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Roots of Independence Polynomials of Well Covered Graphs

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Abstract. Let G be a *well covered* graph, that is, all maximal independent sets of G have the same cardinality, and let i_k denote the number of independent sets of cardinality k in G. We investigate the roots of the *independence polynomial* $i(G, x) = \sum i_k x^k$. In particular, we show that if G is a well covered graph with independence number β, then all the roots of $i(G, x)$ lie in in the disk $|z| \leq \beta$ (this is far from true if the condition of being well covered is omitted). Moreover, there is a family of well covered graphs (for each β) for which the independence polynomials have a root arbitrarily close to $-\beta$.

Keywords: graph, independence, polynomial, root, well covered

1. Introduction

For a graph G with independence number β and nonnegative integer k, let i_k denote the number of independent sets of cardinality k in G ($k = 0, 1, \ldots, \beta$). Several papers exist (c.f. [2, 10, 11, 19] exploring such problems on general graphs (or their complements). In fact in [2] it was shown that for any permutation σ of $\{1, \ldots, \beta\}$, there is a graph with independence number β such that $i_{\sigma(1)} < i_{\sigma(2)} < \cdots < i_{\sigma(k)}$. That is, there are graphs for which $i_1, \ldots, i_k$ is as 'shuffled' as we like.

One highly structured class of graphs with respect to independence are those in which all maximal independent sets have the same cardinality; these are called *well covered* graphs. Well covered graphs have attracted considerable attention (c.f. [27, 28]).

It appears that the independence vectors $(i_0, i_1, \ldots, i_\beta)$ for well covered graphs are not as badly behaved as those for general graphs, and we propose the following:

Conjecture 1.1 The independence vector $(i_0, i_1, \ldots, i_\beta)$ of a well covered graph G is *unimodal*, i.e., there is a $k \in \{0, \ldots, \beta\}$ such that $i_0 \leq i_1 \leq \cdots \leq i_k \geq i_{k+1} \geq \cdots \geq i_\beta$.

Unimodality conjectures permeate combinatorics in such diverse areas as matroids [25, 31], ordered sets [8, 30], chromatic theory [29], network reliability [6] and Gaussian polynomials [23].

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One approach to unimodality of combinatorial sequences has been via the roots of the associated generating polynomials. Newton (c.f. [9]) showed that if a polynomial $f(x) = \sum_{i=0}^{d} a_i x^i$ with positive coefficients has all real roots, then the sequence $a_0, a_1, \ldots, a_d$ is unimodal (and in fact has a stronger property of being *log concave*). More generally, it is not hard to see that if the roots of such an f lie in the sector $2\pi/3 \leq \arg(z) \leq 4\pi/3$, then the sequence of coefficients is still unimodal.

These approaches have led us to investigate the nature and location of the roots of the *independence polynomial*

$$i(G, k) = \sum i_k x^k$$

of G, particularly when G is well covered (some results on independence polynomials can be found in [15–18, 21]). The roots of other graph polynomials, such as chromatic polynomials [29], matching polynomials [13, 14], characteristic polynomials [13] and reliability polynomials [5, 6], to mention a few, have attracted considerable attention in their own right. Indeed, the roots of this polynomial shed light on how independence vectors of well covered graphs differ from those for all graphs. In the next section, we shall explore real roots of independence polynomials of well covered graphs, and in particular show that every well covered graph can be embedded as an induced subgraph in another well covered graph (with the same independence number) whose polynomial has all its roots simple and real. The following section shows that for any noncomplete well covered graph G with independence number β, its roots lie in the disc $|z| < \beta$, and that this upper bound is essentially the best possible.

Throughout, all graphs are assumed to be simple, that is, without loops or multiple edges. The order of a graph (usually denoted by n) is its number of vertices. We denote the complement of graph G by $\bar{G}$. The independence number of graph G, $\beta(G)$ (or often simply β), is the cardinality of the largest independent set in G. Given a vertex v of G, $N[v]$ denotes the closed neighbourhood of v, i.e., $\{x : vx \text{ is an edge of } G\} \cup \{v\}$. For other standard graph theoretic terminology, we follow [7].

