\quad = (\mathrm{rk}\ker d_1 + \mathrm{rkIm}\, d_1) + \mathrm{rkIm}\, d_3 - (\mathrm{rk}\ker d_2 + \mathrm{rkIm}\, d_2) - \mathrm{rk}\ker d_0 \\ &\quad \geq \mathrm{rk} E^r_{n,i} - \mathrm{rk} E^r_{n-1,i-r} - \mathrm{rk} E^r_{n+1,i+r}. \end{aligned} \tag{3.9}$$

Comment. *We use here the fact that if G is an abelian group and H is a subgroup of G then* $\mathrm{rk}(G) = \mathrm{rk}(H) + \mathrm{rk}(G/H)$, see e.q. [10, exercise 7.2.2.].

Summing over all $i \in \mathbb{Z}$ in (3.9) we obtain

$$\begin{aligned} &\sum_{i\in\mathbb{Z}} \mathrm{rk} E^r_{n,i} - \sum_{i\in\mathbb{Z}} \mathrm{rk} E^r_{n-1,i} - \sum_{i\in\mathbb{Z}} \mathrm{rk} E^r_{n+1,i} \\ &\quad \leq \sum_{i\in\mathbb{Z}} \mathrm{rk} E^{r+1}_{n,i} - \sum_{i\in\mathbb{Z}} \mathrm{rk} E^{r+1}_{n-1,i} - \sum_{i\in\mathbb{Z}} \mathrm{rk} E^{r+1}_{n+1,i}, \end{aligned} \tag{3.10}$$

hence using formula (3.2) we obtain

$$\begin{aligned} \beta_k(P) - \beta_{k-1}(P) - \beta_{k+1}(P) &= \sum_{i\in\mathbb{Z}} \mathrm{rk} E^\infty_{n,i} - \sum_{i\in\mathbb{Z}} \mathrm{rk} E^\infty_{n-1,i} - \sum_{i\in\mathbb{Z}} \mathrm{rk} E^\infty_{n+1,i} \\ &\geq \sum_{i\in\mathbb{Z}} \mathrm{rk} E^1_{n,i} - \sum_{i\in\mathbb{Z}} \mathrm{rk} E^1_{n-1,i} - \sum_{i\in\mathbb{Z}} \mathrm{rk} E^1_{n+1,i}. \end{aligned} \tag{3.11}$$

□

3.2. A class of filtrations

In this subsection we consider all homology groups with coefficients in **F**, where **F** is either a field or the ring of integers. In fact, everything prior to (3.13) is valid for **F** being an arbitrary ring.

Let us describe a special class of filtrations on the chain complex for $\Delta(P)$. This class is somewhat more general than the one considered in [9]. First of all one chooses the following data: J a subposet of $\bar{P}$ and a function $f : J \cup \{\hat{0}\} \to \mathbb{N}$, such that $f(\hat{0}) = 0$, and $x < y$ implies $f(x) \neq f(y)$, in other words the preimage of each element in $\mathbb{N}$ forms an antichain in J. The most frequent choices of the function f are the rank function on J (when it exists) and an arbitrary linear extension of the partial order on J. The choice of J is more subtle and usually depends heavily on the structure of the poset P. For example in [8] the case $P = \mathcal{D}_{n,k}$, where $\mathcal{D}_{n,k}$ is the intersection lattice of the k-equal arrangement of type $\mathcal{D}$, has been considered. In this situation it turned out to be appropriate to take J to be the set of all the elements without unbalanced component. Phil Hanlon, in [9], considers the case when J is a lower order ideal and f is the rank function (he considers pure posets only).

Having chosen f and J, we will define an increasing filtration on the chain complex for $\Delta(P)$. Let $\Gamma = \hat{0} < x_0 < \cdots < x_k < \hat{1}$ be a chain (not necessarily maximal) in P. Define the **pivot** of Γ, $\mathrm{piv}(\Gamma)$, to be the element of $\Gamma \cap J$ with the highest value of the function f. Since the preimages under f of each natural number form an antichain, we know that f takes different values on different elements in $\Gamma \cap J$ and hence the notion of pivot is well defined. Furthermore, let the **weight** of Γ, $\omega(\Gamma)$, be the value of f on the pivot, i.e., $\omega(\Gamma) = f(\mathrm{piv}(\Gamma))$. If $\Gamma \cap J = \emptyset$, we take $\hat{0}$ as a pivot and say that the chain Γ has weight 0. This assignment of weights gives us the filtration of the chain complex $C_*(P)$:

$$\begin{aligned} F_i(C_k(P)) &= \langle \{\Gamma = \hat{0} < x_0 < \cdots < x_k < \hat{1} \mid \omega(\Gamma) \le i\}\rangle_{\mathbf{F}}, \quad \text{for } k \ge 0, i \ge 0, \\ F_{-1}(C_k(P)) &= \{0\}, \quad \text{for } k \ge 0, \end{aligned}$$

with $\langle\cdot\rangle_{\mathbf{F}}$ denoting the linear span of the given chains with coefficients in $\mathbf{F}$.

Recall that by the definition of the nerve of a poset,

$$\partial(\hat{0} < x_0 < \cdots < x_k < \hat{1}) = \sum_{i=0}^{k} (-1)^i (\hat{0} < x_0 < \cdots < \hat{x}_i < \cdots < x_k < \hat{1}).$$

Omitting an element other than the pivot does not alter the weight of the chain, omitting $\mathrm{piv}(\Gamma)$ turns another element into the pivot, on which f takes a lower value than on the former pivot, so the resulting chain has a strictly lower weight. Hence $\partial(F_i(C_*)) \subseteq F_i(C_*)$, i.e., the differential operator ∂ respects the filtration. By construction, the filtration is bounded from below.

By definition

$$\begin{aligned} E^0_{k,i} &= F_i(C_k(P))/F_{i-1}(C_k(P)) \\ &= \langle \{\Gamma : \hat{0} < x_0 < \cdots < x_k < \hat{1} \mid \omega(\Gamma) = i\}\rangle_{\mathbf{F}}, \quad \text{for } k \ge 0, i \ge 0. \end{aligned}$$

The differential $d^0 : E^0_{k,i} \to E^0_{k-1,i}$ is induced by the simplicial boundary operator. Let $\Gamma = \hat{0} < x_0 < \cdots < x_{j-1} < \mathrm{piv}(\Gamma) < x_{j+1} < \cdots < x_k < \hat{1}$ be a generator of $E^0_{k,i}$, then

$$d^0(\Gamma) = [\partial(\Gamma)] = \left[\sum_{p=0, p\neq j}^{k} (-1)^p (\hat{0} < x_0 < \cdots < \hat{x}_p < \cdots < x_k < \hat{1}) \right],$$

since the weight of a chain is lowered by the omission of an element if and only if it is the pivot which is removed.

Now we shall replace the chain complexes $(E^0_{*,i}, d^0)$ (bidegree $d^0 = (-1, 0)$!) by chain isomorphic complexes. The latter allow us to give an explicit description of the tableau $E^1_{*,*}$ in terms of the homology groups of certain subposets of P. First we need some notations: for $a \in J$, let $S_a = (\bar{P} \setminus J) \cup \{b \in J \mid f(b) < f(a)\}$.

There is an obvious isomorphism between the following chain complexes "dividing" each chain Γ in P with pivot $a > \hat{0}$ into two chains, namely its subchains below and above the pivot:

$$\begin{aligned} \varphi : E^0_{k,i} &\longrightarrow \bigoplus_{a \in f^{-1}(i)} (\tilde{C}_*((\hat{0}, a) \cap S_a) \otimes \tilde{C}_*((a, \hat{1}) \cap S_a))_{k-2} \\ (\hat{0} < \cdots < x_{j-1} < a < x_{j+1} < \cdots < \hat{1}) \\ &\longmapsto (\hat{0} < \cdots < x_{j-1} < a) \otimes (a < x_{j+1} < \cdots < \hat{1}), \end{aligned}$$

with $\tilde{C}_*$ denoting the augmented simplicial chain complex of the respective intervals. We need to use augmented complexes including the empty chain in order to get proper counterparts for chains which have the pivot as maximal element below $\hat{1}$ or as minimal element above $\hat{0}$.

