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The Design of Optimal Insurance Contracts: A Topological Approach

SANDRINE SPAETER AND PATRICK ROGER
University Louis Pasteur, LARGE-38 Boulevard d'Anvers, 67 070 Strasbourg, France and CREST/ENSAE, Malakoff, France

Abstract

This article deals with the optimal design of insurance contracts when the insurer faces administrative costs. If the literature provides many analyses of risk sharing with such costs, it is often assumed that these costs are linear. Furthermore, mathematical tools or initial conditions differ from one paper to another. We propose here a unified framework in which the problem is presented and solved as an infinite dimensional optimization program on a functional vector space equipped with an original norm. This general approach leads to the optimality of contracts lying on the frontier of the indemnity functions set. This frontier includes, in particular, contracts with a deductible, with total insurance and the null vector. Hence, we unify the existing results and point out some extensions.

Key words: infinite dimensional problem, angular norm, coinsurance, deductible, increasing costs

1. Introduction

In insurance theory, optimal risk sharing between two agents, one at least being risk averse, has been the subject of many papers for the last thirty years[1]. Under different initial conditions about the behavior of the economic agents, the information they get, the type of damage (such as fire, housebreaking, or civil liability), or on the form of the administrative costs borne by the insurance company, the most frequently observed contracts consist of a deductible, coinsurance, or an upper limit.

The insurer administrative costs comprise the financial charges he bears in addition to the indemnities he pays when a damage occurs. Hence, the design of the insurance policy closely depends on the assumptions related to these costs (such as convexity, concavity, linearity, or existence of fixed costs).

Arrow [1963], in his seminal paper on this subject, shows that the indemnity function involves a deductible with full insurance beyond it when costs are linear and the insurer is risk neutral. Raviv [1979] also gets a straight deductible but with coinsurance above it[2]. His result is essentially due to the strict convexity of the cost function and the insurance constraint.

Gollier [1987] uses the same framework as Raviv—that is, the optimal control. The cost function he considers is linear with respect to indemnities and admits some sunk costs and fixed costs per claim. The latter are equal to zero if no indemnity is paid; they are positive and constant otherwise. Hence, a new design of optimal contract appears in Gollier's article: the partially disappearing deductible[3]. Furthermore, he obtains the same result as Arrow [1963] when the per-claim fixed costs equal zero and also one of Raviv's conclusions: there is no deductible when costs are constant. Let us note that the costs that have been considered by the authors quoted above are individual ones: the insurer acts as if he had only one customer. He takes into account only the indemnity function of one client and not the sum of claims he would have to pay his customers as a whole. This remark is especially important when we take an interest in Huberman, Mayers, and Smith's work [1983]. These authors, by using the calculus of variations and by assuming costs are concave, get a disappearing deductible that *completely* disappears. The insured can obtain complete hedging above a specified amount of damage. This result holds because concavity induces some kind of economies of scale. Economies of scale simply mean that the marginal cost decreases when indemnities increase. But this property does not imply aggregate economies of scale[4] (for the cost function of the company). Moreover, Huberman, Mayers, and Smith stress the fact that the disappearing deductible is optimal only if there is no moral hazard. This hypothesis has been made in all articles quoted above.

Furthermore, all these works have been realized in the expected utility maximization framework and indemnity functions differ from one another by the pattern of the administrative costs taken into account. However, different mathematical tools or initial conditions have been used and there is no classification of optimal insurance contracts with respect to the design of costs. This point constitutes the aim of our theoretical study. We want to end with a uniform presentation of all existing results by using a unique optimality condition. To do so, we embed, in the second section, the indemnity functions set, $\mathcal{I}$, in an adequate topological vector space, endowed with the angular norm topology. In this context, $\mathcal{I}$ is shown to be closed and convex and optimal contracts lie on the frontier of $\mathcal{I}$. This result is obtained under usual assumptions, such as risk aversion of the insured and risk neutrality of the insurer. $\mathcal{I}$'s frontier contains, among others, indemnity functions involving either a deductible, or full coverage on a nonempty interval. The optimality of a nonzero deductible when costs are strictly increasing is also pointed out. Section 3 concludes this article.