2. Real roots of independence polynomials

We start by considering the real roots of independence polynomials. Of course, such roots must necessarily be negative as independence polynomials have positive coefficients.

Let's begin by considering well covered graphs with small independence number. For $\beta = 1$, the only (well covered) graph of order n is K_n, and $i(K_n, x) = 1 + nx$. Clearly this independence polynomial has $-1/n$ as its only root.

For $\beta = 2$, the situation is a bit more complex. In general, a graph H with independence number $\beta(H)$ is well covered if and only if $\bar{H}$ is $K_{\beta(H)+1}$-free and every clique of cardinality less than $\beta(H)$ is contained in a clique of order $\beta(H)$. The complement of a graph G with independence number 2 is of course triangle-free. Thus if $\bar{G}$ has n vertices and m edges, then $i(G, x) = 1 + nx + mx^2$. The roots of this are of course $(-n \pm \sqrt{n^2 - 4m})/2m$, which lie in $[(-n - \sqrt{n^2 - 4m})/2, 0]$ since Turan's Theorem implies for the triangle-free graph $\bar{G}$ that $m \leq n^2/4$, and so that all the roots are real. This argument holds for any graph with independence number 2. Now for general graphs with $\beta = 2$, m can be as small as 1, and hence $i(G, x)$ can have a root at $\frac{-n-\sqrt{n^2-4}}{2} \sim -n$. However, if the graph is well covered, then there are no isolated vertices in the complement, so that $m \geq n/2$. It follows that the roots in fact all lie in $(-2, 0)$.

For $\beta \geq 3$, the roots of the independence polynomial of a well covered graph need not be real. For example, consider the k-partite graph $K_{\beta,\beta,\ldots,\beta}$. It is a well covered graph with independence number β and has independence polynomial

$$\begin{aligned} i(K_{\beta,\beta,\ldots,\beta}, x) &= 1 + \sum_{i=0}^{\beta-1} k \binom{\beta}{i} x^i \\ &= k(x+1)^\beta - (k-1) \\ &= k((x+1)^\beta - (1 - 1/k)) \end{aligned}$$

which clearly has at least $\beta - 2$ nonreal roots for $k \geq 2$.

Note in the example above that the independence polynomial does have a real root. In fact, such is always the case.

Theorem 2.1 *For any graph G (not necessarily well covered), a root of the independence polynomial of G of smallest modulus is real.*

Proof: In [12], it was shown that if

$$f_G(x) = \sum_{i \geq 0} (-1)^i c_i x^i,$$

where c_i is the number of complete subgraphs of G of cardinality i, then f has a root of smallest modulus that is real. Clearly $i(G, x) = f_{\bar{G}}(-x)$, so the result follows. □

In fact, it follows from Theorem 3 of [12] that the largest real root of $i(G, x)$ lies in $[-\frac{\beta}{n}, 0)$. On the other hand, we'll see in the next section that this root is less than or equal to $-1/n$.

Now while there are many well covered graphs (for $\beta \geq 3$) with nonreal roots, we can embed every well covered graph in another well covered graph (with the same independence number) such that the independence polynomial of the latter has all real roots. We shall first need an easy recursive formula for calculating independence polynomials.

Proposition 2.2 [17, 18, 21, 26] *For any vertex v of graph G,*

$$i(G, x) = x \cdot i(G - N[v], x) + i(G - v, x).$$ □

We now prove our main result of this section.

Theorem 2.3 *For any well covered graph G, there is a well covered graph H with $\beta(H) = \beta(G)$ such that G is an induced subgraph of H and the independence polynomial of H has all its roots simple and real.*

Proof: We define an *expansion* of a graph G to be a graph formed from G by replacing each vertex by a complete graph; that is, for each vertex u of G, we replace u by a new complete graph K_u, and add in edges between all vertices in K_u and K_v whenever uv is an edge of G. It is easy to see that the result of such an operation does not change the independence number, and moreover, if the original graph is well covered, so is any expansion. We induct on $\beta = \beta(G)$, and show that any well covered graph G has an expansion whose independence polynomial has all simple, real roots.