Let

$$\tilde{\partial}_\otimes = \tilde{\partial}_{(\hat{0},a)\cap S_a} \otimes \text{ id } + \text{ id } \otimes \tilde{\partial}_{(a,\hat{1})\cap S_a},$$

with the usual sign conventions, namely $\tilde{\partial}_\otimes(c_p \otimes c_q) = \tilde{\partial} c_p \otimes c_q + (-1)^p c_p \otimes \tilde{\partial} c_q$, for $c_p \in \tilde{C}_p((\hat{0}, a) \cap S_a)$, $c_q \in \tilde{C}_q((a, \hat{1}) \cap S_a)$. One can see that the isomorphism commutes with the boundary operators d^0 and $\tilde{\partial}_\otimes$, respectively. Hence φ is actually a bijective chain map and we get

$$\begin{aligned} E^1_{k,i} &= H_k(E^0_{*,i}, d^0) \\ &\cong \bigoplus_{a \in f^{-1}(i)} H_{k-2}(\tilde{C}_*((\hat{0}, a) \cap S_a) \otimes \tilde{C}_*((a, \hat{1}) \cap S_a), \tilde{\partial}_\otimes). \end{aligned}$$

For $i = 0$ we simply have

$$E^1_{k,0} = H_k(P \setminus J). \tag{3.12}$$

In case $\mathbf{F}$ is a field, or $\mathbf{F} = \mathbb{Z}$ and at least one of the subposets $(\hat{0}, a) \cap S_a$ and $(a, \hat{1}) \cap S_a$ has free homology groups, we can apply the algebraic Künneth theorem and deduce

$$E^1_{k,i} \cong \bigoplus_{a \in f^{-1}(i)} (\tilde{H}_*((\hat{0}, a) \cap S_a) \otimes \tilde{H}_*((a, \hat{1}) \cap S_a))_{k-2}. \tag{3.13}$$

In this setting, Proposition 3.1 specializes to

$$\mu_P(\hat{0}, \hat{1}) = \mu(\bar{P} \setminus J) + \sum_{a \in J} \mu((\hat{0}, a) \cap S_a) \cdot \mu((a, \hat{1}) \cap S_a). \tag{3.14}$$

Special cases of formula (3.14) can be found in for example [18]. Observe that when P is a lattice and $J = \bar{P} \setminus \bar{P}_{\geq x}$, for some $x \in \bar{P}$, and f is an arbitrary order preserving function on $J \cup \{\hat{0}\}$, then (3.14) gives Weisner's theorem:

$$\mu_P(\hat{0}, \hat{1}) = -\sum_{a \vee x = \hat{1}} \mu_P(\hat{0}, a).$$

For the explicit derivation of the $E^1_{*,*}$-tableau in this case see Theorem 4.1. For more information on Weisner's theorem itself the reader may want to consult [18, Corollary 3.9.3].

When J is a lower ideal and f is an order-preserving function, i.e. if $x > y$ then $f(x) > f(y)$, the formula (3.13) specializes to

$$E^1_{k,i} \cong \bigoplus_{a \in f^{-1}(i)} (\tilde{H}_*(\hat{0}, a) \otimes \tilde{H}_*((a, \hat{1}) \cap (P \setminus J)))_{k-2}. \tag{3.15}$$

4. Applications for general posets

Let P be a pure poset. Form a spectral sequence by choosing $J = \bar{P}$ and $f(x) = \mathrm{rk}(x)$, then, according to (3.15) and (3.12),

$$E^1_{k,i} = \bigoplus_{\mathrm{rk}\,(a)=i} \tilde{H}_{k-1}(\hat{0}, a),$$
$$E^1_{-1,0} = \mathbb{Z}, \quad E^1_{k,0} = 0, \quad \text{for } k \geq 0.$$

We can read off the so-called *Whitney homology groups* of P (first introduced and studied by Baclawski in [1]) from the $E^1_{*,*}$-tableau:

$$W_k(P) := \bigoplus_{i=0}^{\infty} E^1_{k,i} = \bigoplus_{a \in P} \tilde{H}_{k-1}(\hat{0}, a), \quad k \in \mathbb{Z}.$$

Let now P be a CM poset, then $E^1_{k,i} = 0$ for $i \neq k+1$, hence $W_k(P) = E^1_{k,k+1}$, $k \in \mathbb{Z}$. Moreover $d^r = 0$ for $r \geq 2$, and $E^2_{k,i} = 0$ for $k \neq \mathrm{rk}\,(P) - 2$. It means that under the first differential d^1 all of the groups $W_k(P)$, except for the highest one, cancel in some intriguing way. *It would be of a great interest to clarify the combinatorial nature of these cancellations.*

Theorem 4.1 *Let P be a finite lattice, x an atom in P. Then the following inequalities hold*:

$$\beta_k(\hat{0}, \hat{1}) \leq \sum_{y \vee x = \hat{1}} \beta_{k-1}(\hat{0}, y), \tag{4.1}$$

$$\sum_{y\vee x=\hat{1}}(\beta_{k-1}(\hat{0},y)-\beta_{k-2}(\hat{0},y)-\beta_k(\hat{0},y))\leq\beta_k(\hat{0},\hat{1})-\beta_{k-1}(\hat{0},\hat{1})-\beta_{k+1}(\hat{0},\hat{1}). \tag{4.2}$$

In particular, if $\beta_{k-2}(\hat{0},y)=\beta_k(\hat{0},y)=0$, *for all* $y\in P$, *such that* $y\vee x=\hat{1}$, *then*

$$\beta_k(\hat{0},\hat{1})=\sum_{y\vee x=\hat{1}}\beta_{k-1}(\hat{0},y).$$

Proof: Let $J=\bar{P}\setminus\bar{P}_{\geq x}$ and let $x_1,\ldots,x_k$ be any linear extension of J. Consider the spectral sequence E which is associated to the ideal J, where we filtrate using the given linear extension of J. Observe first that $\bar{P}\setminus J=\bar{P}_{\geq x}$ is contractible. Also, for any $a\in J$, we have $(a,\hat{1})\cap(\bar{P}\setminus J)=(a,\hat{1})\cap\bar{P}_{\geq x}=\bar{P}_{\geq x\vee a}$.

This means that $(a,\hat{1})\cap(\bar{P}\setminus J)$ is contractible (actually a cone with apex $x\vee a$) unless $x\vee a=\hat{1}$. When $x\vee a=\hat{1}$ we get $(a,\hat{1})\cap(\bar{P}\setminus J)=\emptyset$. So, using formulae (3.12) and (3.15), we obtain $E^1_{n,0}=0$, for all n, and

$$E^1_{n,i}=\begin{cases}\tilde{H}_{n-1}(\hat{0},x_i), & \text{if } x_i\vee x=\hat{1};\\ 0, & \text{otherwise.}\end{cases}$$

The inequalities (4.1) and (4.2) follow from inequalities (3.7), resp. (3.8). □

Applications of Theorem 4.1 will be given in the next section. The following theorem may be occasionally useful.

Theorem 4.2 *Let* P *be a pure poset of rank* r. *Suppose that there exists a subposet* J *of* P *such that*
(1) $P\setminus J$ *is CM and* $\mathrm{rk}(P\setminus J)=r$;
(2) *for any* $a\in J$, *both* $[\hat{0},a]$ *and* $[a,\hat{1}]_J$ *are CM and* $\mathrm{rk}[a,\hat{1}]_J=\mathrm{rk}[a,\hat{1}]$, *where* $[a,\hat{1}]_J=[a,\hat{1}]\cap(P\setminus J)$.
Then $\tilde{H}_i(P)=0$, *for* $i\neq r-2$, *and*

$$H_{r-2}(P)=\left(\bigoplus_{a\in J}\tilde{H}_{\mathrm{rk}(a)-2}(\hat{0},a)\otimes\tilde{H}_{\mathrm{rk}[a,\hat{1}]-2}(a,\hat{1})\right)\oplus H_{r-2}(P\setminus J). \tag{4.3}$$

Proof: Construct the spectral sequence $(E^r_{*,*})^\infty_{r=0}$ with the subposet J as in the proof of the Theorem 4.1 and with $f(x)=\mathrm{rk}(x)$. Then it follows from the formulae (3.13) and (3.12) that $E^1_{k,0}=H_k(P\setminus J)$, and

$$E^1_{k,i}\cong\bigoplus_{\mathrm{rk}(a)=i,a\in J}(\tilde{H}_*(\hat{0},a)\otimes\tilde{H}_*(a,\hat{1})_J)_{k-2},\quad \text{for } i\geq 1.$$

Using the fact that $P \setminus J$, $[\hat{0}, a]$ and $[a, \hat{1}]_J$ are CM and that $\mathrm{rk}(P \setminus J) = \mathrm{rk}(P) = r$, $\mathrm{rk}[a, \hat{1}]_J = \mathrm{rk}[a, \hat{1}]$, we obtain $E^1_{k,i} = 0$, for $k \neq r - 2$. The spectral sequence collapses here, hence (4.3) and $\tilde{H}_i(P) = 0$, for $i \neq r - 2$, follow from (3.2). □

Let us recall a theorem proved in [6].