2. The insured optimization problem

An insurance contract is defined as a pair $(I(X), P(I(X)))$; X is the random loss initially borne by the insured. $I(X)$ is an indemnity function and $P(I(X))$ is the insurance premium. This function I must verify the insurance constraint: indemnities are neither negative nor greater than the realization of the damage X. Let us consider a risk-neutral insurer who acts on a perfectly competitive insurance market. This hypothesis implies that the premium $P(I(X))$ is defined by

$$P(I(X)) = E_P[I(X) + c(I(X))], \tag{1}$$

where $c(I(X))$ is the random administrative cost the insurer faces when the indemnity equals $I(X)$. E_P is the expectation operator with respect to a probability distribution P defined on a measurable space $(\Omega, \mathcal{A})$. Function c is supposed to be continuous, strictly increasing, and twice differentiable. The agent who wants to take out insurance maximizes his expected utility of final wealth $E_P[U(\tilde{w}_f)]$ where U is a Von Neumann-Morgenstern utility function, strictly increasing, strictly concave, and twice differentiable. We note w the certain initial wealth of the insured with $w \in \mathbb{R}^{*+}$. The risk of damage X is supposed to be nonnegative and bounded by a finite constant T; it belongs to $\mathcal{L}^2(\Omega, \mathcal{A}, P)$, the space of square-integrable random variables on $(\Omega, \mathcal{A}, P)$. The insured faces only one insurable risk[5].

The standard optimization program is

$$\max_I E_P[U(w - P(I(X)) + I(X) - X)]$$

$$\text{subject to} \begin{cases} P(I(X)) = E_P[I(X) + c(I(X))] \\ 0 \leq I(X) \leq X \end{cases}.$$

This optimization problem is not a trivial one, since the decision variable is actually a function, $I(X)$, depending on a random variable X and moving in an infinite dimensional space. Accordingly, there exists an infinite number of available indemnity functions that verify the insurance constraint. Therefore, to be able to get some general results about the design of optimal contracts, one has to take into account all these functions. First, we propose to rigorously specify the set of available solutions and its topological properties. These will be adapted to the problem we focus on and will permit, in a second stage, to generalize the results existing in the literature thanks to a unique optimality condition.

We note[6] $C^0\,[0, T]$ the space of numerical functions defined on $[0, T]$, continuous on this interval, equal to zero at zero, continuously differentiable at zero and having a finite number of nondifferentiability points. These heavy but natural assumptions preclude pathological cases such as continuous but nowhere differentiable functions. Since we are interested in indemnity functions, these hypotheses are not really restrictive. In fact, contracts in the literature or on insurance markets exhibit at most two nondifferentiability points (for example in case of a completely disappearing deductible).

Let $\mathcal{I} = \{f \in C^0\,[0, T]/0 \leq f(x) \leq x\}$ be the set of all available indemnity functions. $\mathcal{I}$ is a subset of $C^0\,[0, T]$ and the angular norm we define on this space, in Lemma 1 hereafter, reflects the insurance constraint: $0 \leq f(x) \leq x$. According to Lemma 2, $\mathcal{I}$ is a closed, bounded, and convex subset of $C^0\,[0, T]$: these properties are particularly well-timed for working in an infinite dimensional space.

Lemma 1: *The mapping defined on $C^0\,[0, T]$ with values on $\mathbb{R}^+$, which associates $\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{x}$ to f, is a norm.*

The proof is given in the appendix.