If $\beta = 1$ then the result follows trivially, as the unique root of $i(K_n, x)$ is real, so we may assume that $\beta \geq 2$ (and hence $n \geq 2$).

Let v be a vertex in a maximal independent set and set $K = G - N[v]$; note that K is a well covered graph with independence number $\beta - 1$. By induction, K has an expansion L whose independence polynomial has all its roots simple and real, say $a_d < a_{d-1} < \cdots < a_1 < 0$ ($d = \beta(L) = \beta(K) = \beta(G) - 1$). Choose a set of $d + 1$ real numbers $b_d, b_{d-1}, \ldots, b_0 < 0$ such that $b_d < a_d < b_{d-1} < a_{d-1} < \cdots < b_1 < a_1 < b_0$ (that is, the b_i's interlace the a_i's). Now as the roots are simple and $i(L, x)$ has positive constant term, $i(L, b_i)$ has sign $(-1)^i$. Let H_r be the graph formed from $G - v$ by replacing $G - N[v]$ by L, $N(v)$ is untouched and v is replaced by K_r (with vertex set S). Now, by applying Proposition 2.2 r times we obtain

$$i(H_r, x) = r \cdot x i(L, x) + i(H_r - S, x).$$

Consequently, we can choose r large enough (say $r = R$) so that

$$\text{sign}(i(H_R, b_i)) = -\text{sign}(i(L, b_i)) = (-1)^{i+1}$$

and so $i(H_R, x)$ has a root in each interval (b_{i+1}, b_i). In addition, since $\text{sign}(i(H_R, b_0)) = -1$ and $i(H_R, 0) > 0$, there is another root in $(b_0, 0)$. That is, we have found $d + 1 = \beta(H_R)$ many distinct real roots of $i(H_R, x)$, a polynomial of degree $d + 1$, and we are done. □

We mentioned earlier a result of Newton's that stated that if a polynomial with positive coefficients has all its roots real, then its coefficients form a unimodal sequence.

Corollary 2.4 *For any well covered graph G, there is a well covered graph H with $\beta(H) = \beta(G)$ and G being an induced subgraph of H such that the independence vector $(i_0(H), i_1(H), \ldots, i_\beta(H))$ of H is unimodal.* □

Now we have seen that the real roots of the independence polynomials of well covered graphs can be arbitrarily close to 0, so the question remains how large can they be in absolute value? For general graphs with independence number β, there can be real roots of fairly large absolute value. For example, consider the complete $(l - \beta + 1)$-partite graph $G_\beta = K_{\beta,\beta-1,\beta-1,\ldots,\beta-1}$ (for $\beta \geq 2$). It is not hard to see that the independence polynomial of G_β is given by

$$i(G_\beta, x) = x^\beta + lx^{\beta-1} + S \tag{1}$$

where

$$S = \left(\sum_{i=1}^{\beta-2} \left[\binom{\beta}{i} + (l-\beta)\binom{\beta-1}{i} \right] x^i \right) + 1. \tag{2}$$

Let's fix $\varepsilon \in (0, 1)$. Provided l is large enough, one can see from (1) and (2) that

$$\operatorname{sign} i(G, -l) = (-1)^{\beta-2}$$

and

$$\operatorname{sign} i(G, -\varepsilon l) = (-1)^{\beta-1}.$$

Thus $i(G_\beta)$ has a root in $(-l, -\varepsilon l)$. Now let n be the order of G_β, as usual. Then $n = (l-\beta)(\beta-1)+\beta$, so that $l \sim \frac{n}{\beta-1}$ (for fixed $\beta \geq 2$). It follows that there are graphs with independence number β with real roots close to $-\frac{n}{\beta-1}$.

We shall show in the next section that the situation is quite different for well covered graphs, in that all the real roots must be greater than $-\beta$.

3. Location of the roots in the complex plane

It is natural to ask for regions of the complex plane that contain all the roots of families of polynomials (similar questions have been investigated for matching polynomial [13], chromatic polynomials [29] and network reliability [5], to name but a couple). We have seen that the independence polynomials of arbitrary graphs can have a real root $\sim -\frac{n}{\beta-1}$. In this section we provide a much tighter bound on the modulus of all the roots of independence polynomials of well covered graphs.