Theorem 4.3 (*Complementation Theorem*) *If L is a bounded lattice, $s \in \bar{L}$, and the complements of s form an antichain, then $\bar{L} \simeq \underset{x \perp s}{\mathrm{wedge}}\ \mathrm{susp}\ ((\hat{0}, x) * (x, \hat{1}))$.*

Remark 4.4 In the special case, when the complements of an atom $x \in P$ form an antichain, the spectral sequence above allows us to derive the homology counterpart of the Complementation Theorem 4.3.

Reason. If the complements of x form an antichain one can choose the function f so that it takes the same value v on all of the complements of x. Then there will be only one non-zero row in $E^1_{*,*}$, namely

$$E^1_{n,v} = \bigoplus_{y \vee x = \hat{1}} \tilde{H}_{n-1}(\hat{0}, y), \qquad E^1_{n,i} = 0 \quad \text{for } i \neq v.$$

All the differentials d^r will be zero maps for $r \geq 1$, so we obtain

$$H_k(P) = \sum_{i \in \mathbb{Z}} E^1_{k,i} = \bigoplus_{y \vee x = \hat{1}} \tilde{H}_{k-1}(\hat{0}, y).$$

5. Applications to partition-type posets

5.1. Orbit arrangements

A **subspace arrangement** $\mathcal{A}$ is a finite collection of affine subspaces $\{K_1, \ldots, K_t\}$ in the Euclidean space $\mathbb{R}^n$. Let $\mathcal{A}$ be a central subspace arrangement (all the subspaces pass through the origin) and take all possible non-empty intersections $K_{i_1} \cap \cdots \cap K_{i_p}$, $1 \leq i_1 < \cdots < i_p \leq t$, ordered by reverse inclusion, that is $x \leq y \Leftrightarrow y \subseteq x$. This is a partially ordered set, which is actually a lattice. The bottom element is $\hat{0} = \mathbb{R}^n$ and the top element is $\hat{1} = \cap \mathcal{A} = K_1 \cap \cdots \cap K_t$. This lattice is called the **intersection lattice** and is often denoted by $\mathcal{L}_{\mathcal{A}}$.

We use the notation $\lambda = (\lambda_1, \ldots, \lambda_p)$ for the partition of the number $n = \sum_{i=1}^{p} \lambda_i$ into blocks of sizes $\lambda_1, \ldots, \lambda_p$ and we always have these blocks ordered in non-increasing order, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. By Π_n we denote the **partition lattice** of the set $[n]$. It is the poset with elements all different partitions of $[n]$ ordered under refinement.

The following class of subspace arrangements was first introduced in [4, subsection 3.3]. If $\pi = (B_1, \ldots, B_p)$ is a nontrivial partition of the set $[n]$, let

$$K_\pi = \{x \in \mathbb{R}^n \mid i, j \in B_k \Rightarrow x_i = x_j, \text{ for all } 1 \leq i, j, \leq n, 1 \leq k \leq p\}.$$

The type of π is the number partition of n given by the block sizes $|B_i|$. Given a non-trivial number partition $\lambda \vdash n$, let

$$\mathcal{A}_\lambda = \{K_\pi \mid \pi \in \Pi_n \text{ and type } (\pi) = \lambda\}.$$

$\mathcal{A}_\lambda$ is called an **orbit arrangement**, expressing the fact that $\mathcal{A}_\lambda$ is the orbit of any single subspace K_π under the natural action of S_n on $\mathbb{R}^n$. Let $\Pi_\lambda = \mathcal{L}_{\mathcal{A}_\lambda}$. Note that $\Pi_n = \Pi_{(2,1,\ldots,1)}$.

Theorem 5.1 *Consider a partition* $\lambda = (\lambda_1, \ldots, \lambda_p, 2, 1^m)$, $p \geq 0$, $m \geq 1$, (*this notation means that we have m blocks of size* 1). *Let*

$$t = \min_{1 \leq i \leq p+1} \left\lceil \frac{m + \lambda_i + \cdots + \lambda_p + 1}{\lambda_i - 1} \right\rceil,$$

where $\lambda_{p+1} = 2$. *Then* Π_λ *is* $(t-3)$*-acyclic.*

Remark For this bound to be useful, we should have much larger m than λ_i's. For example, for $\lambda = (3, 2, 1^m)$ we get that Π_λ is $(\lceil m/2 \rceil - 1)$-acyclic.

Proof: Take a coatom $x = (1, \ldots, n-1)(n)$ and consider the spectral sequence associated with the ideal $J = \Pi_\lambda \setminus (\Pi_\lambda)_{\leq x}$ and $f(x) = \text{rk}(\Pi_\lambda) - \text{rk}(x)$. We have $E^1_{n,0} = 0$, and, for $i > 0$,

$$E^1_{n,i} = \bigoplus_{y \in f^{-1}(\hat{1}),\, y \wedge x = \hat{0}} \tilde{H}_{n-1}(y, \hat{1}).$$

Let d be the number of blocks in y, then $[y, \hat{1}] \simeq \Pi_d$ (here we use that 2 occurs as a block size in λ). We shall show that $d \geq t$. Let y have blocks of sizes $s_1, \ldots, s_d$. The set $\{s_1, \ldots, s_d\}$ gives a number partition of n, $y \in \Pi_\lambda$ means that λ is a refinement of $\{s_1, \ldots, s_d\}$. The condition $x \wedge y = \hat{0}$ means that there exists a block of y, without loss of generality we can assume it is s_d, such that λ is not a refinement of $\{s_1, s_2, \ldots, s_d - 1, 1\}$. It means it is impossible to pack disjointly blocks of sizes $\lambda_1, \ldots, \lambda_p, \lambda_{p+1}$, where $\lambda_{p+1} = 2$, into blocks of sizes $s_1, s_2, \ldots, s_d - 1$.

We will attempt to perform such a packing with a version of a greedy algorithm. Let us start with packing λ_1 into some of the blocks $s_1, \ldots, s_d - 1$. If it is possible continue with λ_2 and so on. At some point we will have to stop. Say we stopped at λ_i, i.e., it is impossible to pack λ_i into the rest (after packing $\lambda_1, \ldots, \lambda_{i-1}$) of the blocks $s_1, \ldots, s_d - 1$. Then the rests of the blocks $s_1, \ldots, s_d - 1$ have at most $\lambda_i - 1$ elements each, it gives us an inequality

$$d \cdot (\lambda_i - 1) + \lambda_1 + \cdots + \lambda_{i-1} + 1 \geq n = \lambda_1 + \cdots + \lambda_p + 2 + m$$

or

$$d \cdot (\lambda_i - 1) \geq \lambda_i + \cdots + \lambda_p + 1 + m,$$

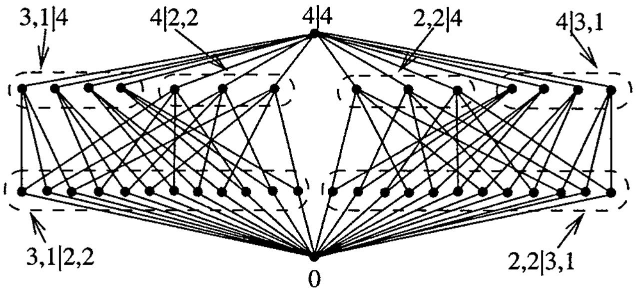

Figure 1.

which implies

$$d \geq \frac{m + \lambda_i + \cdots + \lambda_p + 1}{\lambda_i - 1}, \tag{5.1}$$

hence $d \geq t$.