This norm is well suited for the problem studied here. In fact, let e be the identity function defined by $e(x) = x$, and consider the sets $\mathcal{J}_1$ and $\mathcal{J}_2$, which verify

$$\begin{cases} \mathcal{J}_1 = \{f \in \mathcal{I}/\|f\| = 1\} \\ \mathcal{J}_2 = \{f \in \mathcal{I}/(e - f) \in \mathcal{J}_1\} \end{cases}.$$

The set $\mathcal{J}_1$ contains all functions displaying at least one strictly positive amount of damage that is completely indemnified. $\mathcal{J}_2$ contains the functions that admit at least one strictly positive point of noninsurance: in particular, functions with a strictly positive deductible. Thus, the frontier of $\mathcal{I}$ is defined by the gathering of both sets: $\partial\mathcal{I} = \mathcal{J}_1 \cup \mathcal{J}_2$ (as shown in Lemma 2 hereafter). Accordingly, functions having only zero as common point with functions of $\mathcal{J}_1$ or $\mathcal{J}_2$ display some pure coinsurance; in other words, they characterize contracts without deductible or full insurance. They never touch the first bisecting line or the horizontal axis, except at zero. The following lemma formalizes this reasoning.

Lemma 2: *The set $\mathcal{I}$ is closed, bounded, and convex. Its interior, denoted $\mathcal{I}^0$, is defined by*

$$\mathcal{I}^0 = (\mathcal{J}_1 \cup \mathcal{J}_2)^C.$$

The properties of $\mathcal{I}$ are demonstrated in the appendix.

Then, contracts usually encountered in insurance theory can be defined and characterized with respect to the angular norm.

Definition 1: *Given a continuous indemnity function I that verifies the insurance constraint*,

1. *I admits at least one point of full insurance if* $\|I\| = 1$.
2. *I displays pure coinsurance if* $\|I\| \times \|e - I\| < \min(\|I\|; \|e - I\|)$.
3. *A deductible function I is such that*: $\|e - I\| = 1$.
4. *A completely disappearing deductible verifies*: $\|I\| \times \|e - I\| = 1$.

Figure 1 shows some examples of standard indemnity functions.

Definition 1.3 calls for a comment. Despite a standard deductible verifies $\|e - I\| = 1$, the opposite is not always true. Indeed, a function verifying this property displays at least one positive point of noninsurance, but this does not mean that it squares with a deductible. There may be several nonoverlapping intervals on which the indemnity equals zero, since no monotonicity assumption has been made. Thus, the standard deductible, as defined by Arrow [1963], is a special case of our definition. Definition 1.2 specifies that functions with coinsurance do not belong to the frontier of $\mathcal{I}$.

We are now ready to "localize" optimal contracts in the set $\mathcal{I}$ of indemnity functions by solving the general optimization program of the insured. The so-defined properties of $\mathcal{I}$ allow us to use some mathematical tools adapted to the infinite dimensional problem we are interested in.

In our framework, the problem can be written

$$\max_{I \in \mathcal{I}} E_P[U(w - X - P(I(X)) + I(X))] = \max_{I \in \mathcal{I}} F(I). \tag{2}$$

As stated by Lemma 2, $\mathcal{I}$ is closed and bounded; in such a set, optimality conditions usually distinguish between solutions in the interior of $\mathcal{I}$ (first-order conditions are equalities) and corner solutions (inequalities). To formulate these conditions in our general approach, we briefly recall the notion of the G-differential of a functional, which plays the role of the gradient in infinite dimensional problems[7].

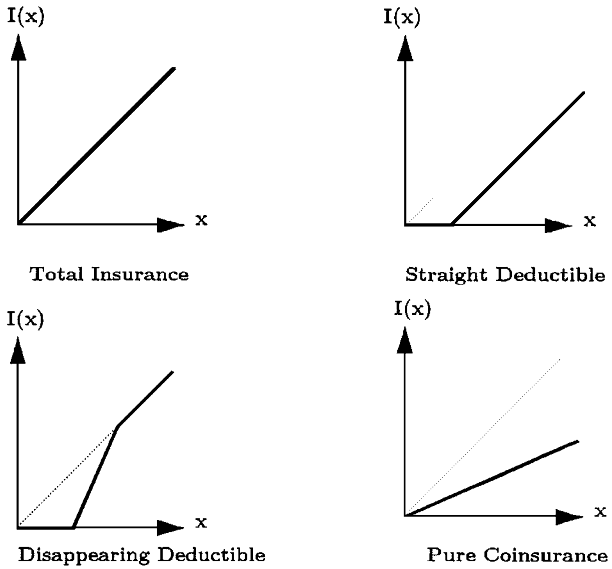

Figure 1. Standard indemnity functions.