We begin with a lemma relating independence numbers of well covered graphs.

Lemma 3.1 *For a well covered graph G, and for $1 \leq k \leq \beta(G)$*

$$i_{k-1} \leq k i_k$$

(where i_j is the number of independent sets of size j in G).

Proof: Consider all independent sets I of size k in G. For any $x \in I$, the set $I - \{x\}$ is an independent set of size $k-1$ of G. Moreover, we count each independent set of size $k-1$ at least once in this way, as all maximal independent sets have the same size, namely β. It follows that $i_{k-1} \leq k i_k$. □

We now present our general bound on the roots of independence polynomials of well covered graphs.

Theorem 3.2 *For a well-covered graph G the roots of $i(G, x)$ lie in the annulus*

$$\frac{1}{n} \leq |z| \leq \beta(G).$$

Furthermore, there is a root on the boundary if and only if G is complete.

Proof: We shall utilize the well known *Eneström-Kakeya Theorem* (c.f. [4]), which states that if $f(x) = a_0 + a_1 x + \cdots + a_d x^d$ has positive coefficients then the roots of f lie in the annulus

$$r \leq |z| \leq R,$$

where

$$r = \left\{ \min \left\{ \left| \frac{a_i}{a_{i+1}} \right| : 0 \leq i \leq d-1 \right\} \right\},$$

and

$$R = \left\{ \max \left\{ \left| \frac{a_i}{a_{i+1}} \right| : 0 \leq i \leq d-1 \right\} \right\}.$$

We first calculate that for $i(G, x) = \sum i_k x^k$, we have from the previous lemma that $i_{k-1} \leq k i_k$, so that $\frac{i_{k-1}}{i_k} \leq k \leq \beta$. As k is arbitrary, we see that (with the notation above) $R \leq \beta$. On the other hand, $(n - (k-1)) i_{k-1} \geq i_k$, since any independent set of size k can be formed by taking some independent set of size $k-1$ and adding a vertex to it (there are at most $n - (k-1)$ choices for the latter). Thus $\frac{i_{k-1}}{i_k} \geq \frac{1}{n-(k-1)} \geq \frac{1}{n}$, so that $r \geq \frac{1}{n}$ (in fact, $r = \frac{1}{n}$, as one gets equality when $k = 1$). It follows immediately from the Eneström-Kakeya Theorem that all the roots lie in the annulus $\frac{1}{n} \leq |z| \leq \beta$.

All that remains to be shown is that no roots lie on the boundary of the annulus. Let's consider first the circle $|z| = \beta$. If $\beta = 1$, then $G = K_n$ and $i(G, x) = 1 + nx$, so $i(G, x)$ has a root on $|z| = 1$ if and only if $G = K_1$. Thus we can assume that $\beta \geq 2$. It was shown in [3] that (with the notation above) a polynomial $f(x) = a_0 + a_1 x + \cdots a_d x^d$ with positive coefficients has some of its roots on

$$|z| = R = \left\{ \max \left\{ \left| \frac{a_j}{a_{j+1}} \right| : 0 \leq j \leq d-1 \right\} \right\}$$

only if $\gcd(\{j = 1, 2, \ldots, d+1 : a_{d-j} < R a_{d+1-j}\}) > 1$, where $a_{-1} \equiv 0$. Now we have seen that $i_{k-1} \leq k i_k$, and as $k \leq \beta$, we see that equality can hold only if $k = \beta$. Thus $\{k = 1, 2, \ldots, d+1 : i_{\beta-k} < R i_{\beta+1-k}\} \supseteq \{2, 3, \ldots, \beta+1\}$, so that we have $\gcd(\{k = 1, 2, \ldots, d+1 : i_{\beta-k} < R i_{\beta+1-k}\}) = 1$, and we conclude that there is no root $|z| = \beta$.