The lattice Π_d has nonzero homology group only in dimension $d-3$, so $E^1_{k,i} = 0$ if $k \leq t-3$ and hence, using (3.6), we can conclude that Π_λ is $(t-3)$-acyclic. □

Often spectral sequences can be used for a direct computation of the poset homology groups. We will give here an informative example.

Let $\lambda = (3, 2, 2, 1)$. We shall compute the homology groups of $\Pi_{3,2,2,1}$. The poset $\Pi_{3,2,2,1}$ is pure and ranked by the function $\text{rk}(x) = 5-$(the number of blocks in x). Let $J = \bar{\Pi}_{3,2,2,1}$, $f(x) = \text{rk}(x)$, and construct the corresponding spectral sequence.

As was described in Section 4 we obtain Whitney homology groups. It is easy to see that $(\hat{0}, a)$ is CM for all $a \in \bar{\Pi}_{3,2,2,1}$ except for the case when a has partition type $(4, 4)$. These intervals are schematically shown in figure 1.

The Betti numbers of intervals $(\hat{0}, a)$ are given in the Table 1.

Observe that we can use formulae (3.15) and (3.12), since the intervals $(\hat{0}, a)$ have torsion free homology groups for $a \in \bar{P}$. Hence, the $E^1_{*,*}$-tableau for $J = \Pi_{3,2,2,1}$, $f(x) = \text{rk}(x)$, can be easily computed. The only non-zero entries are

$$E^1_{-1,0} = \mathbb{Z}, \quad E^1_{0,1} = \mathbb{Z}^{840}, \quad E^1_{1,2} = \mathbb{Z}^{4102}, \quad E^1_{1,3} = \mathbb{Z}^{35}, \quad E^1_{2,3} = \mathbb{Z}^{6588}.$$

First, it is straightforward that $d^1 : E^1_{0,1} \to E^1_{-1,0}$ is surjective, hence $E^\infty_{-1,0} = 0$. Furthermore, it is easy to check that the first two rank levels of $\Pi_{3,2,2,1}$ form a connected poset, hence d^1 is exact in $E^1_{0,1}$. It means that $E^2_{0,1} = 0$ and so $E^\infty_{1,3} = E^1_{1,3} = \mathbb{Z}^{35}$. Already this shows that $H_1(\Pi_{3,2,2,1}) \neq 0$.

It is not difficult to show that d^1 is exact in $E^1_{1,2}$ too (this will be done later). Hence the associated spectral sequence collapses at its second term, and the non-zero entries of the

Table 1.

Type of a	Number	$\tilde{\beta}_{-1}$	$\tilde{\beta}_0$	$\tilde{\beta}_1$
3221	840	1	0	0
332	280	0	5	0
431	280	0	2	0
422	210	0	3	0
521	168	0	9	0
71	8	0	0	155
62	28	0	0	90
53	56	0	0	43
44	35	0	1	12

tableau $E^2_{*,*}$ are:

$$E^2_{1,3} = \mathbb{Z}^{35}, \qquad E^2_{2,3} = \mathbb{Z}^{3325}.$$

Hence,

$$\tilde{\beta}_0(\Pi_{3,2,2,1}) = 0, \quad \tilde{\beta}_1(\Pi_{3,2,2,1}) = 35, \quad \tilde{\beta}_2(\Pi_{3,2,2,1}) = 3325.$$

In [11 Theorem 4.1] it has been proved that Π_λ is shellable if λ has no primitive partition identities. This of course does not apply to $\Pi_{3,2,2,1}$, since $\lambda = (3, 2, 2, 1)$ has the identity $2+2 = 3+1$. It is however not difficult to adapt the proof of [11, Theorem 4.1] to show that $P = \Pi_{3,2,2,1} \setminus \{\text{elements of type } 4, 4\}$ is shellable. This adaptation is technical and requires to go into the details of the 4-pages proof of the mentioned theorem, so we shall omit this argument. Alternatively, one could show that P is shellable by a direct argument.

Now, associate a spectral sequence $(\tilde{E}^r_{*,*})^\infty_{r=1}$ to P in the same way as above. The Whitney homology groups of P are subgroups of the Whitney homology groups of $\Pi_{3,2,2,1}$. On the other hand, since P is shellable, d^1 must be exact in $\tilde{E}^1_{1,2}$. Then, of course, d^1 is also exact in $E^1_{1,2}$.

5.2. Partitions with restricted block sizes

Let $\Pi_{n,1,\ldots,k}$ denote the poset consisting of partitions with block sizes from the set $\{1, \ldots, k, n\}$, ($\Pi_{n,1,\ldots,k} = \Pi_n$, if $k = n$). These lattices were considered in [21] in connection with certain relative subspace arrangements. It is believed that $\Pi_{n,1,\ldots,k}$ is torsion-free. We can obtain some information on the homology groups of these lattices from the following proposition.

Proposition 5.2 $\Pi_{n,1,\ldots,k}$ *is* $(k-3)$*-acyclic for* $k < n$.

Proof: $\Pi_{n,1,\ldots,k}$ is a lower ideal of the partition lattice Π_n. Π_n is a CM poset and $\Pi_{n,1,\ldots,k}$ contains the first $k-1$ rank levels of Π_n. Let J be a subposet of $\Pi_{n,1,\ldots,k}$ consisting of the complement of the first $k-1$ rank levels of Π_n, $f(x) = \mathrm{rk}(x) - k + 1$. Then the formulae (3.13) specialize to

$$E^1_{t,i} \simeq \bigoplus_{\mathrm{rk}(a)=i+k-1} \tilde{H}_{t-1}(\hat{0}, a),$$

since $(a, \hat{1}) \cap S_a = \emptyset$ for all $a \in J$.

Every interval $(\hat{0}, a)$ is a CM poset of rank $\mathrm{rk}(a) \geq k$, also $P \setminus J$ is CM of rank k, hence

$$E^1_{t,i} = 0, \quad \text{for } t \leq k-3, i \in \mathbb{Z}.$$

Using (3.2) we conclude that $\tilde{H}_t(\Pi_{n,1,\ldots,k}) = 0$ for $t \leq k-3$ and so $\Pi_{n,1,\ldots,k}$ is $(k-3)$-acyclic. □

Remark 5.3 It was communicated to the author by the referee that this and more general results can be found in the preprint [20]. The author was unaware of that work and is grateful to the referee for this comment.

6. $\mathcal{S}_n$-Quotient of the complex of directed forests

In this section we shall assume the following notions to be known: *directed graph*, *a subgraph of a directed graph*, *directed tree*, *directed forest*. If needed the reader may consult any textbook on graph theory for the definitions. We shall use $V(G)$, resp. $E(G)$, to denote the sets of vertices, resp. edges, of a directed graph G. We think of $E(G)$ as a subset of $(V(G) \times V(G)) \setminus \{(x, x) \mid x \in V(G)\}$. Since all the graphs considered in this section are directed, we will often omit this word.

Following a hint of Stanley [19], the following simplicial complexes were considered in [12].

Definition 6.1 Let G be an arbitrary directed graph. Construct a simplicial complex $\Delta(G)$ as follows: the vertices of $\Delta(G)$ are given by the edges of G and k-simplices are all directed forests with $k+1$ edges which are subgraphs of G.

Let G_n be the complete directed graph on n vertices, i.e., a graph having exactly one edge in each direction between any pair of vertices, all together $n(n-1)$ edges. It was shown in [12] that $\Delta(G_n)$ is shellable, thus all its homology groups are 0 except for the top one, and one can show that $\beta_{n-2}(\Delta(G_n)) = (n-1)^{(n-1)}$.

