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On Distance-Regular Graphs with Height Two, II

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Abstract. Let Γ be a distance-regular graph with diameter $d \geq 3$ and height $h = 2$, where $h = \max\{i : p^d_{d,i} \neq 0\}$. Suppose that for every α in Γ and every β in $\Gamma_d(\alpha)$, the induced subgraph on $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is isomorphic to a complete multipartite graph $K_{t\times 2}$ with $t \geq 2$. Then $d = 4$ and Γ is isomorphic to the Johnson graph $J(10, 4)$.

Keywords: distance-regular graph, height, Johnson graph, complete multipartite graph

1. Introduction

Let Γ be a connected finite undirected graph without loops or multiple edges. We identify Γ with the set of vertices. For vertices u and v, let $\partial(u, v)$ denote the distance between u and v, i.e., the length of a shortest path connecting u and v in Γ. Let $d = d(\Gamma)$ denote the *diameter* of Γ, i.e., the maximal distance of two vertices in Γ. We set

$$\Gamma_i(u) = \{v \in \Gamma : \partial(u, v) = i\} \quad (0 \leq i \leq d).$$

Γ is said to be *distance-regular* if the cardinality of the set $\Gamma_i(x) \cap \Gamma_j(y)$ depends only on i, j and the distance between x and y. In this case we write

$$p^l_{i,j} = p^l_{i,j}(\Gamma) = |\Gamma_i(x) \cap \Gamma_j(y)| \quad (0 \leq i, j, l \leq d),$$

where $l = \partial(x, y)$. Let

$$k_i = k_i(\Gamma) = p^0_{i,i} = |\Gamma_i(u)| \quad (0 \leq i \leq d).$$

In particular $k = k_1$ is the *valency* of Γ. Let

$$c_i = c_i(\Gamma) = p^i_{1,i-1},\ a_i = a_i(\Gamma) = p^i_{1,i},\ b_i = b_i(\Gamma) = p^i_{1,i+1} \quad (0 \leq i \leq d).$$

They are called the *intersection numbers* of Γ, and

$$\iota(\Gamma) = \left\{ \begin{array}{cccccccc} * & c_1 & c_2 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_i & \cdots & b_{d-1} & * \end{array} \right\}$$

is called the *intersection array* of Γ.

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The following are basic properties of intersection numbers, which we use implicitly in this paper.
(1) $c_i + a_i + b_i = k \quad (0 \le i \le d)$,
(2) $1 = c_1 \le c_2 \le c_3 \le \cdots \le c_{d-1} \le c_d \le k$,
(3) $k = b_0 > b_1 \ge b_2 \ge \cdots \ge b_{d-2} \ge b_{d-1} \ge 1$,
(4) $p^l_{i,j} = p^l_{j,i} \quad (0 \le i, j, l \le d)$,
(5) $p^l_{i,j} = 0$ if $l > i + j$ or $l < |i - j|$,
(6) $p^l_{i,j} \ne 0$ if $l = i + j$ or $l = |i - j|$,
(7) $p^l_{i,0} + p^l_{i,1} + \cdots + p^l_{i,d} = k_i \quad (0 \le i, l \le d)$,
(8) $k_l p^l_{i,j} = k_i p^i_{l,j} = k_j p^j_{l,i} \quad (0 \le i, j, l \le d)$,
(9) $k_i b_i = k_{i+1} c_{i+1} \quad (0 \le i \le d - 1)$,
(10) $c_i \le b_j$ if $i + j \le d$.

A graph is said to be *strongly regular* if it is distance-regular with diameter 2.

A graph is called a *clique* when any two of its vertices are adjacent. A *coclique* is a graph in which no two vertices are adjacent.

Information about the general theory of distance-regular graphs is given in [1], [2] and [3].

For some positive integers n and e with $n \ge 2e$, let X be a finite set of cardinality n and $V = \{T \subset X : |T| = e\}$. The *Johnson graph* $J(n, e)$ is a graph whose vertex set is V and two vertices x and y are adjacent if and only if $|x \cap y| = e - 1$. It is well known that $J(n, e)$ is a distance-regular graph with diameter e.

The *complete multipartite graph* $K_{m_1,m_2,\cdots,m_t}$ is a graph whose vertex set is partitioned into t parts $M_1, M_2, \cdots, M_t$, where $|M_i| = m_i$ $(1 \le i \le t)$, and two vertices are adjacent if and only if they belong to different parts. We write $K_{t\times m}$ if $m_1 = m_2 = \cdots = m_t = m$.

In this paper we identify a subset A of Γ with the induced subgraph on A and define the following terminology.

A subgraph A of Γ is called *μ-closed* if for every pair of vertices x and y in A with $\partial(x, y) = 2$ in Γ, $\Gamma_1(x) \cap \Gamma_1(y) \subseteq A$, and *$\lambda$-closed* if for all adjacent vertices x and y in A, $\Gamma_1(x) \cap \Gamma_1(y) \subseteq A$.

Let $h = \max\{i : p^d_{d,i} \ne 0\}$ be the *height* of Γ. By definition, it is easy to see that $h \le d$. Most known distance-regular graphs satisfy $h = d$. The Johnson graphs $J(n, d)$ $(n < 3d)$ are examples which satisfy $h < d$. A distance-regular graph Γ is of height 0 if and only if Γ is an antipodal 2-cover, and is of height 1 if and only if $\Gamma_d(\alpha)$ is a nontrivial clique for every α in Γ. So if the height of Γ is 1, Γ is the distance-2 graph of a generalized odd graph. (See Proposition 4.2.10 of [3] and Theorem III.4.2 of [1].) The next question is what kind of distance-regular graphs are of height 2. This question is not easy in general, but it is very interesting, because there are several intersection arrays for which no graphs are known. (See Chapter 14 of [3].) Also, K. Nomura conjectured that there is no bipartite distance-regular graph with diameter $d \ge 4$ and height $h = 2$. (Conjecture 1.2 of [5].)

In [13] the author showed the following partial result on distance-regular graphs of height 2.

Theorem 1.1 *Let Γ be a distance-regular graph with diameter $d \geq 3$ and height $h = 2$. Suppose that for every α in Γ and every β in $\Gamma_d(\alpha)$, $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is isomorphic to a clique. Then $d = 3$ and Γ is isomorphic to the Johnson graph $J(8, 3)$.*

In this theorem, we give a condition that $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is isomorphic to a clique for every $\alpha, \beta \in \Gamma$ with $\partial(\alpha, \beta) = d$. A clique is one of the natural graphs, because each vertex is adjacent to all the other vertices. Instead of a clique, we consider a graph in which each vertex is adjacent to all the other vertices but except one. That is a complete multipartite graph $K_{t\times 2}$. In this paper, we show the following theorem.

Theorem 1.2 *Let Γ be a distance-regular graph with diameter $d \geq 3$ and height $h = 2$. Suppose that for every α in Γ and every β in $\Gamma_d(\alpha)$, $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is isomorphic to a complete multipartite graph $K_{t\times 2}$ with $t \geq 2$. Then $d = 4$ and Γ is isomorphic to the Johnson graph $J(10, 4)$.*

Let Γ be a distance-regular graph with height 2 and suppose $\Gamma_d(\alpha)$ is connected for every $\alpha \in \Gamma$. The diameter of $\Gamma_d(\alpha)$ may be greater than 2. We consider the case the diameter of $\Gamma_d(\alpha)$ is 2. For example, if Γ is isomorphic to the Hamming graphs $H(2, q)$ $(q \geq 3)$, the Johnson graphs $J(n, 2)$ $(n \geq 6)$ or $J(2d + 2, d)$ $(d \geq 2)$, then $\Gamma_d(\alpha)$ becomes strongly regular. The known examples in which the diameter $d \geq 3$ and the diameter of $\Gamma_d(\alpha)$ is 2 are $J(2d + 2, d)$. For a given graph Δ whose diameter is 2, is it possible to classify the distance-regular graphs Γ whose antipodal structures $\Gamma_d(\alpha)$ are Δ? It is known that there are finitely many distance-regular graphs in which $\Gamma_d(\alpha) \simeq \Delta$ for every $\alpha \in \Gamma$. (See [6] and [7].)

Let Δ be a graph with diameter 2. Suppose $\Gamma_d(\alpha) \simeq \Delta$ for every $\alpha \in \Gamma$. Then the height of Γ becomes 2. In this situation, it is easy to see that Δ is distance-degree regular, i.e., $|\Delta_1(\beta)| = p^d_{d,1}$, $|\Delta_2(\beta)| = p^d_{d,2}$ do not depend on the choice of β in Δ. So we have the following corollary of Theorem 1.2.

Corollary 1.3 *Let Γ be a distance-regular graph with diameter $d \geq 3$, and Δ a distance-degree regular graph with diameter 2 such that $\Delta_2(\beta)$ is isomorphic to $K_{t\times 2}$ with $t \geq 2$ for every β in Δ. Suppose $\Gamma_d(\alpha)$ is isomorphic to Δ for every α in Γ. Then $d = 4$, Γ is isomorphic to $J(10, 4)$ and Δ is isomorphic to $J(6, 2)$.*

We note that there are many distance-degree regular graphs Δ such that $d(\Delta) = 2$ and $\Delta_2(\beta) \simeq K_{t\times 2}$ for every $\beta \in \Delta$. The complements of strongly regular graphs with $a_1 = 1$ are in this class. It is not hard to construct graphs in this class which are not strongly regular. For example, let a graph Λ be in this class, then we can construct a new graph Δ in this class from Λ. The construction can be done as follows. Take a positive integer s and consider the s copies of each vertex in Λ. Let $K^u = \{u_1, u_2, \cdots, u_s\}$ $(u \in \Lambda)$. Δ is a graph whose vertex set is $\cup_{u\in\Lambda} K^u$ and two distinct vertices $u_i \in K^u$ and $v_j \in K^v$ are adjacent if and only if $u = v$, or u and v are adjacent in Λ, or u and v are not adjacent in Λ and $i \neq j$. (In general, let Λ be a graph with diameter 2 and Δ the graph constructed from Λ as above. Then $\Lambda_2(\alpha) \simeq \Delta_2(\alpha_i)$ for every $\alpha \in \Lambda$ and $\alpha_i \in \Delta$.)

Though there are infinitly many graphs in this class, only $J(6,2)$ can be the antipodal structrue of a distance-regular graph, and only $J(10,4)$ has the antipodal structure in this class.

2. Preliminaries

We shall introduce the intersection diagrams of rank l which we use as our main tool.

Let u, $v \in \Gamma$ with $\partial(u,v) = l$. Set

$$D^i_j = D^i_j(u,v) = \Gamma_i(u) \cap \Gamma_j(v) \quad (0 \le i, j \le d).$$

It is easy to see the following.
(1) $D^i_j = \phi$ if $l > i + j$ or $l < |i - j|$,
(2) $D^i_{l-i} \ne \phi$ if $0 \le i \le l$,
$D^i_{l+i} \ne \phi$ if $0 \le i \le d - l$,
(3) There is no edge between D^i_j and D^f_g if $|i - f| > 1$ or $|j - g| > 1$.

An intersection diagram of rank l with respect to (u,v) is the collection $\{D^i_j\}_{i,j}$ with lines between D^i_j's and D^f_g's. We draw a line

$$D^i_j \text{———} D^f_g$$

if there is the possibility of existence of edges between D^i_j and D^f_g, and we erase the line when we know there is no edge between D^i_j and D^f_g. We also erase D^i_j when we know $D^i_j = \phi$. We say rank l diagram instead of intersection diagram of rank l.

For subsets A and B of Γ, let $e(A,B)$ denote the number of edges between A and B, and $\partial(A,B) = min\{\partial(x,y) : x \in A, y \in B\}$. Let $e(\gamma, B) = e(\{\gamma\}, B)$. We write $\alpha \sim \beta$, when $\beta \in \Gamma_1(\alpha)$, and $\alpha \not\sim \beta$, otherwise.

The following are straightforward and useful for determining the form of the intersection diagrams.