Finally, let's consider the inside boundary, $|z| = 1/n$. For $\beta = 1$, the independence polynomial has a root at $-1/n$. For $\beta \geq 2$, again from [3], we see that a root can exist on

$|z| = 1/n$ only if $\gcd(\{k = 1, 2, \ldots, \beta + 1 : i_{k-1} > ri_k\}) > 1$, where $i_{\beta+1} \equiv 0$. However, we have seen that $(n - (k - 1))i_{k-1} \geq i_k$ and hence $ni_{k-1} > i_k$ unless $k = 1$ (in which case equality holds as $i_0 = 1$ and $i_1 = n$). Thus $\gcd(\{k = 1, 2, \ldots, \beta + 1 : i_{k-1} > ri_k\}) = \gcd(\{2, 3, \ldots, \beta + 1\}) = 1$, so there is no root on $|z| = 1/n$. □

A tantalizing question is whether the bound $|z| \leq \beta$ is best possible. The rest of this section will show that indeed one cannot improve upon the upper bound.

Fix $\beta \geq 1$ and let $[1, \beta]$ denote the set $\{1, \ldots, \beta\}$. Form the graph L^k_β on vertex set $[1, \beta]^k$ with two k-tuples forming an edge if and only if they agree in at least one coordinate. The graph L^k_β is a well covered graph with independence number β. Consider an independent set I of size j in L^k_β. If we project down each of its coordinates, we have a subset of $[1, \beta]$ of size j, and for any such choice, if we order the first coordinates as the beginning of our k-tuples, we can arrive at all independent sets with these projections by assigning each of the other j symbols in each of the coordinates to one of the k-tuples. It follows that the independence polynomial $i(L^k_\beta, x)$ is given by

$$i\left(L^k_\beta, x\right) = \sum_{j=0}^{\beta} \binom{\beta}{j}^k (j!)^{k-1} x^j, \tag{3}$$

for $k \geq 1$, $\beta \geq 1$.

First we show the following.

Proposition 3.3 *The zeros of $i(L^k_\beta, x)$ are all real and negative for any $k \geq 1$, $\beta \geq 1$.*

Proof: We rewrite (3) as

$$\begin{aligned} i\left(L^k_\beta, x\right) &= (\beta!)^{k-1} \sum_{j=0}^{\beta} \binom{\beta}{j} \frac{x^j}{[(\beta - j)!]^{k-1}} \\ &= (\beta!)^{k-1} \sum_{j=0}^{\beta} \binom{\beta}{\beta - j} \frac{x^{\beta - j}}{(j!)^{k-1}} \end{aligned}$$

or

$$i\left(L^k_\beta, x\right) = (\beta!)^{k-1} x^\beta \sum_{j=0}^{\beta} \binom{\beta}{j} \frac{(1/x)^j}{\Gamma(j+1)^{k-1}}. \tag{4}$$

According to a theorem of Hurwitz [22] the sum in this last expression has only real zeros as a polynomial in $z = 1/x$; hence $i(L^k_\beta, x)$ has only real zeros. It is clear that they can only be negative. □

With (4) in mind, we now consider the normalized polynomials

$$g^{(k)}_\beta(x) = (\beta!)^{1-k} \left(\frac{x}{\beta}\right)^\beta i\left(L^k_\beta, \frac{\beta}{x}\right), \tag{5}$$

that is,

$$g_\beta^{(k)}(x) = \sum_{j=0}^{\beta} \frac{\beta(\beta-1)\cdots(\beta-j+1)}{\beta^j} \frac{x^j}{(j!)^k}. \tag{6}$$