Definition 2: *Let V be a normed vector space and J a function on V. J displays a directional derivative (or a Gâteaux-differential) at $v \in V$ in the direction $\phi \in V$, if*

$$\frac{J(v + \alpha\phi) - J(v)}{\alpha}$$

has a limit when $\alpha \to 0$ (in $\mathbb{R}$). This limit is written $\partial J(v, \phi)$. If $\forall \phi \in V : \partial J(v, \phi)$ exists, then J is said to be Gâteaux-differentiable (G-differentiable) on $v \in V$.

A necessary optimality condition can then be written by means of directional derivatives.

Lemma 3: *Let $J(v)$ be a function on V G-differentiable at v^* in the direction $\phi \in V$ and $U \subset V$ convex. A necessary condition for $v^* \in U$ to be a maximum of J on U is, for $\alpha \in [0, 1]$,*

$$\partial J(v^*, \phi) = \lim_{\alpha \downarrow 0} \frac{J(v^* + \alpha\phi) - J(v^*)}{\alpha} \leq 0, \quad \forall \phi \in V \text{ such that } \partial J(v^*, \phi) \text{ exists.}$$

This inequality means that v^* is optimal if every available perturbation ϕ induces a value of J smaller than or equal to the one induced by v^*.

The following proposition translates Lemma 3 in the context of our maximization problem and describes the optimality condition. To simplify the notations, we will note $\theta(I, X) = w - X - P(I(X)) + I(X)$.

Proposition 1:

(a) *Let $F'(I,\phi)$ be the G-differential of F at point I in the direction ϕ. One gets*

$$F'(I,\phi) = -E_P[\{E_P[\phi(X)(1+c'(I(X)))] - \phi(X)\}U'(\theta(I,X))].$$

(b) *The first-order optimality condition reduces to*

$$\begin{aligned}&\forall\phi \in C^0\,[0,T]\ \textit{such that} F'(I^*,\phi)\ \textit{exists}\\ &E_P[\phi(X)(1+c'(I^*(X)))] \geq \frac{E_P[\phi(X)U'(\theta(I^*,X))]}{E_P[U'(\theta(I^*,X))]}.\end{aligned} \tag{3}$$

Proof. Point (a) can be proved with few direct calculations:

$$\begin{aligned}&F(I+\alpha\phi) = E_P[U(w - X - P(I(X)+\alpha\phi(X)) + I(X) + \alpha\phi(X))]\\ \Leftrightarrow\ &F(I+\alpha\phi) = E_P[U(\theta(I,X) - P(I(X)+\alpha\phi(X)) + P(I(X)) + \alpha\phi(X))]\\ \Leftrightarrow\ &F(I+\alpha\phi) = E_P[U(\theta(I,X) - E_P[\alpha\phi(X) + c(I(X)+\alpha\phi(X)) - c(I(X))]\\ &\qquad\qquad + \alpha\phi(X))].\end{aligned}$$

A first-order Taylor series expansion of $c(I(X)+\alpha\phi(X))$ in the neighborhood of $I(X)$ leads to[8]

$$c(I(X)+\alpha\phi(X)) = c(I(X)) + \alpha\phi(X)c'(I(X)) + \epsilon(\alpha),$$

from which we deduce

$$F(I+\alpha\phi) = E_P[U(\theta(I,X) - E_P[\alpha\phi(X)(1+c'(I(X))) + \epsilon(\alpha)] + \alpha\phi(X))].$$