Furthermore, there is an action of $\mathcal{S}_n$ on $\Delta(G_n)$ induced by the permutation action of $\mathcal{S}_n$ on $[n]$, thus one can form the topological quotient $X_n = \Delta(G_n)/\mathcal{S}_n$, see figure 2 for the case $n = 3$. It was asked in [12, Section 6, Question 2] what $H_*(X_n, \mathbb{Z})$ is. The answer to that turned out to be more complex than we thought. In this section we show that the groups $H_*(X_n, \mathbb{Z})$ are, in general, not free, and also give a formula for $\beta_{n-2}(X_n, \mathbb{Q})$.

A combinatorial description for the cell structure of X_n. Clearly, the action of $\mathcal{S}_n$ on $\Delta(G_n)$ is not free. What is worse, the elements of $\mathcal{S}_n$ may fix the simplices of $\Delta(G_n)$ without fixing them pointwise: for example for $n = 3$ the permutation (23) "flips" the 1-simplex given by the directed tree $2 \leftarrow 1 \rightarrow 3$. Therefore, one does not have a bijection between the orbits of simplices of $\Delta(G_n)$ and simplices of X_n. To rectify the situation, consider the barycentric subdivision $B_n = \mathrm{Bsd}(\Delta(G_n))$. We have a simplicial $\mathcal{S}_n$-action on B_n induced from the $\mathcal{S}_n$-action on $\Delta(G_n)$ and, clearly, $B_n/\mathcal{S}_n$ is homeomorphic to X_n. Furthermore, if an element of $\mathcal{S}_n$ fixes a simplex of B_n then it fixes it pointwise. In this situation, it is well-known, e.g. see [7], that the quotient projection $B_n \rightarrow X_n$ induces a simplicial structure on X_n, in which simplices of X_n correspond to $\mathcal{S}_n$-orbits of the simplices of B_n with appropriate boundary relation.

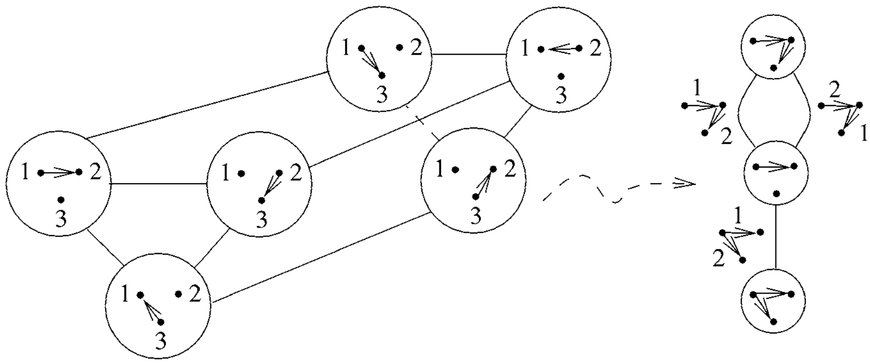

Figure 2.

Let us now give a combinatorial description of the $\mathcal{S}_n$-orbits of the simplices of B_n. Let σ be a simplex of B_n, then σ is a chain $(T_1, T_2, \ldots, T_{\dim(\sigma)+1})$ of forests on n labeled vertices, such that T_i is a subgraph of T_{i+1}, for $i = 1, \ldots, \dim(\sigma)$. One can view this data in a slightly different way: it is a forest with $\dim(\sigma) + 1$ integer labels on edges (labels on different edges may coincide). Indeed, given a chain of forests as above, take the forest $T_{\dim(\sigma)+1}$ and put label 1 an all edges of the forest T_1, label 2 on all edges of T_2, which are not labeled yet, etc. Vice versa, given a forest T with a labeling, let T_1 be the forest consisting of all edges of T with the smallest label, let T_2 be the forest consisting of all edges of T with one of the two smallest labels, etc. To make the described correspondence a bijection, one should identify all labeled forests on which labelings produce the same order on edges.

Formally: *the p-simplices of B_n are in bijection with the set of all pairs (T, ϕ^T), where T is a directed forest on n labeled vertices and $\phi^T : E(T) \rightarrow \mathbb{Z}$, such that $|\mathrm{Im}\,\phi^T| = p + 1$, modulo the following equivalence relation*: $(T_1, \phi^{T_1}) \sim (T_2, \phi^{T_2})$ *if* $T_1 = T_2$ *and there exists an order-preserving injection $\psi : \mathbb{Z} \rightarrow \mathbb{Z}$, such that $\phi^{T_1} \circ \psi = \phi^{T_2}$.*

The boundary operator is given by: for a p-simplex (T, ϕ^T), $p \geq 1$, we have $\partial(T, \phi^T) = \sum_{i=1}^{p+1} (-1)^{p+i+1} (T_i, \phi^{T_i})$. Here, for $i = 1, \ldots, p$, we have $T_i = T$ and ϕ^{T_i} takes the same values as ϕ^T except for the edges on which ϕ^T takes ith and $(i+1)$st largest values (say a and b), on these edges ϕ^{T_i} takes value a. Furthermore, T_{p+1} is obtained from T be removing the edges with the highest value of ϕ^T, $\phi^{T_{p+1}}$ is the restriction of ϕ^T. Of course, this description of the boundary map is just a rephrasing of the deletion of the ith forest from the chain of forests in the original description. However, we will find it more convenient to work with the labeled forests rather than the chains of forests.

The orbits of the action of $\mathcal{S}_n$ can be obtained by forgetting the numbering of the vertices. Thus, using the fact that simplices of X_n and $\mathcal{S}_n$-orbits of simplices of B_n are the same thing, we get the following description.

The p-simplices of X_n are in bijection with pairs (T, ϕ^T), where T is a directed forest on n unlabeled vertices and ϕ^T is an edge labeling of T with $p+1$ labels, modulo a certain equivalence relation. This equivalence relation and the boundary operator are exactly as in the description of simplices of B_n.

On figure 2 we show the case $n = 3$. On the left hand side we have $\Delta(G_3)$, on the right hand side is $X_3 = \Delta(G_3)/\mathcal{S}_3$. The labeled forests next to the edges indicate the bijection described above, labeling on the forests corresponding to the vertices in X_3 is omitted. $\mathcal{S}_3$ acts on $\Delta(G_3)$ as follows: 3-cycles act as rotations around the line which goes through the middles of the triangles, each transposition acts as a central symmetry on one of the quadrangles, and as a "flip" on the edge which is parallel to that quadrangle.

Filtration. There is a natural filtration on the chain complex associated to the simplicial structure on X_n described above. Let F_i be the union of all simplices (T, ϕ^T) where T has at most i edges. Clearly, $\emptyset = F_0 \subset F_1 \subset \cdots \subset F_{n-1} = X_n$.

The description of the E^1 tableau. Recall that $E^1_{p,k} = H_p(F_k, F_{k-1})$, here we use the indexing from Section 3. In other words, the homology is computed with "truncated" boundary operator: the last term, where some edges are deleted from the forest, is omitted. Clearly,

$$E^1_{p,k} = \bigoplus_T H_p(E_T), \tag{6.1}$$

where the sum is over all forests with k edges and E_T is a chain complex generated by the simplices (T, ϕ^T), for various labelings ϕ^T, with the truncated boundary operator as above.

Let us now describe a simplicial complex whose reduced homology groups, after a shift in the index by 1, are equal to the nonreduced homology groups of E_T. The arrangement of $k(k-1)/2$ hyperplanes $x_i = x_j$ in $\mathbb{R}^k$ cuts the space $S^{k-1} \cap H$ into simplices, where H is the hyperplane given by the equation $x_1 + x_2 + \cdots + x_k = 0$. Denote this simplicial complex A_k. The permutation action of $\mathcal{S}_k$ on $[k]$ induces an $\mathcal{S}_k$-action on A_k. It is easy to see that if an element of $\mathcal{S}_k$ fixes a simplex of A_k, then it fixes it pointwise. Hence, for any subgroup $\Gamma \subseteq \mathcal{S}_k$, the Γ-orbits of the simplices of A_k are in a natural bijection with the simplices of A_k/Γ.

Let T be an arbitrary forest with n vertices and k edges. Assume that vertices, resp. edges, are labeled with numbers $1, \ldots, n$, resp. $1, \ldots, k$. $\mathcal{S}_n$ acts on $[n]$ by permutation, let $\mathrm{Stab}(T)$ be stabilizer of T under this action, that is the maximal subgroup of $\mathcal{S}_n$ which fixes T. Then $\mathrm{Stab}\,(T)$ acts on $E(T)$, i.e., we have a homomorphism $\chi : \mathrm{Stab}(T) \to \mathcal{S}_k$. Let $\mathcal{S}(T) = \mathrm{Im}\chi$.