Lemma 3.4 *Let $2^{k-1} \geq \beta \geq 1$. Then the largest zero $x_\beta^{(k)}$ of $g_\beta^{(k)}(x)$ lies in the interval*

$$-1 - 2^{-k} < x_\beta^{(k)} < -1. \tag{7}$$

Proof: From Theorem 3.2 and Proposition 3.3 it follows that all zeroes of $g_\beta^{(k)}(x)$ are real and less than -1. Fix β and k, and let a_j be the absolute value of the coefficient of x^j in

$$g_\beta^{(k)}(-x) = 1 - x + \left(1 - \frac{1}{\beta}\right)\frac{x^2}{(2!)^k} - \left(1 - \frac{1}{\beta}\right)\left(1 - \frac{2}{\beta}\right)\frac{x^3}{(3!)^k} + \cdots \tag{8}$$

$$+ (-1)^\beta \frac{\beta!}{\beta^\beta} \frac{x^\beta}{(\beta!)^k}. \tag{9}$$

It is easy to see that for $0 \leq j \leq \beta - 1$ and $x > 0$ we have

$$a_j x^j > a_{j+1} x^{j+1}$$

if and only if

$$x < \frac{\beta}{\beta - j}(j+1)^k,$$

or, for $j \geq 1$, when

$$x < \frac{\beta}{\beta - 1} 2^k.$$

Hence, by a basic fact on alternating sums we have

$$g_\beta^{(k)}(-x) > 1 - x \geq 0$$

when $0 < x \leq 1$, while

$$\begin{aligned} g_\beta^{(k)}(-1-2^{-k}) &< 1 - (1 + 2^{-k}) + \left(1 - \frac{1}{\beta}\right) 2^{-k}(1 + 2^{-k})^2 \\ &= 2^{-k}\left(-\frac{1}{\beta} + 2^{-k+1} + 2^{-k+1}\left(-\frac{1}{\beta} + 2^{-k-1}\right) - \frac{1}{\beta} 2^{-2k}\right) \\ &< 0 \end{aligned}$$

for $\beta \leq 2^{k-1}$, and this proves the lemma. □

The following result is now an immediate consequence of the previous lemma, together with (5).

Theorem 3.5 *For $\beta, k \geq 2$ let L_β^k denote the well covered graph on $[1, \beta]^k$ in which two k-tuples are joined by an edge if and only if they agree in a coordinate (L_β^k has independence number β). If $2^{k-1} \geq \beta \geq 1$ then the smallest zero $y_\beta^{(k)}$ of $i(L_\beta^k, x)$ lies in the interval*

$$-\beta < y_\beta^{(k)} < -\beta(1 - 2^{-k}). \tag{10}$$

Proof: The Lemma and (5) show that $y_\beta^{(k)}$ lies in the interval

$$-\beta < y_\beta^{(k)} < \frac{-\beta}{1 + 2^{-k}}. \tag{11}$$

The second inequality in (10) now follows from the inequality $1/(1 + 2^{-k}) > 1 - 2^{-k}$. □

It follows that for any $\varepsilon > 0$ (by choosing k large enough) there are well covered graphs (for each $\beta \geq 2$) with a real root in $(-\beta, -\beta + \varepsilon)$, so indeed the bound of $|z| < \beta$ is (in terms of a constant bound) best possible.

We conclude with a few remarks:

- A qualitative version of Lemma 3.4 and of Theorem 3.5 follows easily from (6) (or (9)). It is clear that

$$g_n^{(k)}(z) \to 1 + z \tag{12}$$

as $k \to \infty$, uniformly on any compact subset of $\mathbb{C}$. Hence by another (better known) theorem of Hurwitz (see, e.g., [24, p. 3]), one zero of $g_n^{(k)}(z)$ converges to the zero $z = -1$, as $k \to \infty$.
- The $i(L_\beta^k, x)$, for $k = 1, 2, 3$, are related to well-known classes of polynomials. It is obvious from (3) that

$$i\left(L_\beta^1, x\right) = (1 + x)^\beta. \tag{13}$$

Next, from (4) we see that

$$i\left(L_\beta^2, x\right) = \beta! x^\beta L_\beta\left(\frac{-1}{x}\right), \tag{14}$$

where $L_k(x)$ is the kth Legendre polynomial, one of the "classical orthogonal polynomials", which has the explicit expansion

$$L_k(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{x^j}{j!} \tag{15}$$

(see, e.g., [1, p. 775]). Finally, it follows from (4) and from identity (10.37.4) in [20, p. 199] that

$$i\left(L_\beta^3, x\right) = (\beta!)^2 x^\beta J_\beta^{[0,-1/2]}\left(\frac{i}{\sqrt{x}}\right), \tag{16}$$

where $J_n^{[\alpha,\rho]}(x)$ are the "Bateman polynomials". Through the connection with these special functions one could easily deduce differential equations, recurrence relations and other properties satisfied by the appropriate polynomials $i(L_\beta^k, x)$.