In the same way, the Taylor series expansion of U in the neighborhood of $\theta(I,X)$ is

$$\begin{aligned}&U(\theta(I,X) - E_P[\alpha\phi(X)(1+c'(I(X))) + \epsilon(\alpha)] + \alpha\phi(X))\\ &\quad = U(\theta(I,X)) + \{-E_P[\alpha\phi(X)(1+c'(I(X))) + \epsilon(\alpha)] + \alpha\phi(X)\}U'(\theta(I,X)) + \gamma(\alpha).\end{aligned}$$

Finally, we have

$$\begin{aligned}F(I+\alpha\phi) = E_P[U(\theta(I,X)) - \{&E_P[\alpha\phi(X)(1+c'(I(X))) + \epsilon(\alpha)]\\ &- \alpha\phi(X)\}U'(\theta(I,X)) + \gamma(\alpha)],\end{aligned}$$

with $\epsilon(\alpha) = o(\alpha)$ and $\gamma(\alpha) = o(\alpha)$.[9]

Consequently, according to the definition of $F'(I,\phi)$, we can write

$$F'(I,\phi) = -E_P[\{E_P[\phi(X)(1+c'(I(X)))] - \phi(X)\}U'(\theta(I,X))], \tag{4}$$

and point (a) is proved.

The second part of the proposition is quite obvious. The first-order condition of the problem is

$$F'(I^*, \phi) \leq 0 \quad \text{for every } \phi \in C^0[0, T] \text{ such that } F'(I^*, \phi) \text{ exists.} \tag{5}$$

Relations (4) and (5) lead to

$$E_P[\phi(X)(1 + c'(I^*(X)))] \geq \frac{E_P[\phi(X)U'(\theta(I^*, X))]}{E_P[U'(\theta(I^*, X))]}.$$

This concludes the proof of Proposition 1. □

Inequality (3) turns to an equality for any ϕ if I^* belongs to the interior of $\mathcal{I}$. Stated as (3), the first-order condition does not seem very intuitive. But it is equivalent to

$$E_P[\phi(X)(1 + c'(I^*(X)))E_P[U'(\theta(I^*, X))]] \geq E_P[\phi(X)U'(\theta(I^*, X))], \tag{6}$$

where the left member is the expected marginal cost of insurance and the right one squares with the expected marginal benefit, for any perturbation ϕ. Both are expressed in terms of utility. Finally, the insured simply wants to equalize his marginal cost and his marginal benefit, which is a well-known behavior in insurance theory. This tradeoff between marginal cost and marginal benefit reflects the agent process of choice since it takes simultaneously into account both effects of the indemnity variation on the expected utility: the premium and the net loss move together. It also leads here to a unique condition of optimality that holds in the frame of constant or linear costs as well as in the one of nonlinear costs.

Furthermore,

Lemma 4: *Equation* (6) *holds in equality for any* ϕ *if and only if*

$$(1 + c'(I^*(X)))E_P[U'(\theta(I^*, X))] = U'(\theta(I^*, x)), \quad \forall x > 0. \tag{7}$$

The proof is given in the appendix.

This result means that the marginal cost must equal the marginal benefit at any strictly positive amount of damage if coinsurance is optimal ($I^* \in \mathcal{I}^0$). Actually, equation (7) never holds whatever the design of costs may be. The solution, if it exists, belongs to the frontier of $\mathcal{I}$. This is formalized in the following proposition:

Proposition 2: *Let* I^* *be a solution of the optimization problem* (2): *it verifies* $I^* \in \partial\mathcal{I}$. *More precisely*,

(a) $c(\cdot) \equiv a, a \in \mathbb{R}^+ \Rightarrow \exists\, \bar{a} \in \mathbb{R}^{*+} \Big/ \begin{cases} a < \bar{a} \Rightarrow I^* \in \mathcal{J}_1(I^*(X) = X) \\ a > \bar{a} \Rightarrow I^* \in \mathcal{J}_2(I^*(X) = \vec{O}) \end{cases}$

(b) $c'(\cdot) > 0 \Rightarrow I^* \in \mathcal{J}_2(I^*(x) = 0, \forall x < D;\ D > 0)$.