For each $\gamma \in D^i_j$, we have the following.
(4) $c_i = e(\gamma, D^{i-1}_{j+1}) + e(\gamma, D^{i-1}_j) + e(\gamma, D^{i-1}_{j-1})$,
$c_j = e(\gamma, D^{i+1}_{j-1}) + e(\gamma, D^i_{j-1}) + e(\gamma, D^{i-1}_{j-1})$,
(5) $a_i = e(\gamma, D^i_{j+1}) + e(\gamma, D^i_j) + e(\gamma, D^i_{j-1})$,
$a_j = e(\gamma, D^{i+1}_j) + e(\gamma, D^i_j) + e(\gamma, D^{i-1}_j)$,
(6) $b_i = e(\gamma, D^{i+1}_{j+1}) + e(\gamma, D^{i+1}_j) + e(\gamma, D^{i+1}_{j-1})$,
$b_j = e(\gamma, D^{i+1}_{j+1}) + e(\gamma, D^i_{j+1}) + e(\gamma, D^{i-1}_{j+1})$.

For the properties and applications of intersection diagrams, see for example [5], [6], [8], [9] and [13].

Let α, $\beta \in \Gamma$ with $\partial(\alpha, \beta) = d$. We mainly use intersection diagrams of rank d with respect to (α, β). So in this paper rank d diagram means rank d diagram with respect to (α, β).

Let Γ be a distance-regular graph with height $h = 2$. We determine the shapes of some diagrams of Γ and prove some lemmas.

First we consider the rank d diagram. Suppose there is a vertex $x \in D^i_j$, for some i, j with $i + j \geq d + 3$. Then we can take $y \in \Gamma_d(\alpha) \cap \Gamma_{d-i}(x)$. Since $\beta, y \in \Gamma_d(\alpha)$ and $h = 2$, $\partial(\beta, y) \leq 2$. On the other hand, $\partial(\beta, y) \geq \partial(\beta, x) - \partial(x, y) \geq 3$, which is impossible. So we get $D^i_j = \phi$ for i, j with $i + j \geq d + 3$. Therefore the rank d diagram becomes as in Figure 1.

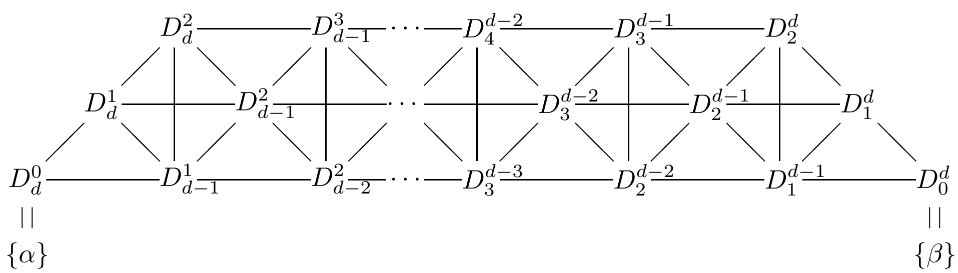

Figure 1.

Take any $\gamma \in D^2_d$, then $\Gamma_j(\alpha) \cap \Gamma_{j-2}(\gamma) \subseteq D^j_{d-j+2}$ for $2 \leq j \leq d$. As $p^2_{j,j-2} \neq 0$, we get

$$D^j_{d-j+2} \neq \phi \text{ and } p^d_{j,d-j+2} \neq 0 \text{ for } 2 \leq j \leq d.$$

Since $k_j p^j_{d,d-j+2} = k_d p^d_{j,d-j+2} \neq 0$, we have

$$p^j_{d,d-j+2} \neq 0 \text{ for } 2 \leq j \leq d.$$

Next take any $u, v \in \Gamma$ with $\partial(u, v) = i$ for $2 \leq i \leq d$, and consider the rank i diagram with respect to (u, v). Suppose there is $x \in D^f_g$ for some $f + g \geq 2d - i + 3$. Take $y \in \Gamma_d(u) \cap \Gamma_{d-f}(x)$. Then $y \in D^d_j$, where $j = \partial(v, y) \geq \partial(v, x) - \partial(x, y) \geq d - i + 3$. We take $w \in D^d_{d-i}$, since $p^i_{d,d-i} \neq 0$. So $\partial(y, w) \geq 3$. This contradicts $y, w \in \Gamma_d(u)$ and $h = 2$. So we have $D^f_g = \phi$ for $f + g \geq 2d - i + 3$. Hence the rank i diagram with respect to (u, v) becomes as in Figure 2. As $p^i_{d-i+2,d} \neq 0$, take $z \in D^{d-i+2}_d$. Then

$$\Gamma_j(z) \cap \Gamma_{d-i+2+j}(u) \subseteq D^{d-i+2+j}_{d-j} \text{ for } 0 \leq j \leq i - 2.$$

Since $p^{d-i+2}_{j,d-i+2+j} \neq 0$, we have

$$D^{d-i+2+j}_{d-j} \neq \phi \text{ and } p^i_{d-i+2+j,d-j} \neq 0 \text{ for } 0 \leq j \leq i - 2.$$

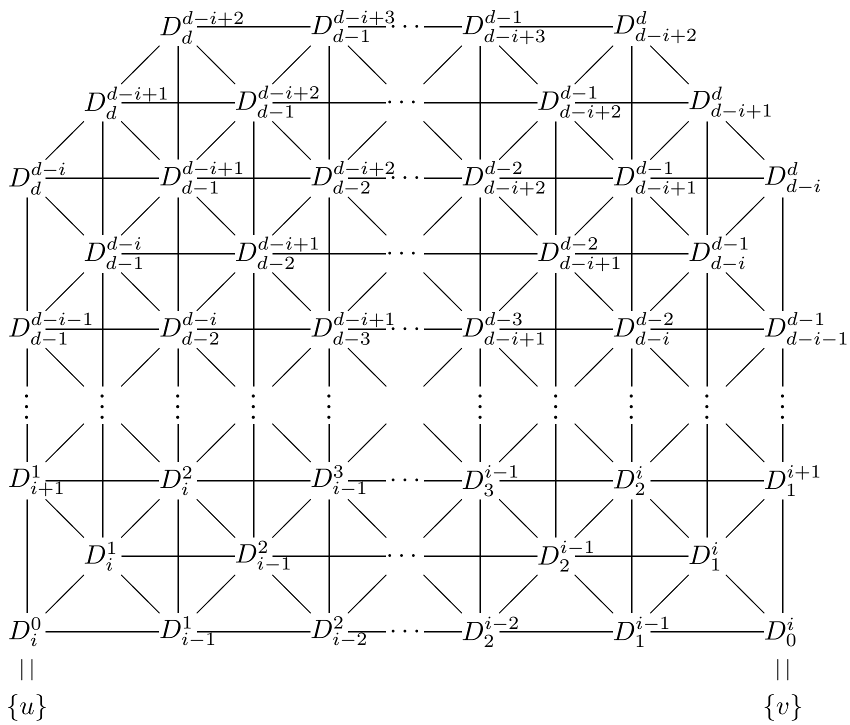

Figure 2.

Lemma 2.1 *The following hold.*
(1) *For* $0 \le i \le d-1$*, we have* $p^{i+1}_{d,d-i-1} \le p^{i}_{d,d-i}$*, and equality holds if and only if* $b_i = c_{d-i}$.
(2) *For* $2 \le i \le d$*, we have* $p^{d-i+2}_{d,i-2} \le p^{i}_{d,d-i+2}$.
(3) *For* $2 \le i \le d-1$*, we have* $p^{i}_{d,d-i+2} \le p^{i+1}_{d,d-i+1}$ *and* $b_{d-i+1} \le c_{i+1}$*. Moreover* $p^{i}_{d,d-i+2} = p^{i+1}_{d,d-i+1}$ *if and only if* $b_{d-i+1} = c_{i+1}$ *if and only if* $e(D^{d-i+1}_{i+1}, D^{d-i+1}_{i}) + e(D^{d-i+1}_{i+1}, D^{d-i}_{i}) = 0$ *in the rank* d *diagram.*
(4) *For* $0 \le j \le i-2 \le d-2$*, take* $u, v \in \Gamma$ *with* $\partial(u,v) = i$*. Then there is* $z \in \Gamma_{d-i+2+j}(u)$ *such that* $\Gamma_d(u) \cap \Gamma_{d-i}(v) \subseteq \Gamma_d(u) \cap \Gamma_{i-j}(z)$.

Proof: (1) By Lemma 4.1.7 of [3]

$$p^{i}_{d,d-i} = \frac{b_i b_{i+1} b_{i+2} \cdots b_{d-1}}{c_1 c_2 \cdots c_{d-i-1} c_{d-i}} = \frac{b_i}{c_{d-i}} p^{i+1}_{d,d-i-1}.$$

Since $b_i \geq c_{d-i}$, we get (1).

Let $u, v \in \Gamma$ such that $\partial(u,v) = i$, and consider the rank i diagram with respect to (u,v).
(2) For any $x \in D_d^{d-i+2}$,

$$\Gamma_d(u) \cap \Gamma_{i-2}(x) \subseteq D_{d-i+2}^d.$$

Since $\partial(u,x) = d-i+2$, we get $p_{d,i-2}^{d-i+2} \leq p_{d,d-i+2}^i$.
(3) Take $y \in D_1^{i+1}$. For each $z \in D_{d-i+2}^d$, $\partial(z,y) \geq \partial(v,z) - \partial(v,y) = d-i+1$. Since $p_{d,j}^{i+1} = 0$ if $j \geq d-i+2$, we have $\partial(z,y) = d-i+1$. So

$$D_{d-i+2}^d \subseteq \Gamma_d(u) \cap \Gamma_{d-i+1}(y),$$

and we have $p_{d,d-i+2}^i \leq p_{d,d-i+1}^{i+1}$.

Next we use the rank d diagram. For any $w \in D_{i+1}^{d-i+1}$, we get

$$\begin{aligned} b_{d-i+1} &= e(w, D_i^{d-i+2}), \\ c_{i+1} &= e(w, D_i^{d-i+2}) + e(w, D_i^{d-i+1}) + e(w, D_i^{d-i}). \end{aligned}$$

So we have $b_{d-i+1} \leq c_{i+1}$, and equality holds if and only if
$e(D_{i+1}^{d-i+1}, D_i^{d-i+1}) + e(D_{i+1}^{d-i+1}, D_i^{d-i}) = 0$

Since

$$\begin{aligned} b_i k_i p_{d,d-i+2}^i &= b_i k_d p_{d-i+2,i}^d, \\ b_i k_i p_{d,d-i+1}^{i+1} = c_{i+1} k_{i+1} p_{d,d-i+1}^{i+1} &= c_{i+1} k_d p_{d-i+1,i+1}^d, \end{aligned}$$

$p_{d,d-i+2}^i = p_{d,d-i+1}^{i+1}$ if and only if $b_i p_{d-i+2,i}^d = c_{i+1} p_{d-i+1,i+1}^d$.