- More generally, the $i(L_\beta^k, x)$, for arbitrary positive integers k and β, can be written in terms of hypergeometric functions. Using this machinery, one could derive further properties, such as ODEs satisfied by the polynomials $i(L_\beta^k, x)$.
- As the polynomial $i(L_\beta^k, x)$ has all real roots, its coefficients are unimodal. In fact, we can show that if θ is the unique root of $f(x) = (\beta - x)^k - x - 1$ on $[0, \beta - 1]$, then a peak occurs at $\lceil\theta\rceil$, which tends to $\beta - 1$ as $k \to \infty$.

4. Concluding remarks

We point out that while the disc $|z| \leq \beta$ holds the roots of all well covered graphs with independence number β, the expansion operation can push the roots into the unit disc.

Theorem 4.1 *Every graph G is an induced subgraph of a graph H whose roots lie in $|z| \leq 1$.*

Proof: This follows by expanding G to H by replacing every vertex of G by the same, suitably large, complete graph K_r. Then $i(H, x) = i(G, rx) = \sum_{k=0}^{d} r^k i_k x^d$. If r is large enough then the coefficients of $i(H, x)$ can be made to be increasing, and by the Eneström-Kakeya Theorem, all the roots will lie in $|z| \leq 1$. □

Now as independence polynomials have positive coefficients, of course all real roots are negative. We do not know of a single well covered graph whose independence polynomial has a root with a positive real part. In fact, we pose the following.

Conjecture 4.2 For any well covered graph G with independence number β, all the roots of $i(G, \beta)$ lie in the disc $|z + \frac{\beta}{2}| < \frac{\beta}{2}$.

This conjecture is clearly true if all the roots are real, as the roots, as noted above, are negative and by Theorem 3.2, they are greater than $-\beta$. Thus the conjecture holds for $\beta = 1$ and 2. We finish by showing that indeed it holds for $\beta = 3$.

Theorem 4.3 *If G is a well covered graph with $\beta = 3$, then all of the roots of $i(G, x)$ lie in the disc $|z + 3/2| < 3/2$.*

Proof: We shall make use of the *Schur-Cohn Criterion* (c.f. [4, p. 181]) that the monic polynomial $x^3 + bx^2 + cx + d$ has all its roots within the unit disc if and only if

$$|bd - c| < 1 - d^2 \text{ and } |b + d| < |1 + c|.$$

Let the independence polynomial of G be

$$f(x) = tx^3 + mx^2 + nx + 1.$$

Here, n, m and t denote respectively the number of vertices, edges and triangles in $\bar{G}$ (which is a K_4-free graph, as G has independence number 3); note that $t \geq 1$ as $\beta = 3$. If we set $g(x) = \frac{1}{t} f((3/2)(x-1))$, then f has all its roots in the disc of radius $3/2$ centered at $z = -3/2$ if and only if g has all its roots in the unit disc. Now a simple calculation will show that

$$g(x) = \frac{27}{8}x^3 + \left(\frac{9m}{4t} - \frac{81}{8}\right)x^2 + \left(-\frac{9m}{2t} + \frac{81}{8} + \frac{3n}{2t}\right) + \frac{1}{t} + \frac{9m}{4t} - \frac{3n}{2t} - \frac{27}{8}$$

so that

$$\begin{aligned} k(x) &= \frac{8}{27}g(x) \\ &= x^3 + \frac{8}{27}\left(\frac{9m}{4t} - \frac{81}{8}\right)x^2 + + \frac{8}{27}\left(-\frac{9m}{2t} + \frac{81}{8} + \frac{3n}{2t}\right) + \frac{8}{27t} \\ &\quad + \frac{2m}{3t} - \frac{4n}{9t} - 1. \end{aligned}$$