Clearly $\mathcal{S}(T)$ does not depend on the choice of the labeling of vertices. However, relabeling the edges changes $\mathcal{S}(T)$ to a conjugate subgroup. Therefore, for a forest T without labeling on vertices and edges, $\mathcal{S}(T)$ can be defined, but only up to a conjugation.

Proposition 6.2 *The chain complex of $A_k/\mathcal{S}(T)$ and E_T (with a shift by 1 in the indexing) are isomorphic. In particular, $\tilde{H}_p(A_k/\mathcal{S}(T)) = H_{p+1}(E_T)$.*

Proof: Label the k edges of T with numbers $1, \dots, k$. As mentioned above, the p-simplices of E_T are in bijection with labelings of the edges of T with numbers $1, \dots, p+1$ (using each number at least once). Taking in account the chosen labeling of the edges, this is the same as to divide the set $[k]$ into an ordered tuple of $p+1$ non-empty sets, modulo the symmetries of $[k]$ induced by the symmetries of T. Clearly, these symmetries of $[k]$ are precisely the elements of $\mathcal{S}(T)$.

The $(p-1)$-simplices of A_k are in bijection with dividing $[k]$ into an ordered tuple of $p+1$ non-empty sets: by the values of the coordinates. Therefore we conclude that the p-simplices of E_T are in a natural bijection with the $(p-1)$-simplices of $A_k/\mathcal{S}(T)$. Here the unique 0-simplex of E_T, $(T, \mathbf{1})$, ($\mathbf{1}$ is the constant function taking value 1), corresponds in $A_k/\mathcal{S}(T)$ to the empty set, which is a (-1)-simplex. One verifies immediately that the boundary operators of E_T and $A_k/\mathcal{S}(T)$ commute with the described bijection. Therefore E_T and $A_k/\mathcal{S}(T)$ are isomorphic as chain complexes (after a shift in the indexing). In particular, $\tilde{H}_p(A_k/\mathcal{S}(T)) = H_{p+1}(E_T)$. □

$\mathbb{Q}$ **coefficients.** Proposition 6.2 allows us to give a description of $E^1_{*,*}$-entries in the case when the homology groups are computed with rational coefficients. Indeed, it is well known that, when a finite group Γ acts on a finite simplicial complex X, one has $\tilde{H}_i(X/\Gamma, \mathbb{Q}) = \tilde{H}_i^{\Gamma}(X, \mathbb{Q})$, where $\tilde{H}_i^{\Gamma}(X, \mathbb{Q})$ is the maximal vector subspace of $\tilde{H}_i(X, \mathbb{Q})$ on which Γ acts trivially (more generally $\mathbb{Q}$ can be replaced with a field whose characteristic does not divide $|\Gamma|$). Since A_k is homeomorphic to S^{k-2} we have $\tilde{H}_{k-2}(A_k, \mathbb{Q}) = \mathbb{Q}$ and $\tilde{H}_i(A_k, \mathbb{Q}) = 0$ for $i \neq k-2$. It is easy to compute $\tilde{H}_{k-2}^{\mathcal{S}(T)}(A_k, \mathbb{Q})$. In fact, for $\pi \in \mathcal{S}_k$, $\alpha \in \tilde{H}_{k-2}(A_k, \mathbb{Q})$, one has $\pi(\alpha) = (-1)^{\operatorname{sgn}\pi}\alpha$, where sgn denotes the sign homomorphism $\operatorname{sgn} : \mathcal{S}_k \to \{-1, 1\}$. Therefore

$$\tilde{H}_{k-2}(A_k/\mathcal{S}(T), \mathbb{Q}) = \tilde{H}_{k-2}^{\mathcal{S}(T)}(A_k, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } \mathcal{S}(T) \subseteq \mathcal{A}_k, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{A}_k$ is the alternating group, $\mathcal{A}_k = \operatorname{sgn}^{-1}(1)$.

Combined with the Proposition 6.2 this gives $H_i(E_T, \mathbb{Q}) = \mathbb{Q}$, if $i = |E(T)| - 1$ and $\mathcal{S}(T) \subseteq \mathcal{A}_{|E(T)|}$, and $H_i(E_T, \mathbb{Q}) = 0$ in all other cases. Therefore it follows from (6.1) that $\operatorname{rk} E^1_{k-1,k} = f_{k,n}$, where $f_{k,n}$ is equal to the number of forests T with k edges and n vertices, such that $\mathcal{S}(T) \subseteq \mathcal{A}_k$. $\operatorname{rk} E^1_{p,k} = 0$ for $p \neq k-1$. Note that $\beta_i(X_n, \mathbb{Q}) = 0$, for $i \neq n-2$, because $\beta_i(\Delta(G_n), \mathbb{Q}) = 0$, for $i \neq n-2$ (shown in [12]), and $\beta_i(X_n, \mathbb{Q}) = \beta_i^{\mathcal{S}_n}(\Delta(G_n), \mathbb{Q})$. In particular, by computing the Euler characteristic of X_n in two different ways, we obtain

Theorem 6.3 *For $n \geq 3$, $\beta_{n-2}(X_n, \mathbb{Q}) = \sum_{k=2}^{n-1}(-1)^{n+k+1} f_{k,n}$.*

The first values of $f_{k,n}$ are given in the table below. Note that there are zeroes on and below the main diagonal and that the rows stabilize at the entry $(k, 2k-1)$ (for $k \geq 2$).

$k\backslash n$	1	2	3	4	5	6
1	0	**1**	1	1	1	1
2	0	0	**1**	1	1	1
3	0	0	0	2	**3**	3
4	0	0	0	0	4	7
5	0	0	0	0	0	8

$\mathbb{Z}$ **coefficients.** The case of integer coefficients is more complicated. In general, we do not even know the entries of the first tableau. However, we do know that it is different from the rational case, i.e., torsion may occur.

For example, let T be the forest with 8 vertices and 6 edges depicted on figure 3. Clearly, $\mathcal{S}(T) = \{\text{id}, (12)(34)(56)\}$. It is easy to see that $A_6/\mathcal{S}(T)$ is a double suspension (by which we mean suspension of suspension) of $\mathbb{RP}^2$, thus the only nonzero homology group is $\tilde{H}_3(A_6/\mathcal{S}(T), \mathbb{Z}) = \mathbb{Z}_2$. In particular, $E^1_{4,6}$ is not free.

On the positive side, we can describe the values which d^1 takes on the "rational" generators of $E^1_{*,*}$. Let us call a forest *admissible* if $\mathcal{S}(T) \subseteq \mathcal{A}_{|E(T)|}$. For every admissible forest T with k edges we fix some order on the edges, i.e., a bijection $\psi_T : E(T) \to [k]$. This determines uniquely an integer generator e_T of $H_{k-1}(E_T, \mathbb{Z})$ by

$$e_T = \sum_{\mathcal{S}(T)g} \text{sgn}(g)(T, g \circ \psi_T), \tag{6.2}$$

where we sum over all right cosets of $\mathcal{S}(T)$, (we choose one representative for each coset). Observe that the sign of g, resp. the simplex $(T, g \circ \psi_T)$, are the same for different representatives of the same right coset class, because $\mathcal{S}(T) \subseteq \mathcal{A}_k$, resp. by the definition of $\mathcal{S}(T)$.