In the rank d diagram,

$$\begin{aligned} e(D_{i+1}^{d-i+1}, D_i^{d-i+2}) + e(D_{i+1}^{d-i+1}, D_i^{d-i+1}) + e(D_{i+1}^{d-i+1}, D_i^{d-i}) &= c_{i+1} p_{d-i+1,i+1}^d. \\ e(D_{i+1}^{d-i+1}, D_i^{d-i+2}) &= b_i p_{d-i+2,i}^d. \end{aligned}$$

So $b_i p_{d-i+2,i}^d = c_{i+1} p_{d-i+1,i+1}^d$ if and only if $e(D_{i+1}^{d-i+1}, D_i^{d-i+1}) + e(D_{i+1}^{d-i+1}, D_i^{d-i}) = 0$ if and only if $b_{d-i+1} = c_{i+1}$.
(4) Since $p_{d-i+2+j,d-j}^i \neq 0$, take $z \in \Gamma_{d-i+2+j}(u) \cap \Gamma_{d-j}(v)$. In the rank $d-i+2+j$ diagram with respect to (u,z), $v \in D_{d-j}^i$. So $\Gamma_d(u) \cap \Gamma_{d-i}(v) \subseteq D_{i-j}^d = \Gamma_d(u) \cap \Gamma_{i-j}(z)$. □

Lemma 2.2 *Suppose $p_{d,d}^2 = 1$. Then for every pair of adjacent vertices u and v, $\Gamma_d(u) \cap \Gamma_d(v)$ is a clique.*

Proof: Suppose there are distinct vertices x, $y \in \Gamma_d(u) \cap \Gamma_d(v)$ such that $x \not\sim y$. Then $\partial(x,y) = 2$ as $h = 2$. We have $p_{d,d}^2 \geq 2$ because $u, v \in \Gamma_d(x) \cap \Gamma_d(y)$. □

Lemma 2.3 *Suppose* $b_{d-1} = 1$. *Then for every* $\alpha \in \Gamma$, $\Gamma_d(\alpha)$ *is* μ*-closed and* λ*-closed. Moreover* $\Gamma_d(\alpha)$ *becomes strongly regular with the intersection array*

$$\iota(\Gamma_d(\alpha)) = \left\{ \begin{array}{ccc} * & 1 & c_2 \\ 0 & a_1 & a_d - c_2 \\ a_d & a_d - a_1 - 1 & * \end{array} \right\}.$$

Proof: Take any $\alpha \in \Gamma$, and any $\beta, \gamma \in \Gamma_d(\alpha)$. Consider the rank d diagram. Since $b_{d-1} = 1$, we know $e(D_1^{d-1}, D_2^d) + e(D_1^{d-1}, D_1^d) = 0$.

If $\partial(\beta,\gamma) = 1$, then $\gamma \in D_1^d$ and we get $\Gamma_1(\beta) \cap \Gamma_1(\gamma) \subseteq D_1^d \subseteq \Gamma_d(\alpha)$. So we have $\Gamma_d(\alpha)$ is λ-closed. If $\partial(\beta,\gamma) = 2$, then $\gamma \in D_2^d$ and $\Gamma_1(\beta) \cap \Gamma_1(\gamma) \subseteq D_1^d \subseteq \Gamma_d(\alpha)$. Hence $\Gamma_d(\alpha)$ is μ-closed. Therefore $\Gamma_d(\alpha)$ becomes strongly regular with $c_2(\Gamma_d(\alpha)) = c_2$ and $a_1(\Gamma_d(\alpha)) = a_1$. □

Lemma 2.4 *For some* i *with* $1 \leq i \leq d-1$ *and every* $u, v \in \Gamma$ *with* $\partial(u,v) = i$, *suppose* $\Gamma_d(u) \cap \Gamma_{d-i}(v)$ *is a clique. Then the following hold.*
(1) *In the rank* d *diagram,* $e(D_{d-j}^j, D_{d-j+1}^{j+1}) = 0$ *for* $i \leq j \leq d-1$.
(2) $\Gamma_d(\alpha)$ *is* μ*-closed and* $d(\Gamma_d(\alpha)) = 2$ *for every* $\alpha \in \Gamma$.

Proof: Suppose $e(D_{d-j}^j, D_{d-j+1}^{j+1}) \neq 0$ for some j with $i \leq j \leq d-1$. Then there is an edge $x \sim y$ such that $x \in D_{d-j}^j, y \in D_{d-j+1}^{j+1}$. Take $z \in \Gamma_d(\alpha) \cap \Gamma_{d-j-1}(y)$. Then $z \in D_2^d$. So $z \not\sim \beta$. Take $w \in \Gamma_i(\alpha) \cap \Gamma_{j-i}(x)$. Then $z, \beta \in \Gamma_d(\alpha) \cap \Gamma_{d-i}(w)$. This contradicts $\Gamma_d(\alpha) \cap \Gamma_{d-i}(w)$ being a clique. So we have (1). (2) follows from $e(D_1^{d-1}, D_2^d) = 0$. □

Lemma 2.3 *Suppose* $d \geq 3$. *And suppose* $\Gamma_d(x) \cap \Gamma_{d-l+2}(y)$ *is a clique for some* l *with* $3 \leq l \leq d$ *and every* $x, y \in \Gamma$ *with* $\partial(x,y) = l$. *Then* $\Gamma_d(u) \cap \Gamma_1(v)$ *is a clique for every* $u, v \in \Gamma$ *with* $\partial(u,v) = d-1$.

Proof: By Lemma 2.1(4) with $i = d-1$ and $j = l-3$, for every $u, v \in \Gamma$ with $\partial(u,v) = d-1$, there is $z \in \Gamma_l(u)$ such that $\Gamma_d(u) \cap \Gamma_1(v) \subseteq \Gamma_d(u) \cap \Gamma_{d-l+2}(z)$. Since $\Gamma_d(u) \cap \Gamma_{d-l+2}(z)$ is a clique, we get the assertion. □

Lemma 2.6 *Suppose* $d \geq 3$. *Then we have* $c_3 > 1$.

Proof: Suppose $c_3 = 1$. We use the rank d diagram. Then by Lemma 2.1(3) with $i = 2$, $b_{d-1} = c_3 = c_2 = 1$ and $e(D_3^{d-1}, D_2^{d-1}) + e(D_3^{d-1}, D_2^{d-2}) = 0$. By Lemma 2.3, $e(D_1^{d-1}, D_2^d) + e(D_1^{d-1}, D_1^d) = 0$ and $\Gamma_d(\alpha)$ is μ-closed. The rank d diagram becomes as in Figure 3.

Take $\gamma \in D_{d-1}^1$ and let $\Delta = \Gamma_d(\gamma)$.

Claim. $\Delta \subseteq D_3^{d-1} \cup D_2^{d-1} \cup D_2^d \cup D_1^d$, $\Delta \cap D_3^{d-1} \neq \phi$ and $|\Delta \cap D_1^d| = 1$.

Take $x \in \Delta$. We have $\partial(x,\alpha) \geq \partial(x,\gamma) - \partial(\gamma,\alpha) = d-1$. So we get $\Delta \subseteq D_3^{d-1} \cup D_2^{d-1} \cup D_1^{d-1} \cup D_2^d \cup D_1^d$. Since $p_{d,3}^{d-1} \neq 0$, $\Delta \cap D_3^{d-1} = \Gamma_d(\gamma) \cap \Gamma_3(\beta) \neq \phi$.

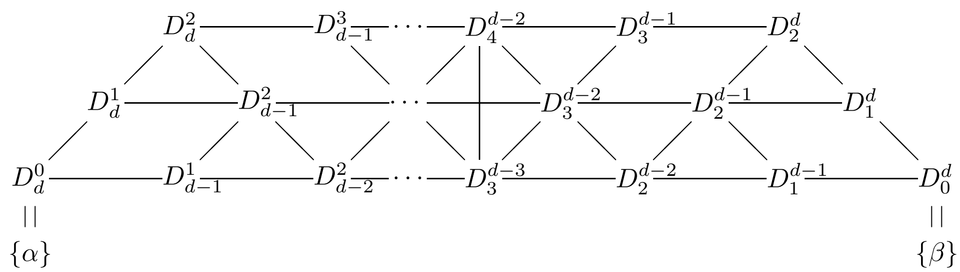

Figure 3.

Take $y \in \Delta \cap D_3^{d-1}$. Suppose there is some vertex $z \in \Delta \cap D_1^{d-1}$. Then $\partial(y,z) \geq 3$, which contradicts $y, z \in \Gamma_d(\gamma)$ and $h = 2$. So we have $\Delta \cap D_1^{d-1} = \phi$. $|\Delta \cap D_1^d| = |\Gamma_d(\gamma) \cap \Gamma_1(\beta)| = b_{d-1} = 1$. Hence we have the claim.

Let $\{u\} = \Delta \cap D_1^d$. Since $|\Delta_1(u)| = a_d > e(u, D_1^d \cup D_2^d)$, there exists $v \in D_2^{d-1} \cap \Delta_1(u)$. Then $\partial(v,y) = 2$ as $h = 2$. By Lemma 2.3, Δ is μ-closed. By Claim, $\Gamma_1(v) \cap \Gamma_1(y) \subseteq D_2^d$. Hence $\{u\} \cup \big(\Gamma_1(v) \cap \Gamma_1(y)\big) \subseteq \Gamma_d(\alpha) \cap \Gamma_1(v)$. This contradicts $b_{d-1} = 1$. □

Lemma 2.7 *Suppose $d \geq 3$ and $\Gamma_d(x) \cap \Gamma_{d-1}(y)$ is a clique for every x, $y \in \Gamma$ with $\partial(x,y) = 3$. Then for every $u \in \Gamma_{d-1}(x) \cap \Gamma_d(y)$ and every $v \in \Gamma_d(x) \cap \Gamma_{d-1}(y)$, we have $u \sim v$. Moreover we have $b_{d-1} = p_{d,d-1}^3$.*

Proof: Use the rank 3 diagram with respect to (x, y).

Then $u \in D_d^{d-1}$ and $v \in D_{d-1}^d$. By way of contradiction, we assume $u \not\sim v$. Since D_{d-1}^d is a clique, we get $\partial(u,v) = 2$.

As $p_{d,d}^2 \neq 0$, we take $z \in \Gamma_d(u) \cap \Gamma_d(v)$. Since x, $z \in \Gamma_d(v)$ and $h = 2$, $\partial(x,z) \leq 2$. Similarly $\partial(y,z) \leq 2$. So $z \in D_2^1 \cup D_2^2 \cup D_1^2$. We may assume $\partial(x,z) = 2$.

Since D_{d-1}^d is a clique, $\Gamma_1(v) \supseteq \Gamma_d(x) \cap \Gamma_1(u)$. By Lemma 2.5 with $l = 3$ and 2.4(2) with $i = d-1$, $\Gamma_d(z)$ is μ-closed. Since u, $v \in \Gamma_d(z)$ with $\partial(u,v) = 2$, we get

$$\begin{aligned} \Gamma_d(z) &\supseteq \big(\Gamma_1(u) \cap \Gamma_1(v)\big) \cup \{v\} \cup \{u\} \\ &\supseteq \big(\Gamma_d(x) \cap \Gamma_1(u)\big) \cup \{v\}. \end{aligned}$$

Hence

$$\Gamma_d(x) \cap \Gamma_d(z) \supseteq \big(\Gamma_d(x) \cap \Gamma_1(u)\big) \cup \{v\}.$$

Claim 1. $b_{d-1} = c_3$.

Suppose there is $\gamma \in D^d_{d-1}$ such that $\gamma \notin \Gamma_d(x) \cap \Gamma_d(z)$. Then $\partial(z, \gamma) = d-1$ because D^d_{d-1} is a clique and $\partial(z, v) = d$. So

$$\begin{aligned} \Gamma_d(z) \cap \Gamma_1(\gamma) &\supseteq \big((\Gamma_d(x) \cap \Gamma_1(u)) \cup \{v\}\big) \cap \Gamma_1(\gamma) \\ &= (\Gamma_d(x) \cap \Gamma_1(u)) \cup \{v\} \end{aligned}$$

In this case $b_{d-1} \geq b_{d-1} + 1$, which is impossible. Hence $\Gamma_d(x) \cap \Gamma_d(z) \supseteq D^d_{d-1}$, i.e., $p^2_{d,d} \geq p^3_{d,d-1}$. By Lemma 2.1(3) with $i = 2$, we have $p^2_{d,d} = p^3_{d,d-1}$ and $b_{d-1} = c_3$.

Claim 2. For any α, $\delta \in \Gamma$ such that $\partial(\alpha, \delta) = 3$, $\Gamma_2(\alpha) \cap \Gamma_1(\delta)$ is a clique.

For any α, $\delta \in \Gamma$ such that $\partial(\alpha, \delta) = 3$, take $\beta \in \Gamma_d(\alpha) \cap \Gamma_{d-1}(\delta)$ and consider the rank d diagram. Then $\delta \in D^3_{d-1}$. By Claim 1 and Lemma 2.1(3) with $i = 2$, we get $e(D^3_{d-1}, D^2_{d-1}) + e(D^3_{d-1}, D^2_{d-2}) = 0$. So $\Gamma_2(\alpha) \cap \Gamma_1(\delta) = \Gamma_d(\beta) \cap \Gamma_1(\delta)$. By Lemma 2.5 with $l = 3$, $\Gamma_d(\beta) \cap \Gamma_1(\delta)$ is a clique. So we get $\Gamma_2(\alpha) \cap \Gamma_1(\delta)$ is a clique.