Hence it remains to show that k has all it roots in the unit disc. We set b, c and d to be respectively the coefficients of x^2, x and 1 in $k(x)$. Then

$$bd - c = \frac{16}{81}\frac{m}{t^2} + \frac{4}{9}\frac{m^2}{t^2} - \frac{8}{27}\frac{mn}{t^2} - \frac{4}{3}\frac{m}{t} - \frac{8}{9}\frac{1}{t} + \frac{8}{9}\frac{n}{t} \tag{17}$$

and

$$1 - d^2 = -\frac{64}{729}\frac{1}{t^2} - \frac{32}{81}\frac{m}{t^2} + \frac{64}{243}\frac{n}{t^2} + \frac{16}{27}\frac{1}{t} - \frac{4}{9}\frac{m^2}{t^2} + \tag{18}$$

$$\frac{16}{27}\frac{mn}{t^2} + \frac{4}{3}\frac{m}{t} - \frac{16}{81}\frac{n^2}{t^2} - \frac{8}{9}\frac{n}{t}. \tag{19}$$

To simplify these, we multiply through by the positive constant $\frac{729}{4}t^2$, and name them k_1 and k_2 respectively:

$$\begin{aligned} k_1 &= \frac{729}{4}t^2(bd - c) \\ &= 36m + 81m^2 - 54mn - 243mt - 162t + 162nt \\ &= 36(m - 3t) + (81m - 54n)(m - 3t) - 54t \end{aligned}$$

and

$$\begin{aligned} k_2 &= \frac{729}{4} t^2 \left(1 - d^2\right) \\ &= -16 - 72m + 48n + 108t - 81m^2 + 108mn + 243mt - 36n^2 - 162nt. \end{aligned}$$

Now for a well covered graph G with $\beta = 3$, $m \leq 3t$ (as in $\bar{G}$, every edge is in some triangle), and $m \geq n$ (as in $\bar{G}$, every vertex is in some triangle, and hence has degree at least 2 in $\bar{G}$). Hence we see that $k_1 < 0$. Thus the first condition of the Schur-Cohn Criterion, $|bd - c| < 1 - d^2$, is equivalent to $k_1 + k_2 > 0$. Now

$$\begin{aligned} k_1 + k_2 &= -36m + 54mn - 54t - 16 + 48n - 36n^2 \\ &\geq -36m + 18mn - 54t - 16 + 48n \end{aligned} \tag{20}$$

as $m \geq n$.

To show that $k_1 + k_2 > 0$, we need to show that the inequality $mn \geq 3t + 2m$ holds. Consider counting ordered pairs (e_i, v_j), where e_i and v_j are respectively an edge and vertex of $\bar{G}$ such that the subgraph induced by v_j and e_i is a triangle in $\bar{G}$. In $\bar{G}$, there are at most $n - 2$ choices of vertices for each choice of edge, and each triangle is counted $3t$ times. Hence $m(n - 2) \geq 3t$, which is equivalent to $mn \geq 3t + 2m$, and in particular

$$18mn \geq 54t + 36m.$$

From this and (20) we see that

$$k_1 + k_2 \geq 48n - 16 > 0$$

(as $n \geq 3$). This shows that the first condition of the Schur-Cohn Criterion holds.

For the second condition, $|b + d| < |1 + c|$, a calculation will show that

$$l_1 = \frac{27}{4} t\,(b + d) = 9m - 27t - 3n + 2$$

and

$$l_2 = \frac{27}{4} t\,(1 + c) = 27t - 9m + 3n.$$

From $m \leq 3t$, we see that $l_1 < 0$ (and hence $b + d < 0$) and $l_2 > 0$ (and hence $1 + c > 0$). Thus we need to show that $-l_1 < l_2$, that is $l_1 + l_2 > 0$. However,

$$l_1 + l_2 = 2 > 0$$

so it follows that the second condition of the Schur-Cohn Criterion is also satisfied. We conclude that all the roots of $g(x)$ lie in the unit disc, and hence the independence polynomial of G lies in the disc of radius $3/2$ centered at $-3/2$. □

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