Proposition 6.4 *For an admissible forest T, we have*

$$d^1(e_T) = \sum_{\alpha} \text{sgn}\big(\tilde{\psi}_{T,\alpha} \circ \psi_T^{-1}\big)\lambda_{T,\alpha} e_{T\backslash\alpha}, \tag{6.3}$$

Figure 3.

where the sum is over $\mathcal{S}(T)$-orbits of $E(T)$, for which there exists a representative α, such that $T \setminus \alpha$ is admissible, we choose one representative for each orbit; note that the admissibility of $T \setminus \alpha$ depends only on the $\mathcal{S}(T)$-orbit of α, not on the choice of the representative. Notation in the formula: $T \setminus \alpha$ denotes the forest obtained from T by removing the edge α; $\tilde{\psi}_{T,\alpha} : E(T) \to [k]$ is defined by $\tilde{\psi}_{T,\alpha}|_{T\setminus\alpha} = \psi_{T\setminus\alpha}$ and $\tilde{\psi}_{T,\alpha}(\alpha) = k$; $\lambda_{T,\alpha} = [\mathcal{S}(T \setminus \alpha) : \tilde{\mathcal{S}}(T)]$, where $\tilde{\mathcal{S}}(T)$ consists of those permutations of edges of $T \setminus \alpha$ which can be extended to T by fixing the additional edge.

Proof: For an admissible forest T with k edges and a bijection $\phi : E(T) \to [k]$, let $(\tilde{T}, \tilde{\phi})$ denote a face simplex of (T, ϕ), where $\tilde{T}$ is obtained from T by removing the edge with the highest label, $\tilde{\phi}$ is the restriction of ϕ to $\tilde{T}$. In our notations $(\tilde{T}, \tilde{\phi}) = (T \setminus \phi^{-1}(k), \phi \mid_{E(T\setminus\phi^{-1}(k))})$. However, for convenience, we use the notation "tilde" in the rest of the proof.

According to the general theory for spectral sequences, $d^1(e_T) = \partial(e_T)$, where ∂ denotes the usual boundary operator, and we view $\partial(e_T)$ as embedded into the relative homology group $H_{k-2}(F_{k-1}, F_{k-2})$. $\partial(e_T)$ is a linear combination of simplices which are obtained from the simplices $(T, g \circ \psi_T)$ by either merging two labels, or omitting the edge with the top label. $e_T \in H_{k-1}(F_k, F_{k-1})$ means that the application of the "truncated" boundary operator to e_T gives 0, therefore all the simplices obtained by merging two labels will cancel out. Furthermore, since $\partial(e_T) \in H_{k-2}(F_{k-1}, F_{k-2})$, $\dim F_{k-1} = k - 2$, and the group $H_{k-2}(F_{k-1}, F_{k-2})$ is freely generated by e_U, where U is an admissible forest with $k - 1$ edges, we can conclude that also the contributions $(\tilde{T}, \tilde{\phi})$, where $\tilde{T}$ is not admissible, will cancel out. Combining these arguments with (6.2) we obtain:

$$d^1(e_T) = \sum_{\mathcal{S}(T)g} \operatorname{sgn}(g)(\tilde{T}, \widetilde{g \circ \psi_T}), \tag{6.4}$$

where we have only those terms left in the sum, for which $\tilde{T}$ is admissible. After regrouping we get

$$\sum_{\mathcal{S}(T)g} \operatorname{sgn}(g)(\tilde{T}, \widetilde{g \circ \psi_T}) = \sum_{\alpha} \sum_{\mathcal{S}(T)g} \operatorname{sgn}(g)(\tilde{T}, \widetilde{g \circ \psi_T}), \tag{6.5}$$

where in the second term the first sum is taken over all $\mathcal{S}(T)$-orbits of $[k]$, for which $\tilde{T}$ is admissible, while the second sum is taken over all right cosets $\mathcal{S}(T)g$ which have a representative g such that $g \circ \psi_T(\alpha) = k$, we take one representative per coset. To verify (6.5) we just need to observe that the $\mathcal{S}(T)$-orbit of $(g \circ \psi_T)^{-1}(k)$ does not depend on the choice of the representative of $\mathcal{S}(T)g$; this follows from the definition of $\mathcal{S}(T)$.

Finally, one can see that, for α being an edge of T, such that $T \setminus \alpha$ is admissible,

$$\sum_{\mathcal{S}(T)g} \operatorname{sgn}(g)(\tilde{T}, \widetilde{g \circ \psi_T}) = \operatorname{sgn}\left(\tilde{\psi}_{T,\alpha} \circ \psi_T^{-1}\right)\lambda_{T,\alpha} \sum_{\mathcal{S}(T\setminus\alpha)h} \operatorname{sgn}(h)(T \setminus \alpha, h \circ \psi_{T\setminus\alpha}), \tag{6.6}$$

where the sum in the first term is again taken over all right cosets $\mathcal{S}(T)g$ which have a representative g such that $g \circ \psi_T(\alpha) = k$, and the sum in the second term is simply over all right cosets of $\mathcal{S}(T \setminus \alpha)$.

Indeed, on the left hand side we have a sum over all labelings of $E(T)$ with numbers $1, \ldots, k$, such that α gets a label k, and we consider these labelings up to a symmetry of T; each labeling comes in with a sign of the permutation g, which is obtained by reading off this labeling in the order prescribed by ψ_T. On the right hand side the same sum is regrouped, using the observation that to label $E(T)$ with $[k]$, so that α gets a label k, is the same as to label $E(T \setminus \alpha)$ with $[k-1]$. The only details which need attention are the multiplicity and the sign.

Every $\mathcal{S}(T)$-orbit of labelings of $E(T)$ with $[k]$ so that α gets a label k corresponds to $[\mathcal{S}(T \setminus \alpha) : \tilde{\mathcal{S}}(T)]$ of $\mathcal{S}(T \setminus \alpha)$-orbits of labelings of $E(T \setminus \alpha)$ with $[k-1]$, since we identify labelings by the actions of different groups: $\mathcal{S}(T \setminus \alpha) \supseteq \tilde{\mathcal{S}}(T)$. Each of this $\mathcal{S}(T \setminus \alpha)$-orbits comes with the same sign, because $\mathcal{S}(T \setminus \alpha) \subseteq \mathcal{A}_{k-1}$. The sign $\mathrm{sgn}(\tilde{\psi}_{T,\alpha} \circ \psi_T^{-1})$ corresponds to the change of the order in which we read off the edges: instead of reading them off according to ψ_T, we first read off along $\psi_{T \setminus \alpha}$ and then read off the edge α last. Formally: $g \circ \psi_T = \tilde{h} \circ \tilde{\psi}_{T,\alpha}$, and $\mathrm{sgn}\tilde{h} = \mathrm{sgn} h$, hence $\mathrm{sgn}\ g = \mathrm{sgn}\ h\ \mathrm{sgn}\ (\tilde{\psi}_{T,\alpha} \circ \psi_T^{-1})$, where $\tilde{h}$ is defined by $\tilde{h}|_{[k-1]} = h$, $\tilde{h}(k) = k$.

Combining (6.4), (6.5) and (6.6) we obtain (6.3). □

Homology groups of X_n for $n = 2, 3, 4, 5, 6$. X_2 is just a point. As shown in figure 2, $X_3 \simeq S^1$, where $\simeq$ denotes homotopy equivalence. With a bit of labour, one can manually verify that $X_4 \simeq S^2$. Furthermore, one can see that $H_3(X_5, \mathbb{Z}) = \mathbb{Z}^2$ and $\tilde{H}_i(X_5, \mathbb{Z}) = 0$ for $i \neq 3$. We leave this to the reader, while confining ourselves to the case $n = 6$. On figure 4 we have all forests on 6 vertices. We denote some of the forests by two digits. The numbers over the edges denote the order in which we read the labels, i.e., the bijection ψ_T.

It is easy to see that $A_k/\mathcal{S}(T)$ is homeomorphic to S^{k-2} for all admissible T, and is contractible otherwise. The only nontrivial cases are 41, 47, 48, 51, 55, and 59, all of which can be verified directly. Therefore, the only nontrivial entries of $E^1_{*,*}$ ($\mathbb{Z}$ coefficients) will lie on the $(k-1, k)$-diagonal. Thus $H_*(X_6, \mathbb{Z})$ can be computed from the chain complex $0 \leftarrow E^1_{0,1} \stackrel{d^1}{\leftarrow} E^1_{1,2} \stackrel{d^1}{\leftarrow} E^1_{2,3} \stackrel{d^1}{\leftarrow} E^1_{3,4} \stackrel{d^1}{\leftarrow} E^1_{4,5} \leftarrow 0$.