By Claim 2 and Lemma 5.5.2 of [3], we get $c_3 = c_2 = 1$. This contradicts Lemma 2.6. So we have $u \sim v$. Therefore $b_{d-1} = |\Gamma_d(x) \cap \Gamma_1(u)| = |D^d_{d-1}| = p^3_{d,d-1}$. □

Lemma 2.8 *Suppose $d \geq 3$ and $d(\Gamma_d(\alpha)) = 2$ for every $\alpha \in \Gamma$. Then, in the rank d diagram, $e(D^i_{d-i+1}, D^{i+1}_{d-i}) \neq 0$ for $1 \leq i \leq d-1$. In particular, $D^i_{d-i+1} \neq \phi$ for $1 \leq i \leq d$.*

Proof: Suppose $e(D^i_{d-i+1}, D^{i+1}_{d-i}) = 0$ for some i with $2 \leq i \leq d-1$. Take any $x \in D^d_1$. Then $\Gamma_i(\alpha) \cap \Gamma_{d-i}(x) \subseteq D^i_{d-i}$. Since both sizes are $p^d_{i,d-i}$, we have

$$\Gamma_i(\alpha) \cap \Gamma_{d-i}(x) = D^i_{d-i}.$$

Hence for any $y \in D^i_{d-i}$,

$$\Gamma_d(\alpha) \cap \Gamma_{d-i}(y) \supseteq D^d_1 \cup \{\beta\}.$$

Since $d(\Gamma_d(\alpha)) = 2$, for every $z \in D^d_2$ there is $w \in D^d_1$ such that $z \sim w$. So $\partial(y, z) \leq d-i+1$, and $\Gamma_d(\alpha) \cap \Gamma_{d-i+2}(y) = \phi$. This contradicts $p^i_{d,d-i+2} \neq 0$. Hence we have $e(D^i_{d-i+1}, D^{i+1}_{d-i}) \neq 0$ for $2 \leq i \leq d-1$. By symmetry we have $e(D^1_d, D^2_{d-1}) \neq 0$. □

3. Some lemmas

In the rest of this paper, we assume the hypothesis of Theorem 1.2. In the rank d diagram, we have $D^d_2 \simeq K_{t\times 2}$.

Let $\kappa_1 = p^d_{d,1} = a_d$ and $\kappa_2 = p^d_{d,2} = 2t$. Then $k_d = 1 + \kappa_1 + \kappa_2$. Take any $\gamma \in D^d_2$, then we have $e(\gamma, D^d_2) = \kappa_2 - 2$, $e(\gamma, D^d_1) = \kappa_1 - \kappa_2 + 2$ and $e(\gamma, D^{d-1}_1) = c_2 - (\kappa_1 - \kappa_2 + 2)$ by our assumption.

Lemma 3.1 *For every $\alpha \in \Gamma$ and every $\beta, \gamma \in \Gamma_d(\alpha)$ with $\partial(\beta, \gamma) = 2$, we have* $|\{\delta \in \Gamma_d(\alpha) : \partial(\delta, \beta) = \partial(\delta, \gamma) = 2\}| = 1$.

Proof: Consider the rank d diagram. Then $\gamma \in D_2^d$. As $D_2^d \simeq K_{t\times 2}$, we get $|\{\delta \in \Gamma_d(\alpha) : \partial(\delta,\beta) = \partial(\delta,\gamma) = 2\}| = |\{\delta \in D_2^d : \delta \not\sim \gamma\}| = 1$. □

Lemma 3.2 *We have* $b_{d-1} \geq 2$.

To prove Lemma 3.2, we need the following lemma.

Lemma 3.3 *Let* Δ *be a strongly regular graph with* $p^2_{2,2}(\Delta) = 1$. *Then we have* $c_2(\Delta) \geq 2$ *and* $k_1(\Delta) \geq k_2(\Delta)$. *Moreover one of the following holds.*
(1) Δ *is isomorphic to the complete multipartite graph* $K_{3,3}$.
(2) Δ *is isomorphic to the Hamming graph* $H(2,3)$.
(3) $\Delta_1(\alpha)$ *is connected for every* $\alpha \in \Delta$, *i.e.,* Δ *is locally connected.*

Proof: Let $\kappa_1 = k_1(\Delta)$, $\kappa_2 = k_2(\Delta)$ and $\lambda = a_1(\Delta)$. As $p^2_{2,2}(\Delta) = 1$, we have $a_2(\Delta) = \kappa_2 - 2$ and $c_2(\Delta) = \kappa_1 - \kappa_2 + 2$.

If $c_2(\Delta) = 1$, then $\kappa_1 + 1 = \kappa_2 = \kappa_1 b_1(\Delta)$, which is a contradiction. So $c_2(\Delta) \geq 2$ and $\kappa_1 \geq \kappa_2$.

Take any $\alpha \in \Delta$ and $x, y \in \Delta_1(\alpha)$ with $x \not\sim y$. Let $X = \Delta_1(\alpha) \cap \Delta_1(x)$, $Y = \Delta_1(\alpha) \cap \Delta_1(y)$ and $\{z\} = \Delta_2(x) \cap \Delta_2(y)$. Then by $p^2_{2,2}(\Delta) = 1$, $\Delta_1(\alpha) \subseteq \{x, y, z\} \cup X \cup Y$. So we have $\kappa_1 \leq 2\lambda + 3$.

Suppose $\kappa_1 = 2\lambda + 3$. Then $X \cap Y = \phi$ and $\{z\} \subseteq \Delta_1(\alpha)$. Since $\{x\} = \Delta_2(y) \cap \Delta_2(z)$, $X \subseteq \Delta_1(z)$. Similarly $Y \subseteq \Delta_1(z)$. So we have $X \cup Y \subseteq \Delta_1(\alpha) \cap \Delta_1(z)$ and $2\lambda \leq \lambda$. Hence $\lambda = 0$ and $\kappa_1 = 3$. So we have $\Delta \simeq K_{3,3}$

Suppose $\kappa_1 = 2\lambda + 2$. Then since

$$\lambda + 1 = \kappa_1 - \lambda - 1 = b_1(\Delta) = \frac{\kappa_2 c_2(\Delta)}{\kappa_1} = \frac{\kappa_2\big(2(\lambda+1) - \kappa_2 + 2\big)}{2(\lambda+1)}$$

we get

$$\big(\kappa_2 - (\lambda+2)\big)^2 + (\lambda^2 - 2) = 0.$$

Hence $\lambda = 1$, $\kappa_2 = 4$ and we easily get $\Delta \simeq H(2,3)$.

Suppose $\kappa_1 \leq 2\lambda + 1$, then $X \cap Y \neq \phi$. So Δ is locally connected. □

Proof of Lemma 3.2: Suppose $b_{d-1} = 1$. Take $x \in \Gamma$. Then by Lemma 2.3, $\Gamma_d(x)$ becomes strongly regular. Let $\Delta = \Gamma_d(x)$, then by Lemma 3.1, Δ has the property that $p^2_{2,2}(\Delta) = 1$. Hence by Lemma 3.3, $\Delta \simeq K_{3,3}$ or $\Delta \simeq H(2,3)$ or Δ is locally connected. A. Hiraki and H. Suzuki showed that there is no distance-regular graph Γ with diameter $d \geq 3$ such that $\Gamma_d(\alpha) \simeq K_{t\times s}(t \geq 2, s \geq 2)$. (See [13].) So the case $\Delta \simeq K_{3,3}$ does not occur.

Case 1. $\Delta \simeq H(2,3)$.

Since $a_1 = 1$ and $c_2 = 2$, we have $k_2 = k(k-2)/2$. Since $kp^1_{d,d} = k_d p^d_{1,d} = 36$, $k_2 p^2_{d,d} = k_d p^d_{2,d} = 36$ and $4 = a_d < k$, we get $k = 6$ and $k_2 = 12$. Therefore we get $c_d = 2$. This is a contradiction because $c_2 \geq 2$ implies $c_d > c_2$ (see Theorem 5.4.1 of [3]).

Case 2. Δ is locally connected.

Take $y \in \Gamma$ with $\partial(x,y) = i$ $(i = 1,2)$. Suppose $\Gamma_d(x) \cap \Gamma_d(y)$ contains at least two vertices, then it contains an edge $u \sim v$ since it is μ-closed. So $\Delta_1(u) \subseteq \Gamma_d(x) \cap \Gamma_d(y)$ because $\Gamma_d(x) \cap \Gamma_d(y)$ is λ-closed. Since $c_2 > 1$ and $\Gamma_d(x) \cap \Gamma_d(y)$ is λ-closed and μ- closed, $\Delta \subseteq \Gamma_d(x) \cap \Gamma_d(y)$. Hence $p^i_{d,d} = k_d$ and $p^i_{d,d-i} = 0$, which is impossible. Therefore we get $p^1_{d,d} = p^2_{d,d} = 1$ and $k = k_d\kappa_1$, $k_2 = k_d\kappa_2$. By Lemma 3.3, $k \geq k_2$. This also contradicts Lemma 5.1.2 of [3]. $\square$

Lemma 3.4 *Suppose* $d \geq 4$. *For* $3 \leq i \leq d-1$, *take every* u, $v \in \Gamma$ *with* $\partial(u,v) = i$. *Then* $\Gamma_d(u) \cap \Gamma_{d-i}(v)$ *and* $\Gamma_d(u) \cap \Gamma_{d-i+2}(v)$ *are cliques of size at least 2.*

Proof: By Lemmas 2.1 and 3.2, $p^i_{d,d-i} \geq p^{d-1}_{d,1} = b_{d-1} \geq 2$ and $p^i_{d,d-i+2} \geq p^3_{d,d-1} \geq p^{d-1}_{d,1} \geq 2$.

For some u, $v \in \Gamma$ with $\partial(u,v) = i$, suppose there are x, $y \in \Gamma_d(u) \cap \Gamma_{d-i}(v)$ such that $x \not\sim y$. Use the rank i diagram with respect to (u,v). Then x, $y \in D^d_{d-i}$ with $\partial(x,y) = 2$.

Since $p^i_{d,d-i+2} \geq 2$, there are z, $w \in D^d_{d-i+2}$. Clearly $\partial(z,x) = \partial(z,y) = \partial(w,x) = \partial(w,y) = 2$. So in $\Gamma_d(u)$ there are at least two vertices at distance 2 from x and y, which contradicts Lemma 3.1. So $\Gamma_d(u) \cap \Gamma_{d-i}(v)$ is a clique.

Similarly $\Gamma_d(u) \cap \Gamma_{d-i+2}(v)$ is a clique. $\square$

Lemma 3.5 *Suppose* $d \geq 4$. *Then we have the following.*
(1) *In the rank* d *diagram,* $e(D^i_{d-i}, D^{i+1}_{d-i+1}) = 0$ *for* $1 \leq i \leq d-1$.
(2) $\Gamma_d(\alpha)$ *is* μ*-closed and* $d(\Gamma_d(\alpha)) = 2$ *for every* $\alpha \in \Gamma$.
(3) $D^i_{d-i+1} \neq \phi$ *for* $1 \leq i \leq d$.