Proof. First, let us suppose that $I^* \in \mathcal{I}^0$. From Lemma 4, we must have

$$(1 + c'(I^*(X)))E_P[U'(\theta(I^*, X))] = U'(\theta(I^*, x)), \quad \forall x > 0$$

or

$$(1 + c'(I^*(X))) = \frac{U'(\theta(I^*, x))}{E_P[U'(\theta(I^*, X))]}, \quad \forall x > 0.$$

Function $c(I(X))$ is, by assumption, constant or strictly increasing with respect to indemnities for any x. Then $(1 + c'(I(x)))$ is always equal to 1 or greater than 1. However, $\mathcal{K}(I^*, x) = \frac{U'(\theta(I^*, x))}{E_P[U'(\theta(I^*, X))]}$ is lower than 1 for $x = 0$ since $I^* \in \mathcal{I}^0$ and U is concave, and its expected value is worth 1. Hence, $\mathcal{K}(I^*, x)$ is lower than 1 for some states of nature and greater than 1 for others. Finally, c' and U' being continuous functions, one cannot have $(1 + c'(I^*(x))) = \mathcal{K}(I^*, x)$ for any strictly positive x: $I^* \in \partial\mathcal{I}$.

Point (a). The scalar $\bar{a}$ is such that every larger amount of fixed costs induces a marginal cost of insurance higher than the marginal benefit for any damage x. If constant costs are sufficiently small ($a < \bar{a}$), condition (7) states

$$E_P[U'(\theta(I^*, X))] = U'(\theta(I^*, x)), \quad \forall x > 0. \tag{8}$$

Since U is strictly concave and $I(0)$ equals zero, the solution of (8) is $I^*(X) = X$. Consequently, $\|I^*\| = 1$ and $I^* \in \mathcal{J}_1$. On the other hand, if $a > \bar{a}$, the agent does not go on the insurance market because entrance fees are too high. There is no insurance policy that induces a greater expected utility than the noncoverage: $I^* = \vec{O}$ and $I^* \in \mathcal{J}_2$. This proves point (a).

Point (b). Let I^* be an optimal function. Since the insurance constraint must be verified and the utility function is concave, the final wealth of the agent evaluated at $x = 0$ is the largest wealth he can obtain. Consequently, the marginal cost is strictly larger than the marginal benefit at this point when costs are increasing:

$$(1 + c'(0))E_P[U'(\theta(I^*, X))] > U'(w - P(I^*)).$$

$U'(\cdot)$ being continuous by assumption, it exists a strictly positive scalar D such that

$$\begin{cases} (1 + c'(0))E_P[U'(\theta(I^*, X))] > U'(w - P(I^*) - x), & \forall x < D \\ (1 + c'(0))E_P[U'(\theta(I^*, X))] = U'(w - P(I^*) - D) \end{cases}$$

In other words, the marginal cost of a slightly positive indemnity is larger than the marginal benefit it yields in terms of utility. Then the optimal solution must be such that $I^*(x) = 0$, $\forall x \leq D$ and $I^* \in \mathcal{J}_2$. This proves point (b) of Proposition 2. □

To illustrate Proposition 2, we propose the following three-dimensional example (see figure 2). Let us consider a space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and assume, without loss of generality, that $0 < X(\omega_1) < X(\omega_2) < X(\omega_3)$. Let C be the unit cube in $\mathbb{R}^3$ defined by

$$\mathcal{C} = \left\{(y_1, y_2, y_3)/y_i = \frac{I(X(\omega_i))}{X(\omega_i)}; i = 1, 2, 3\right\}.$$

For the fixed cost case, the values of $\frac{I^*(x)}{x}$ (where I^* is the optimal indemnity function) lie on the vertex (1, 1, 1) (point A on figure 2) if a is sufficiently small and on the vertex (0, 0, 0) (point B) if $a > \bar{a}$. For the general case $c'(\cdot) > 0$, one or two components of the vector of indemnities equal zero. Consequently, the solution is on a face or an edge of the cube (points C and D).