From Proposition 6.4 we have

$$
\begin{array}{ccc}
d^1(11) = 0, & d^1(21) = 0, & \\
d^1(31) = 2 \cdot 21, & d^1(32) = 21, & d^1(33) = 21, \\
d^1(41) = 32 - 33, & d^1(42) = 31 - 32 - 33, & d^1(43) = 31 - 2 \cdot 33, \\
d^1(44) = 0, & d^1(45) = 31 - 32 - 33, & d^1(46) = 32 - 33, \\
& d^1(47) = 0, &
\end{array}
$$

$$
\begin{array}{cc}
d^1(51) = 41 - 46, & d^1(52) = -42 + 43 + 44 - 46, \\
d^1(53) = -42 + 45, & d^1(54) = 2 \cdot 41 + 42 - 43 - 46 + 2 \cdot 47, \\
d^1(55) = 41 - 43 + 45, & d^1(56) = 42 + 44 - 45 - 2 \cdot 47, \\
d^1(57) = -43 + 44 + 45 + 46, & d^1(58) = 2 \cdot 44 + 2 \cdot 47,
\end{array}
$$

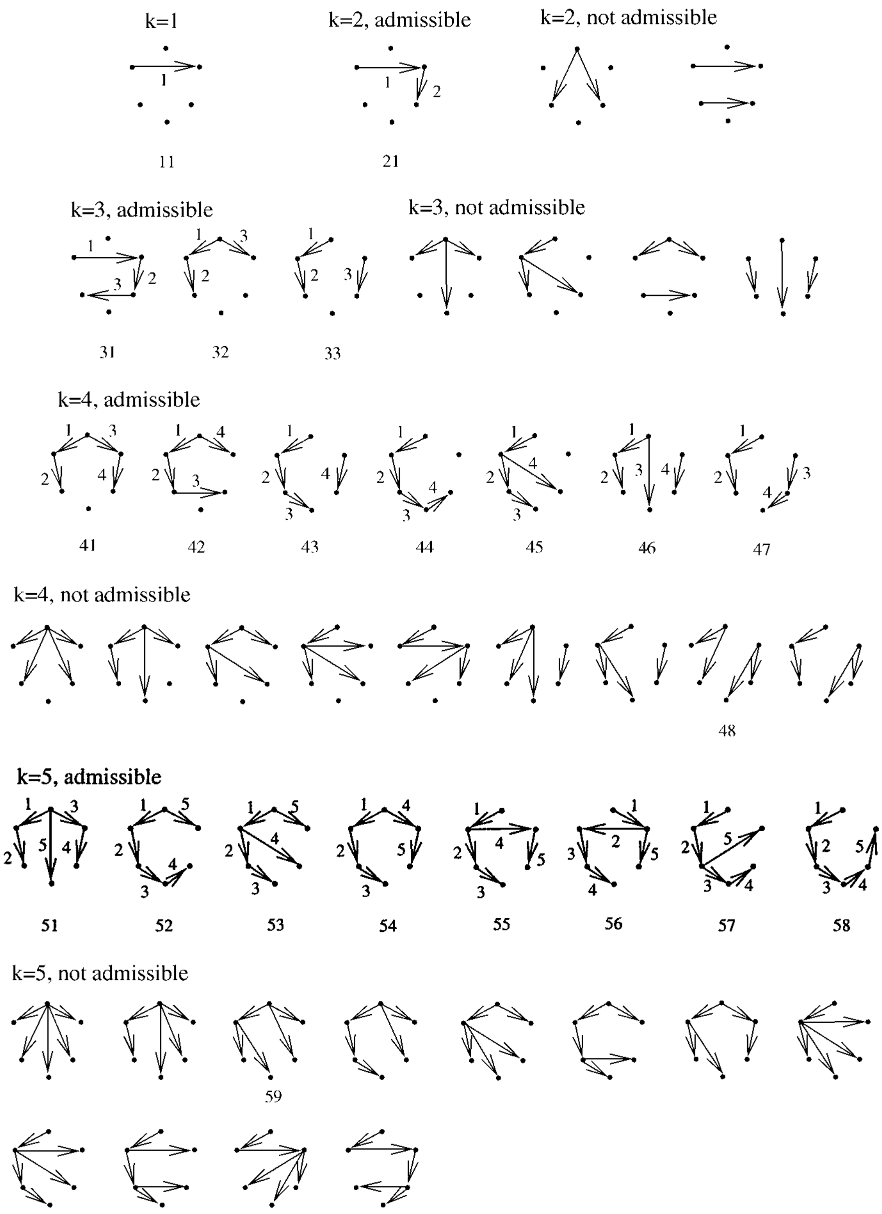

Figure 4.

here the two-digit strings denote the corresponding forests on figure 4. Thus $\tilde{H}_3(X_6, \mathbb{Z}) = \mathbb{Z}_2$, $\tilde{H}_4(X_6, \mathbb{Z}) = \mathbb{Z}^3$ and $\tilde{H}_i(X_6, \mathbb{Z}) = 0$ for $i \neq 3, 4$.

Therefore 6 *is the smallest value of* n, *for which the homology groups* $H_*(X_n, \mathbb{Z})$ *are not free*.

Acknowledgments

I am grateful to the anonymous referee for pointing out a computational mistake, and to Eva-Maria Feichtner for the careful proofreading of the early versions of the paper.

References

1. K. Baclawski, "Whitney numbers of geometric lattices," *Advances in Math*. **16** (1975), 125–138.
2. K. Baclawski, "Galois connections and the Leray spectral sequence," *Advances in Math*. **25** (1977), 191–215.
3. K. Baclawski, "Cohen-Macaulay ordered sets," *J. Algebra* **63** (1980), 226–258.
4. A. Björner, "Subspace arrangements," in *First European Congress of Mathematics, Paris 1992*, A. Joseph et al. (Eds.), Progress in Math., Vol. 119, Birkhäuser, Basel, 1994, pp. 321–370.
5. A. Björner, "Topological methods," in *Handbook of Combinatorics*, R. Graham, M. Grötschel, and L. Lovász (Eds.), North-Holland, Amsterdam, 1995, pp. 1819–1872.
6. A. Björner and J.W. Walker, "A homotopy complementation formula for partially ordered sets," *European J. Combin*. **4** (1983), 11–19.
7. G.E. Bredon, Introduction to Compact Transformation Groups, Academic Press, San Diego, 1972.
8. E.M. Feichtner and D.N. Kozlov, "On subspace arrangements of type $\mathcal{D}$," in *Proceedings of FPSAC'96, Discrete Math.*, Vol. 210, No. 1–3, 2000, pp. 27–54.
9. P. Hanlon, "The generalized Dowling lattices," *Trans. Amer. Math. Soc*. **325** (1991), 1–37.
10. M.I. Kargapolov and Ju.L. Merzljakov, *Fundamentals of the Theory of Groups*, Graduate Texts in Mathematics, Vol. 62, Springer-Verlag, Berlin, 1979 (English translation of Osnovy teorii grupp, Nauka, Moscow, 1977).
11. D.N. Kozlov, "General lexicographic shellability and orbit arrangements," *Ann. Comb*. **1** (1) (1997), 67–90.
12. D.N. Kozlov, "Complexes of directed trees," *J. Comb. Theory A* **88**(1) (1999), 112–122.
13. W.S. Massey, "Exact couples in algebraic topology I, II," *Ann. of Math*. **56** (1952), 363–396.
14. J. McCleary, *User's Guide to Spectral Sequences*, Publish or Perish, Wilmington, 1985.
15. D. Quillen, *Higher Algebraic K-Theory I*, Lecture Notes in Mathematics, Vol. 341, Springer-Verlag, Berlin, 1973, 85–148.
16. D. Quillen, "Homotopy properties of the poset of nontrivial p-subgroups of a group," *Adv. in Math*. **28** (1978), 101–128.
17. E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.
18. R.P. Stanley, *Enumerative Combinatorics*, Vol. I, Wadsworth, Belmont, CA, 1986.
19. R.P. Stanley, private communication, 1997.
20. S. Sundaram, "On the topology of two partition posets with forbidden blocksizes," *J. Pure Appl. Algebra* **155**(2–3) (2001), 271–304.
21. V. Welker, Partition lattices, group actions on subspace arrangements and combinatorics of discriminants, Habilitationsshrift, Essen, 1996.