Proof: (1) From Lemmas 3.4 and 2.4, $e(D^i_{d-i}, D^{i+1}_{d-i+1}) = 0$ for $3 \leq i \leq d-1$. Suppose $e(D^2_{d-2}, D^3_{d-1}) \neq 0$. Then there are $x \in D^2_{d-2}$, $y \in D^3_{d-1}$ such that $x \sim y$. By Lemma 3.4, we take $z, w \in \Gamma_d(\alpha) \cap \Gamma_{d-3}(y) \subseteq D^d_2$. By our assumption, (i.e., that $D^d_2 \simeq K_{t\times 2}$,) for any $\gamma \in \Gamma_d(\alpha)$, $\partial(\gamma,\beta) \leq 1$ or $\partial(\gamma,z) \leq 1$ or $\partial(\gamma,w) \leq 1$. Therefore we get $\Gamma_d(\alpha) \cap \Gamma_d(x) = \phi$. This is impossible because $p^2_{d,d} \neq 0$. So we get $e(D^2_{d-2}, D^3_{d-1}) = 0$. By symmetry, $e(D^1_{d-1}, D^2_d) = 0$.
(2)(3) It is clear from Lemmas 2.4(2), 2.8 and 3.4. $\square$

Lemma 3.6 *Suppose* $d \geq 4$. *Then we have* $b_{d-1} \leq \frac{\kappa_2}{2}$.

Proof: Consider the rank d diagram. For any $\gamma \in D^{d-1}_3$,

$$\Gamma_d(\alpha) \cap \Gamma_1(\gamma) \subseteq D^d_2.$$

By Lemma 3.4 with $i = d-1$, $\Gamma_d(\alpha) \cap \Gamma_1(\gamma)$ is a clique. On the other hand, $D^d_2 \simeq K_{t\times 2}$ by our assumption. Since the maximal cliques of $K_{t\times 2}$ are size t, we have

$$b_{d-1} \leq t = \frac{\kappa_2}{2}.$$

$\square$

4. The case $d \geq 5$

In this section we assume $d \geq 5$ and show this case does not occur.

Proposition 4.1 *Let Γ be a distance-regular graph with diameter d and height $h = 2$. Suppose that for every α in Γ and every β in $\Gamma_d(\alpha)$, $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is isomorphic to $K_{t\times 2}$ with $t \geq 2$. Then $d \leq 4$.*

By Lemma 3.5, the rank d diagram becomes as in Figure 4.

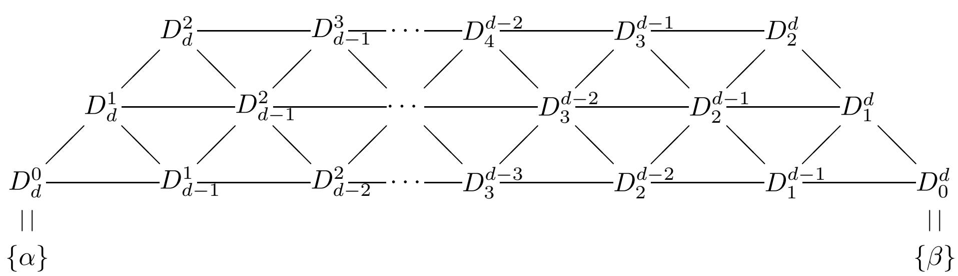

Figure 4.

Lemma 4.2 *We have the following.*
(1) $b_2 \geq c_{d-1}$.
(2) $b_{d-2} = c_3$.

Proof: First we show $b_2 \geq c_{d-1}$ and $b_{d-2} \geq c_3$. From Lemma 2.1(3), we have $b_i \leq c_{d-i+2}$ for $i = 2, d-2$. If $b_i = c_{d-i+2}$, then we get

$$b_i = c_{d-i+2} \geq c_{d-i+1}.$$

So we may assume $b_i < c_{d-i+2}$. Then, in the rank d diagram, there is an edge $x \sim y$ such that $x \in D_i^{d-i+2}$ and $y \in D_i^{d-i+1}$.
Claim 1. $e(y, D_i^{d-i}) = 0$ for $i = 2, d-2$.

Suppose there is $z \in D_i^{d-i}$ such that $y \sim z$. If $i = 2$, then β, $x \in \Gamma_d(\alpha) \cap \Gamma_2(z)$ with $\beta \not\sim x$. This contradicts Lemma 3.4 with $i = d-2$. If $i = d-2$, then from Lemma 3.4 with $i = 4$, we can take u, $v \in \Gamma_d(\alpha) \cap \Gamma_{d-4}(x)$. Then $u, v \in D_2^d$ and $u, v \in \Gamma_{d-2}(z)$. Take any $\gamma \in \Gamma_d(\alpha)$. If $\gamma \in D_1^d$, then $\partial(z, \gamma) \leq d-1$. If $\gamma \in D_2^d$, then by our assumption, $\gamma \sim u$ of $\gamma \sim v$. So we have $\partial(z, \gamma) \leq d-1$. Hence we have $\partial(z, \gamma) \leq d-1$ for any $\gamma \in \Gamma_d(\alpha)$. $\partial(z, \gamma) \leq d-1$. So $\Gamma_d(\alpha) \cap \Gamma_d(z) = \phi$. This contradicts $p_{d,d}^2 \neq 0$. So we have the claim.

By Claim 1, for $i = 2$ and $d-2$, we get

$$c_{d-i+1} = e(y, D_{i+1}^{d-i}) \leq e(y, D_{i+1}^{d-i+1}) + e(y, D_{i+1}^{d-i}) = b_i.$$

So we have (1) and $c_3 \leq b_{d-2}$.

We know $b_{d-3} \geq c_3$. If $b_{d-3} = c_3$, then we get

$$b_{d-2} \leq b_{d-3} = c_3.$$

So we may assume $b_{d-3} > c_3$, then there is an edge $x \sim y$ such that $x \in D_3^{d-3}$ and $y \in D_3^{d-2}$.

Claim 2. $e(y, D_3^{d-1}) = 0$.

Suppose there is $z \in D_3^{d-1}$ such that $y \sim z$. Then by Lemma 3.2, take $u, v \in \Gamma_d(\alpha) \cap \Gamma_1(z) \subseteq D_2^d$. By our assumption, for any $\gamma \in \Gamma_d(\alpha)$, $\partial(x, \gamma) \leq 4$. So $\Gamma_d(\alpha) \cap \Gamma_5(x) = \phi$. This contradicts $p_{d,5}^{d-3} \neq 0$. Hence we have the claim.

By Claim 2, we have

$$b_{d-2} = e(y, D_2^{d-1}) \leq e(y, D_2^{d-1}) + e(y, D_2^{d-2}) = c_3.$$

□

Lemma 4.3 *In the rank d diagram, we have the following.*
(1) *For every $x \in D_{d-1}^3$, $\Gamma_d(x) \subseteq D_3^{d-3} \cup D_3^{d-2} \cup D_2^{d-2} \cup D_3^{d-1} \cup D_2^{d-1} \cup D_1^{d-1}$.*
(2) *For every $y \in D_{d-2}^2$, $\Gamma_d(y) \subseteq D_4^{d-2} \cup D_3^{d-2} \cup D_3^{d-1} \cup D_2^{d-1} \cup D_2^d$. In particular $\Gamma_d(y) \cap (D_2^{d-1} \cup D_2^d)$ is a clique, $\Gamma_d(y) \cap D_2^d \neq \phi$ and $|\Gamma_d(y) \cap D_4^{d-2}| \geq 2$.*

Proof: (1) Take any $z \in \Gamma_d(x)$. Since $\partial(\alpha, x) = 3$ and $p_{i,d}^3 = 0$ if $i \leq d-4$ or $i = d$, we know $d-3 \leq \partial(\alpha, z) \leq d-1$. Similarly $1 \leq \partial(\beta, z) \leq 3$. So we get (1).
(2) Take any $w \in \Gamma_d(y)$. Since $\partial(\alpha, y) = 2$ and $\partial(\beta, y) = d-2$, $\partial(\alpha, w) \geq d-2$ and $\partial(\beta, w) \geq 2$. So we get

$$\Gamma_d(y) \subseteq D_4^{d-2} \cup D_3^{d-2} \cup D_2^{d-2} \cup D_3^{d-1} \cup D_2^{d-1} \cup D_2^d.$$

Claim. $\Gamma_d(y) \cap D_2^d \neq \phi$, $\Gamma_d(y) \cap D_2^{d-2} = \phi$ and $\Gamma_d(y) \cap (D_2^{d-1} \cup D_2^d)$ is a clique.

Take $u \in \Gamma_d(y) \cap \Gamma_d(\alpha)$, then $u \in \Gamma_d(y) \cap D_2^d$. So $\Gamma_d(y) \cap D_2^d \neq \phi$. Since $\Gamma_d(y) \cap (D_2^{d-2} \cup D_2^{d-1} \cup D_2^d) = \Gamma_d(y) \cap \Gamma_2(\beta)$ is a clique by Lemma 3.4 with $i = d-2$, we get the claim.

By Lemma 3.4 with $i = d-2$ we get

$$|\Gamma_d(y) \cap D_4^{d-2}| = |\Gamma_d(y) \cap \Gamma_4(\beta)| \geq 2.$$

□

Lemma 4.4 *In the rank d diagram, we have $\partial(D_{d-2}^2, D_{d-1}^3) \geq 3$.*

Proof: By way of contradiction, assume that there are $x \in D^3_{d-1}$ and $y \in D^2_{d-2}$ such that $\partial(x, y) = 2$. Then we can take $u \in \Gamma_d(x) \cap \Gamma_d(y)$. By Lemma 4.3, $u \in D^{d-1}_2 \cup D^{d-1}_3 \cup D^{d-2}_3$.

Case 1. $u \in D^{d-1}_2$.

Since $\Gamma_d(y) \cap (D^{d-1}_2 \cup D^d_2)$ is a clique and $\Gamma_d(y) \cap D^d_2 \neq \phi$, there is $v \in D^d_2$ such that $u \sim v$. By Lemma 3.4 with $i = 3$, we take an edge $\gamma \sim \delta$ in $\Gamma_d(\alpha) \cap \Gamma_{d-3}(x)$. Then γ, $\delta \in D^d_2$. By our assumption, $v \sim \gamma$ or $v \sim \delta$. Therefore we get $\partial(u, x) \leq d - 1$, which contradicts $u \in \Gamma_d(x)$.

Case 2. $u \in D^{d-1}_3$.

We can take $v \in \Gamma_d(\alpha) \cap \Gamma_1(u)$. Then $v \in D^d_2$. Similar to Case 1, this case does not occur.

Case 3. $u \in D^{d-2}_3$.

Since $\Gamma_d(y) \cap D^d_2 \neq \phi$ and $|\Gamma_d(y) \cap D^{d-2}_4| \geq 2$, take $z \in \Gamma_d(y) \cap D^d_2$ and ε, $\zeta \in \Gamma_d(y) \cap D^{d-2}_4$. Then $\partial(z, \varepsilon) = \partial(z, \zeta) = \partial(z, u) = 2$ and z, ε, ζ, $u \in \Gamma_d(y)$. Hence by Lemma 3.1 $u \sim \varepsilon$ or $u \sim \zeta$. We may assume $u \sim \varepsilon$. Then we take an edge $\eta \sim \theta$ in $\Gamma_d(\beta) \cap \Gamma_{d-4}(\varepsilon)$ and $\xi \in \Gamma_d(\beta) \cap \Gamma_1(x)$. Then ξ, η, $\theta \in D^d_d$, so $\xi \sim \eta$ or $\xi \sim \theta$ and $\partial(x, u) \leq d - 1$. We get a contradiction.

Hence we have the assertion. □

Proof of Proposition 4.1: Suppose $b_{d-2} > c_2$, then there is an $x \sim y$ such that $x \in D^2_{d-2}$ and $y \in D^2_{d-1}$. By Lemma 4.2(1),

$$\begin{aligned} b_2 &= e(y, D^3_{d-1}) + e(y, D^3_{d-2}) \\ \geq c_{d-1} &= e(y, D^3_{d-2}) + e(y, D^2_{d-2}) \geq e(y, D^3_{d-2}) + e(y, x). \end{aligned}$$

So there is $z \in D^3_{d-1}$ such that $y \sim z$, which contradicts Lemma 4.4. Therefore we get $b_{d-2} = c_2$. By Lemma 4.2(2), we get

$$c_3 = b_{d-2} = c_2.$$

$c_3 = c_2$ implies $c_3 = 1$. (See Theorem 5.4.1 of [3].) This contradicts Lemma 2.6. □

5. The case $d = 3$

In this section we assume $d = 3$ and prove the following proposition.

Proposition 5.1 *Let Γ be a distance-regular graph with height $h = 2$. For every α in Γ and every β in $\Gamma_d(\alpha)$, $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is isomorphic to $K_{t\times 2}$ with $t \geq 2$. Then $d \neq 3$.*

Lemma 5.2 *For every u, $v \in \Gamma$ with $\partial(u, v) = 2$, $\Gamma_3(u) \cap \Gamma_3(v)$ is a clique.*

Proof: Suppose there are $x, y \in \Gamma_3(u) \cap \Gamma_3(v)$ such that $x \not\sim y$. Then $\partial(x, y) = 2$. Since $b_2 \geq 2$, there are $z, w \in \Gamma_3(u) \cap \Gamma_1(v)$. Then x, y, z, $w \in \Gamma_3(u)$ and $\partial(z, x) = \partial(z, y) = \partial(w, x) = \partial(w, y) = 2$. This contradicts Lemma 3.1. □

Lemma 5.3 *We have the following.*
(1) $p^2_{3,3} = 1$, $k_2 = \kappa_2(1 + \kappa_1 + \kappa_2)$ *and* $c_3 = \kappa_2 b_2$.
(2) *For every edge* $u \sim v$, $\Gamma_3(u) \cap \Gamma_3(v)$ *is a clique.*

Proof: (1) By way of contradiction, we assume $p^2_{3,3} \geq 2$.

Similar to Lemma 5.2, we have $\Gamma_3(u) \cap \Gamma_1(v)$ is a clique for every u, $v \in \Gamma$ with $\partial(u, v) = 2$.

Consider the rank 3 diagram. By Lemma 2.4 with $i = 2$, $e(D^1_2, D^2_3) = e(D^2_1, D^3_2) = 0$ and $\Gamma_3(\alpha)$ is μ-closed for every $\alpha \in \Gamma$.

Take x, $y \in D^3_2$ such that $\partial(x, y) = 2$. Since $p^2_{3,3} \geq 2$, we take $\gamma \in \Gamma_3(x) \cap \Gamma_3(y)$ with $\gamma \neq \alpha$. Then $\gamma \in D^1_3$ or $\gamma \in D^1_2$ because $\Gamma_3(x) \cap \Gamma_3(y)$ is a clique by Lemma 5.2.

Case 1. $\gamma \in D^1_3$.
Since x, $y \in \Gamma_3(\gamma)$, $D^3_2 \simeq K_{t\times 2}$ and $\Gamma_3(\gamma)$ is μ-closed, we get $D^3_2 \subseteq \Gamma_3(\gamma)$. Since

$$\begin{aligned} p^3_{3,2} = |\Gamma_3(\gamma) \cap \Gamma_2(\beta)| &= |\Gamma_3(\gamma) \cap (D^1_2 \cup D^2_2 \cup D^3_2)| \\ &\geq |\Gamma_3(\gamma) \cap D^3_2| = |D^3_2| = p^3_{3,2}, \end{aligned}$$

we have $\Gamma_3(\gamma) \cap (D^2_2 \cup D^1_2) = \phi$.

Take $z \in \Gamma_3(\gamma) \cap \Gamma_2(\alpha)$. Then $z \in D^2_1$ because $\Gamma_3(\gamma) \cap D^2_2 = \phi$. Since $\partial(x, z) = 2$ and $\Gamma_3(\gamma)$ is μ-closed,

$$\Gamma_1(x) \cap \Gamma_1(z) \subseteq D^3_1 \cap \Gamma_1(z) \subseteq \Gamma_3(\alpha) \cap \Gamma_1(z) \setminus \{\beta\}.$$

So

$$c_2 = |\Gamma_1(x) \cap \Gamma_1(z)| \leq |\Gamma_3(\alpha) \cap \Gamma_1(z) \setminus \{\beta\}| = b_2 - 1.$$

If there is $\delta \in D^2_3$ such that $\gamma \sim \delta$, then we take $\varepsilon \in D^3_2$ such that $\delta \sim \varepsilon$. So $\partial(\gamma, D^3_2) = 2$, which contradicts $\Gamma_3(\gamma) \supseteq D^3_2$. Hence $e(\gamma, D^2_3) = 0$. So we can take $\zeta \in D^2_2$ such that $\gamma \sim \zeta$. Since $\Gamma_3(\gamma) \supseteq D^3_2$, we get $e(\zeta, D^3_2) = 0$ and $b_2 \leq c_2$. This contradicts $c_2 \leq b_2 - 1$. Therefore this case does not occur.

Case 2. $\gamma \in D^1_2$.
Similar to Case 1, we get $D^3_2 \subseteq \Gamma_3(\gamma)$.

Clearly we have $\Gamma_3(\gamma) \cap \Gamma_3(\beta) \subseteq D^2_3$. Since $\Gamma_3(\gamma) \cap \Gamma_3(\beta)$ is a clique and $D^2_3 \simeq K_{t\times 2}$, we take $w \in D^2_3$ such that $w \notin \Gamma_3(\gamma)$. Then $\partial(w, \gamma) = 2$ because $e(D^1_2, D^2_3) = 0$. By $p^2_{3,3} \geq 2$, there are η, θ in $\Gamma_3(\gamma) \cap \Gamma_3(\beta)$. Then η, $\theta \in D^2_3$. Since $D^2_3 \simeq K_{t\times 2}$, we have $w \sim \eta$ or $w \sim \theta$. We may assume $w \sim \eta$. Then

$$\Gamma_3(\alpha) \cap \Gamma_1(w) \subseteq D^3_2 \cap \Gamma_1(w) \subseteq \Gamma_3(\gamma) \cap \Gamma_1(w).$$

So

$$b_2 + 1 = \big|\big(\Gamma_3(\alpha) \cap \Gamma_1(w)\big) \cup \{\eta\}\big| \leq |\Gamma_3(\gamma) \cap \Gamma_1(w)| = b_2.$$

This is a contradiction. Hence this case is also impossible.

Therefore we get $p^2_{3,3} = 1$. So we have $k_2 = k_2 p^2_{3,3} = k_3 p^3_{3,2} = \kappa_2(1 + \kappa_1 + \kappa_2)$. $c_3 = \kappa_2 b_2$ follows from $k_2 b_2 = k_3 c_3$.
(2) It is clear from (1) and Lemma 2.2. □

Lemma 5.4 *For every* $u, v \in \Gamma$ *with* $\partial(u,v) = 2$, $\Gamma_3(u) \cap \Gamma_1(v) \simeq K_{s\times 2}$ *for some* $s \le t$.

Proof: By Lemma 2.1(4) with $d = 3, i = 2, j = 0$ and our assumption $\Gamma_3(u) \cap \Gamma_1(v)$ is a subgraph of $K_{t\times 2}$. So we may assume $b_2 < \kappa_2$.

Claim 1. In the rank 3 diagram, $e(\gamma, D_2^3) \le 1$ for every $\gamma \in D_1^2$.

If $e(\gamma, D_2^3) \ge 2$ for some $\gamma \in D_1^2$, then $\Gamma_3(\alpha) \cap \Gamma_3(\gamma) = \phi$ because $D_2^3 \simeq K_{t\times 2}$. This contradicts $p_{3,3}^2 \ne 0$. So we get the claim.

Suppose $e(\gamma, D_2^3) = 1$ for every α, $\beta \in \Gamma$ with $\partial(\alpha,\beta) = 3$ and $\gamma \in D_1^2$. Then since α and β are arbitary, we get $\Gamma_3(\alpha) \cap \Gamma_1(\gamma) \simeq K_{s\times 2}$ for every α, γ with $\partial(\alpha,\gamma) = 2$. So by way of contradiction, we assume $e(\gamma, D_2^3) = 0$ for some α, $\beta \in \Gamma$ with $\partial(\alpha,\beta) = 3$ and $\gamma \in D_1^2$. Then by Claim 1,

$$p_{3,2}^3\big(c_2 - (\kappa_1 - \kappa_2 + 2)\big) = e(D_1^2, D_2^3) < p_{2,1}^3.$$

So

$$\kappa_2\big(c_2 - (\kappa_1 - \kappa_2 + 2)\big) < c_3 = \kappa_2 b_2.$$

So we have

$$c_2 - (\kappa_1 - \kappa_2 + 2) \le b_2 - 1.$$

Claim 2. $c_2 = (\kappa_1 - \kappa_2 + 2) + (b_2 - 1)$.

In the rank 3 diagram, we can take $x \in D_3^2$, $y \in D_2^3$ such that $\partial(x,y) = 2$ because $b_2 < \kappa_2$. Let $\{z\} = \Gamma_3(x) \cap \Gamma_3(y)$, then $z \in D_2^1 \cup D_2^2 \cup D_1^2$. So we may assume $\partial(\alpha,z) = 2$. We know $\Gamma_3(y) \subseteq \{\alpha\} \cup D_2^1 \cup D_2^2 \cup D_1^2$ because $\Gamma_3(\beta) \cap \Gamma_3(y) = \{\alpha\}$. Since $z \in \Gamma_3(y) \cap \Gamma_2(\alpha) = \Gamma_3(y) \cap (D_2^2 \cup D_1^2) \simeq K_{t\times 2}$, $e(x, \Gamma_3(y) \cap D_2^2) \le 1$. Hence $e(x, D_2^1) \ge e(x, \Gamma_3(y) \cap D_2^1) \ge b_2 - 1$. Therefore we get $c_2 - (\kappa_1 - \kappa_2 + 2) = b_2 - 1$.

Claim 3. For every $\alpha \in \Gamma$ and β, $\gamma \in \Gamma_3(\alpha)$ with $\partial(\beta,\gamma) = 2$, $b_2 - 1$ vertices of $\Gamma_1(\beta) \cap \Gamma_1(\gamma)$ are at distance 2 from α and $\kappa_1 - \kappa_2 + 2$ vertices of $\Gamma_1(\beta) \cap \Gamma_1(\gamma)$ are at distance 3 from α

Consider the rank 3 diagram. Then $\gamma \in D_2^3$. Since $e(\gamma, D_1^2) = b_2 - 1$, by Claim 2, and $e(\gamma, D_1^3) = \kappa_1 - \kappa_2 + 2$, we get the claim.

Claim 4. There are α, $x \in \Gamma$ with $\partial(\alpha,x) = 2$ such that $\Gamma_3(\alpha) \cap \Gamma_1(x)$ is not a clique.

Take $\alpha \in \Gamma$ and β, $\gamma \in \Gamma_3(\alpha)$ with $\partial(\beta,\gamma) = 2$. By Claim 3 and $b_2 \ge 2$, we can take $x \in \Gamma_2(\alpha)$ such that $x \sim \gamma$, $x \sim \beta$.

Claim 5. $b_2 \le t$.

In the rank 3 diagram, take $x \in D_3^2$, $y \in D_2^3$ such that $\partial(x,y) = 2$. Let $\{z\} = \Gamma_3(x) \cap \Gamma_3(y)$. We may assume $\partial(\alpha,z) = 2$. Let

$$A = \{u \in \Gamma_3(\alpha) \cap \Gamma_1(x) : u \sim y\},$$

$$B = \{u \in \Gamma_3(\beta) \cap \Gamma_1(y) : u \sim x\}.$$

Since $\Gamma_3(\alpha) \cap \Gamma_1(x) \subseteq D_2^3 \simeq K_{t\times 2}$ and $y \in D_2^3$, we have $|A| = b_2$ or $b_2 - 1$. Similarly $|B| = b_2$ or $b_2 - 1$. As $\{y\} = \Gamma_3(\alpha) \cap \Gamma_3(z)$, no vertex in $\Gamma_3(\alpha) \cap \Gamma_1(x)$ is at distance 3 from z. So all vertices in A are at distance 2 from z. By Claim 3 with $\alpha = z$, $\beta = x$ and $\gamma = y$, all vertices in B are at distance 3 from z. Hence

$$\Gamma_3(\beta) \cap \Gamma_3(z) \supseteq B \cup \{x\}.$$

By Lemma 5.3, we get $\partial(\beta, z) = 1$ and $B \cup \{x\}$ is a clique of size b_2 or $b_2 + 1$. Since $B \cup \{x\} \subseteq D_3^2 \simeq K_{t\times 2}$ and the maximal cliques of $K_{t\times 2}$ are size t, we have $b_2 \leq t$.

By Claim 4, we take α, $x \in \Gamma$ with $\partial(\alpha, x) = 2$ such that $\Gamma_3(\alpha) \cap \Gamma_1(x)$ is not a clique. Let $\{\beta\} = \Gamma_3(\alpha) \cap \Gamma_3(x)$ and consider the rank 3 diagram, then $x \in D_3^2$. We can take $y \in D_2^3$ such that $\Gamma_3(\alpha) \cap \Gamma_1(x) \subseteq \Gamma_1(y)$ because $\Gamma_3(\alpha) \cap \Gamma_1(x) \subseteq D_2^3 \simeq K_{t\times 2}$, $\Gamma_3(\alpha) \cap \Gamma_1(x)$ is not a clique and $b_2 \leq t$.

Let A, B be as above. Then $A = \Gamma_3(\alpha) \cap \Gamma_1(x)$. Let $\{z\} = \Gamma_3(x) \cap \Gamma_3(y)$. Then $z \in D_2^1 \cup D_2^2 \cup D_1^2$.

If $z \in D_2^1$, then by an arguement similar to that in the proof of Claim 5, $A \cup \{y\} \subseteq \Gamma_3(\alpha) \cap \Gamma_3(z)$. This is impossible because A is not a clique and $\Gamma_3(\alpha) \cap \Gamma_3(z)$ is a clique.

If $z \in D_2^2 \cup D_1^2$, then similarly all vertices in A are at distance 2 from z. Since $|A| = b_2$, this contradicts Claim 3.

Hence we have the assertion. □

Proof of Proposition 5.1: Consider the rank 3 diagram. If D_1^3 ia a clique, then $\{\beta\} \cup D_1^3$ is a clique of size $a_3 + 1$. This contradicts $\Gamma_3(\alpha)$ being diameter 2. So D_1^3 is not a clique.

Take $x \in D_3^1$. Since $\Gamma_3(x) \cap \Gamma_3(\alpha)$ is a clique, $\Gamma_3(x) \cap \Gamma_3(\alpha) \subseteq D_1^3 \cup \{\beta\}$. D_1^3 is not a clique, so we take $y \in D_1^3$ such that $\partial(x, y) = 2$. Let $\{z\} = \Gamma_3(x) \cap \Gamma_3(y)$, then we know $z \in D_2^2$.

Claim. $\Gamma_3(z) \subseteq D_3^1 \cup D_2^1 \cup D_2^2 \cup D_1^2 \cup D_1^3$.

Since $p_{3,3}^2 = 1$, $\{x\} = \Gamma_3(z) \cap \Gamma_3(\beta)$ and $\{y\} = \Gamma_3(z) \cap \Gamma_3(\alpha)$, We get the claim.

By Lemma 5.4, $\Gamma_3(z) \cap \Gamma_1(\alpha) \simeq \Gamma_3(z) \cap \Gamma_1(\beta) \simeq K_{s\times 2}$. So we can take $u \in \Gamma_3(z) \cap \Gamma_1(\alpha)$ such that $x \not\sim u$ and $v \in \Gamma_3(z) \cap \Gamma_1(\beta)$ such that $y \not\sim v$. Then $u \in D_2^1$ and $v \in D_1^2$. So $\partial(u, x) = \partial(u, y) = \partial(v, x) = \partial(v, y) = 2$, which contradicts Lemma 3.1. Therefore we have the assertion. □

6. Proof of Theorem 1.2

In this section we prove our main theorem. By Proposition 4.1 and 5.1, we may assume $d = 4$.

By Lemma 3.5(1), the rank 4 diagram becomes as in Figure 5.

Lemma 6.1 *Take every $x, y \in \Gamma$ with $\partial(x, y) = 3$. For every $u \in \Gamma_3(x) \cap \Gamma_4(y)$ and every $v \in \Gamma_4(x) \cap \Gamma_3(y)$, we have $u \sim v$. Moreover we have $b_3 = p_{4,3}^3$.*

Proof: It is clear from Lemma 3.4 with $i = 3$ and Lemma 2.7. □

Lemma 6.2 *Take every α, $\beta \in \Gamma$ with $\partial(\alpha, \beta) = 4$. For every $x \in \Gamma_2(\alpha) \cap \Gamma_4(\beta)$ and every $y \in \Gamma_4(\alpha) \cap \Gamma_2(\beta)$, we have $\partial(x, y) = 2$. Moreover we have $p_{4,2}^2 = \kappa_2$.*

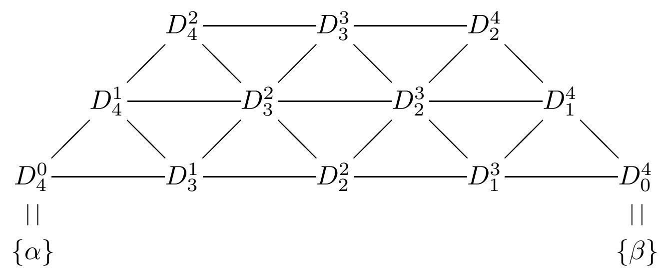

Figure 5.

Proof: Use the rank 4 diagram, then $x \in D_4^2$ and $y \in D_2^4$. Suppose $\partial(x, y) \neq 2$. Since $b_3 \geq 2$, we get $\partial(x, y) = 3$. Consider $\Gamma_4(x)$, then we know

$$\Gamma_4(x) \subseteq D_2^2 \cup D_2^3 \cup D_1^3 \cup D_1^4 \cup \{\beta\}.$$

Similarly, we get

$$\Gamma_4(y) \subseteq D_2^2 \cup D_3^2 \cup D_3^1 \cup D_4^1 \cup \{\alpha\}.$$

Suppose $\Gamma_4(x) \cap \Gamma_1(y) \subseteq \Gamma_4(x) \cap D_1^4$. Then

$$\begin{aligned} p_{4,4}^2 = |\Gamma_4(\alpha) \cap \Gamma_4(x)| &= |\Gamma_4(x) \cap (D_1^4 \cup \{\beta\})| \\ &\geq |\Gamma_4(x) \cap \Gamma_1(y)| + |\{\beta\}| \\ &= b_3 + 1 > p_{4,3}^3. \end{aligned}$$

This contradicts Lemma 2.1(3) with $i = 2$. So there is $z \in D_2^3$ such that $z \in \Gamma_4(x) \cap \Gamma_1(y)$.

Suppose there is no vertex u in $\Gamma_4(x) \cap D_1^3$ such that $\partial(y, u) = 3$. Then

$$\Gamma_4(x) \cap \Gamma_3(y) \subseteq (D_2^2 \cup D_2^3) \cap \Gamma_4(x).$$

From Lemma 6.1, $|\Gamma_4(x) \cap \Gamma_3(y)| = p_{4,3}^3 = b_3 \geq 2$. So there are at least two vertices γ, $\delta \in \Gamma_4(x) \cap \Gamma_3(y)$. Then $\partial(\gamma, \beta) = \partial(\delta, \beta) = 2$ and $\partial(\gamma, z) = \partial(\delta, z) = 2$. This contradicts Lemma 3.1. So there is $u \in \Gamma_4(x) \cap D_1^3$ such that $\partial(u, y) = 3$. Similarly there is $v \in \Gamma_4(y) \cap D_3^1$ such that $\partial(v, x) = 3$. Then $u \not\sim v$.

Now consider the rank 3 diagram with respect to (x, y). Then $u \in D_3^4$, $v \in D_4^3$ with $u \not\sim v$. This contradicts Lemma 6.1.

Finally,

$$p_{4,2}^2 = |\Gamma_4(\alpha) \cap \Gamma_2(x)| = |D_2^4| = \kappa_2.$$

□

Lemma 6.3 *We have the following.*
(1) $p^2_{4,4} = 1$ *and* $k_2 = \kappa_2(1 + \kappa_1 + \kappa_2)$.
(2) $c_2 = \kappa_2 - 2$.
(3) *For every edge* $u \sim v$, $\Gamma_4(u) \cap \Gamma_4(v)$ *is a clique.*

Proof: (1) In the rank 4 diagram take $\gamma \in D^2_2$. We can take $x \in \Gamma_4(\beta) \cap \Gamma_4(\gamma)$ and $y \in \Gamma_4(\alpha) \cap \Gamma_4(\gamma)$. Then $x \in D^2_4$ and $y \in D^4_2$. By Lemma 6.2, $\partial(x, y) = 2$. Hence there is a unique vertex $z \in \Gamma_4(\gamma)$ such that $\partial(x, z) = \partial(y, z) = 2$ by our assumption. Since $z \in \Gamma_4(\gamma) \cap \Gamma_2(y)$ and $\alpha \in \Gamma_2(\gamma) \cap \Gamma_4(y)$, we get $\partial(\alpha, z) = 2$ from Lemma 6.2. Similarly we get $\partial(\beta, z) = 2$. We have $z \in D^2_2$. Suppose $p^2_{4,4} \geq 2$, then we take $w \in \Gamma_4(\alpha) \cap \Gamma_4(\gamma)$ with $y \neq w$. Then $\partial(x, z) = \partial(y, x) = \partial(y, z) = \partial(w, x) = \partial(w, z) = 2$, which contradicts Lemma 3.1. So we have $p^2_{4,4} = 1$. Since $k_2 = k_2 p^2_{4,4} = k_4 p^4_{4,2}$, we have $k_2 = \kappa_2(1 + \kappa_1 + \kappa_2)$.
(2) Since $\Gamma_4(\gamma)$ is μ-closed and $e(D^2_2, D^3_3) = 0$,

$$\Gamma_1(y) \cap \Gamma_1(z) \subseteq \Gamma_4(\gamma) \cap D^3_2 \subseteq \Gamma_4(\gamma) \cap (D^2_2 \cup D^3_2 \cup D^4_2) \subseteq \Gamma_4(\gamma) \cap \Gamma_2(x) \simeq K_{t\times 2}.$$

So we have $c_2 = \kappa_2 - 2$.
(3) It is clear from Lemma 2.2. □

Lemma 6.4 *In the rank* 4 *diagram, for every* $x \in D^2_4 \cup D^3_3$ *and every* $u \in D^3_2$*, we have* $\partial(x, u) \neq 4$.

Proof: Suppose $\partial(x, u) = 4$. If $x \in D^2_4$, then $\alpha \in \Gamma_2(x) \cap \Gamma_4(\beta)$ and $u \in \Gamma_4(x) \cap \Gamma_2(\beta)$. By Lemma 6.2, we have $\partial(\alpha, u) = 2$. This contradicts $\alpha \in D^3_2$. If $x \in D^3_3$, similarly by Lemma 6.1, we have a contradiction. □

Lemma 6.5 *In the rank* 4 *diagram, for any* $u \in D^3_2$, $e(u, D^2_2) \neq 0$.

Proof: Take any $u \in D^3_2$.
Claim 1. $\Gamma_4(u) \subseteq D^1_4 \cup D^1_3 \cup D^2_3 \cup D^2_2 \cup D^3_2$.
Take any $v \in \Gamma_4(u)$, then $1 \leq \partial(\alpha, v) \leq 3$, $2 \leq \partial(\beta, v) \leq 4$. If $v \in D^2_4 \cup D^3_3$, then by Lemma 6.4, $\partial(v, u) \neq 4$. Hence we get the claim.
Claim 2. $\Gamma_4(u) \cap D^2_2 \neq \phi$.
By Lemma 3.6, 6.1 and 6.2,

$$\begin{aligned} |D^3_2 \cap \Gamma_4(u)| &= |\Gamma_3(\alpha) \cap \Gamma_4(u)| = p^3_{4,3} = b_3 \\ |(D^2_2 \cup D^3_2) \cap \Gamma_4(u)| &= |\Gamma_2(\beta) \cap \Gamma_4(u)| = p^2_{2,4} = \kappa_2 > b_3. \end{aligned}$$

Hence we have $\Gamma_4(u) \cap D^2_2 \neq \phi$.
Take $\gamma \in \Gamma_4(u) \cap D^2_2$ and take x, y, $z \in \Gamma_4(\gamma)$ as in the proof of Lemma 6.3. Then $y \sim u \sim z$ because $y, u, z \in \Gamma_4(\gamma) \cap \Gamma_2(x) \simeq K_{t\times 2}$ and $\partial(y, z) = 2$. Therefore we get $e(u, D^2_2) \neq 0$. □

Lemma 6.6 *Take $\alpha, \gamma \in \Gamma$ with $\partial(\alpha,\gamma) = 3$. In rank 3 diagram with respect to (α,γ), for every edge $x \sim y$ in D_1^4, $\Gamma_4(\alpha) \cap \Gamma_2(x) \cap \Gamma_2(y) \subseteq D_3^4$.*

Proof: Suppose there is $\beta \in \Gamma_4(\alpha) \cap \Gamma_2(x) \cap \Gamma_2(y)$ such that $\beta \in D_2^4$. Consider the rank 4 diagram. Then $\gamma \in D_2^3$, x, $y \in D_2^4$, $\gamma \sim x$, $\gamma \sim y$ and $x \sim y$. By Lemma 6.5, there is $\delta \in D_2^2$ such that $\delta \sim \gamma$. Then $\Gamma_4(\alpha) \cap \Gamma_4(\delta) = \phi$, which contradicts $p_{4,4}^2 \neq 0$. □

Lemma 6.7 *For every $\alpha \in \Gamma$ and every edge $\beta \sim \gamma$ in $\Gamma_4(\alpha)$, there is $\delta \in \Gamma_3(\alpha)$ such that $\delta \sim \beta$, $\delta \sim \gamma$.*

Proof: Consider the rank 4 diagram. Then $\gamma \in D_1^4$.
Claim 1. $\Gamma_4(\gamma) \subseteq \{\alpha\} \cup D_4^1 \cup D_3^1 \cup D_3^2$.

For every $x \in \Gamma_4(\gamma)$, $\partial(\alpha, x) \leq 2$ and $\partial(\beta, x) \geq 3$. If $\partial(\beta, x) = 4$, then $\alpha \sim x$ by Lemma 6.3(3). Hence we get the claim.

Take $y \in \Gamma_2(\alpha) \cap \Gamma_4(\gamma)$. Then by claim, $y \in D_3^2$. Since $\gamma \in \Gamma_4(y) \cap \Gamma_1(\beta)$ and $b_3 \geq 2$, we can take $\delta \in \Gamma_4(y) \cap \Gamma_1(\beta)$ such that $\delta \neq \gamma$. Then $\delta \in D_1^3$ because $\{\gamma\} = \Gamma_4(\alpha) \cap \Gamma_4(y)$. By Lemma 3.4 with $i = 3$, $\gamma \sim \delta$. □

Lemma 6.8 *For every $\alpha \in \Gamma$, $\Gamma_4(\alpha)$ is a strongly regular graph with intersection array*

$$\iota(\Gamma_4(\alpha)) = \left\{ \begin{array}{ccc} * & 1 & \kappa_1 - \kappa_2 + 2 \\ 0 & \frac{\kappa_1^2 - \kappa_1 - \kappa_1\kappa_2 + \kappa_2^2 - 2\kappa_2}{\kappa_1} & \kappa_2 - 2 \\ \kappa_1 & \frac{\kappa_2(\kappa_1 - \kappa_2 + 2)}{\kappa_1} & * \end{array} \right\},$$

and $p_{2,2}^1(\Gamma_4(\alpha)) = b_3$.

Proof: Let $\Delta = \Gamma_4(\alpha)$. It is clear that $k(\Delta) = \kappa_1$, $k_2(\Delta) = \kappa_2$, $c_2(\Delta) = \kappa_1 - \kappa_2 + 2$. We only need to show $|\Delta_2(x) \cap \Delta_1(y)|$ is constant for every edge $x \sim y$ in Δ. To show this, we prove $|\Delta_2(x) \cap \Delta_2(y)|$ is constant as $|\Delta_2(x) \cap \Delta_1(y)| + |\Delta_2(x) \cap \Delta_2(y)| = \kappa_2$.

By Lemma 6.7, there is $\gamma \in \Gamma_3(\alpha)$ such that $\gamma \sim x$, $\gamma \sim y$. Consider the rank 3 diagram with respect to (α,γ). Then x, $y \in D_1^4$. By Lemma 6.1 and 6.6,

$$|\Delta_2(x) \cap \Delta_2(y)| = |D_3^4| = b_3.$$

So $|\Delta_2(x) \cap \Delta_2(y)|$ is constant for every edge $x \sim y$ in Δ. >From $k(\Delta)b_1(\Delta) = k_2(\Delta)c_2(\Delta)$, we have $b_1(\Delta) = \frac{\kappa_2(\kappa_1 - \kappa_2 + 2)}{\kappa_1}$. □

Now we complete the proof of Theorem 1.2.

Proof of Theorem 1.2:
By Lemmas 6.3, 3.5(2) and 6.8, $\kappa_2 - 2 = c_2 = c_2(\Delta) = \kappa_1 - \kappa_2 + 2$. So

$$\kappa_1 = 2\kappa_2 - 4.$$

As $\kappa_2 = 2t$, then the intersection array of $\Delta = \Gamma_4(\alpha)$ becomes

$$\iota(\Delta) = \left\{ \begin{array}{ccc} * & 1 & 2(t-1) \\ 0 & 3t-5 & 2(t-1) \\ 4(t-1) & t & * \end{array} \right\}.$$

Claim 1. $c_2 = 2(t-1)$, $a_4 = 4(t-1)$, $k_4 = 6t-3$, $k_2 = 2t(6t-3)$ and $b_3 = t$.

$c_2 = c_2(\Delta)$, $a_4 = k(\Delta)$, $k_4 = 1 + k(\Delta) + k_2(\Delta)$, $k_2 = \kappa_2(1 + \kappa_1 + \kappa_2)$. Since $b_3 = p^1_{2,2}(\Delta) = k_2(\Delta) - p^1_{2,1}(\Delta) = t$, therefore we get the claim.

Claim 2. $b_2 = 4(t-1)$, $c_4 = 8(t-1)$, $k = 12(t-1)$ and $c_3 = t^2$.

By Lemma 6.2, $p^2_{4,2} = 2t$. By Lemma 4.1.7 of [3], $p^2_{4,2} = \frac{b_2 b_3}{c_2}$. So we get $b_2 = 4(t-1)$. By Lemma 6.1, $p^3_{4,1} = p^3_{4,3}$. So $p^4_{1,3} = p^4_{3,3}$. Consider the rank 4 diagram, then

$$e(D^3_3, D^4_2) = p^4_{3,3} b_3 = c_4 b_3$$

and

$$e(D^3_3, D^4_2) = p^4_{4,2} b_2 = 2t b_2.$$

Hence we get $c_4 = 8(t-1)$ and $k = 12(t-1)$. Since $k_2 b_2 b_3 = k_4 c_4 c_3$, we get $c_3 = t^2$.

Claim 3. $a_1 = 5t - 7$, $b_1 = 7t - 6$ and $t = 3$.

In the rank 4 diagram, take any $\gamma \in D^4_1$. Then

$$e(D^3_1, D^4_1) = p^4_{3,1}(b_3 - 1) = 8(t-1)^2$$

and

$$e(D^3_1, D^4_1) = p^4_{4,1}\big(a_1 - a_1(\Delta)\big) = 4(t-1)\big(a_1 - (3t-5)\big).$$

So we get $a_1 = 5t - 7$ and $b_1 = 7t - 6$. Since $kb_1 = k_2 c_2$, we get $t = 3$.

By these claims, we know all the intersection numbers of Γ. By the uniqueness ([4] and [12]), we get

$$\Gamma \simeq J(10, 4).$$

Therefore we have completed the proof of Theorem 1.2. □

7. Remarks

In Theorem 1.2 we assume $t \geq 2$, i.e., $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is not a coclique of size 2. If the assumption $t \geq 2$ is removed, does any other graph but $J(10, 4)$ appear? More generally, for every $\alpha, \beta \in \Gamma$ with $\partial(\alpha, \beta) = d$, which distance-regular graph satisfies $d \geq 3$, $h = 2$ and $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is a coclique of size s? In this situation we do not have a complete answer, but in the case $s = 2$, i.e., $t = 1$ in Theorem 1.2, there is only one graph.

Lemma 7.1 *Let Γ be a distance-regular graph with $d \geq 3$ and height $h = 2$. Suppose $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is a coclique of size s for every $\alpha, \beta \in \Gamma$ with $\partial(\alpha, \beta) = d$. Then $\Gamma_d(\alpha)$ is a coclique of size $s + 1$ for every $\alpha \in \Gamma$.*

Proof: Suppose $a_d = p^d_{d,1} \neq 0$. Then in the rank d diagram, for every $\gamma \in D^d_2$ and every $\delta \in D^d_1$, $\gamma \sim \delta$. So we know $\Gamma_d(\alpha) \simeq K_{\tau\times(s+1)}$ with $\tau \geq 2$ for every $\alpha \in \Gamma$. A. Hiraki and H. Suzuki showed that there is no such graph. (See Appendix of [13].)

We may assume $a_d = 0$. In this case we get $\Gamma_d(\alpha)$ is a coclique of size $s+1$ for every $\alpha \in \Gamma$. □

Lemma 7.2 *Let Γ be a distance-regular graph with $d \geq 3$ and height $h = 2$. Suppose $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is a coclique of size 2 for every α, $\beta \in \Gamma$ with $\partial(\alpha,\beta) = d$. Then the intersection array of Γ becomes*

$$\iota(\Gamma) = \left\{ \begin{array}{cccc} * & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & * \end{array} \right\}.$$

Proof: By the previous lemma, $\Gamma_d(\alpha)$ is a coclique of size 3 for every $\alpha \in \Gamma$. Since $2 < k < k_2$ and k_2 divides $k_d p^d_{d,2} = 6$, we get $k_2 = 6$. So $k = 3$ or 4 or 5. Since $kb_1 = k_2c_2 = 6c_2$ and $c_2 < k$, $k \neq 5$. Since $kb_1 \cdots b_{d-1} = k_dc_dc_{d-1}\cdots c_1$ and $c_d = k$, 3 divides $b_1 \cdots b_{d-1}$. So $k \neq 3$. Hence $k = 4$. Then from $kb_1 = k_2c_2$, we have $b_1 = 3$ and $c_2 = 2$. Since $c_2 = 2$, $c_i \geq 3$ $(3 \leq i \leq d-1)$. (See 5.4.1 of [3].) By using $k_2b_2\cdots b_{d-1} = k_dc_dc_{d-1}\cdots c_3$, $c_i \geq 3$ $(3 \leq i \leq d-1)$ and $b_i \leq 2$ $(2 \leq i \leq d-1)$, we have $d = 3$ and $b_2 = 2$. Thus we have the assertion. □

Remarks. 1. The array in Lemma 7.2 uniquely determines a graph. (See [11] or Theorem 7.5.3 of [3].)
2. Every bipartite distance-regular graph Γ with $h = 2$ satisfies $\Gamma_d(\alpha) \cap \Gamma_2(\beta)$ is a coclique for every α, $\beta \in \Gamma$ with $\partial(\alpha,\beta) = d$.
3. K. Nomura conjectured that there is no bipartite distance-regular graph Γ with diameter $d \geq 4$ and height $h = 2$. (Conjecture 1.2 of [5].) But the following counter example is known for $d = 4$. A graph with the array

$$\iota(\Gamma) = \left\{ \begin{array}{ccccc} * & 1 & 5 & 12 & 15 \\ 0 & 0 & 0 & 0 & 0 \\ 15 & 14 & 10 & 3 & * \end{array} \right\}$$

satisfies $h = 2$. (See Section 6 of [8].) In [10] H. Suzuki showed that this conjecture is true if d is odd. If d is even, then each bipartite half of Γ has height 1.

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