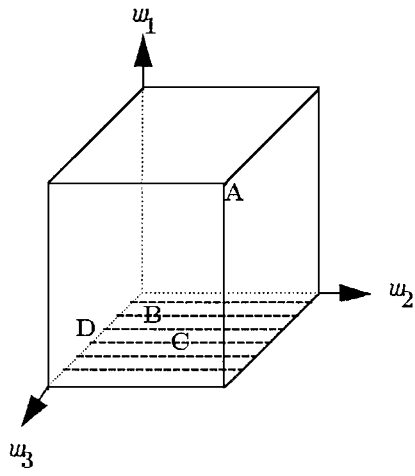

Figure 2. Example with three states of nature.

It is important to note that the characterization of the optimal solution is closely linked to the topological structure of $\mathcal{I}$. But $\mathcal{I}$ does not depend on the function to be optimized—here, the standard expected utility. This subset is closed, bounded, and convex whatever the decision rule may be. Its interior and frontier are well-defined: the standard optimization conditions lead to interior solutions (coinsurance) or corner ones (deductibles, full insurance, no insurance).

Let us go back to Proposition 2. Finally, it implies that pure coinsurance is never optimal when the insurer is risk neutral and when there is no moral hazard. Moreover, since the existing results have been, up to now, demonstrated in different ways and with different assumptions from one model of the literature to another, our approach leads to more general conclusions. They have been achieved thanks to restrictions as small as possible on the design of available indemnity functions or on the type of cost functions. In the absence of administrative costs, Mossin's result [1968] is obtained: full insurance is optimal when no loading factor is applied to the premium. Function I^* verifies $\|I^*\| = 1$ and belongs to $\mathcal{J}_1$. When costs are positive but constant, the optimal function displays either full insurance or no insurance at all, depending on whether the agent considers these fixed costs rather fair or not to take out some insurance. If he accepts them, the process of choice works as if there were no charge and his initial certain wealth equals $w - a$. Fixed costs can be considered as a lump sum entrance fee, as stated by Gollier [1987], and they have no direct incidence on the optimal insurance contract. Only a wealth effect persists and the optimal indemnity function involves full insurance when the agent decides to cover his risk: $I^* \in \mathcal{J}_1$.

Raviv's [1979] result related to the optimality of a strictly positive deductible when costs are increasing is also obtained thanks to the unique optimality condition: $I^* \in \mathcal{J}_2$.

Let us give the economic interpretation of these conclusions. The nonoptimality of full insurance, when the insurer faces increasing administrative costs, is a well-known result in insurance theory. In such a context, the marginal cost of insurance is strictly positive and the expected costs are borne by the insured through the premium. He will never recover them by means of indemnities. Thus, he prefers to bear some proportion of the risk for the benefit of a lower insurance premium. Full insurance having been moved aside, the agent must choose between the deductible contract and the pure coinsurance policy. Let us consider two damage amounts x_1 and x_2 such that x_1 (x_2) is supposed to be small (large) compared to the agent's initial wealth. He will be more sensitive to a one unit increase of the compensation when the damage equals x_2. Indeed, his final wealth, free of indemnities, decreases more, compared to his initial wealth, when the loss is worth x_2 instead of x_1. As administrative costs are increasing, he attaches a higher weight to the costs increase induced by the increase of insurance when the damage is not important. Consequently, he prefers not to be indemnified for small losses. The optimal contract displays a strictly positive deductible, the value of which depends on the risk aversion degree of the insured and on his initial wealth[10].

Finally, by devoting our attention only to the first derivative of the cost function, we are able to restrict the set of available indemnity functions to those with a deductible, when administrative costs are increasing. Nevertheless, at this stage of analysis, an infinite number of indemnity functions still remains available when the marginal cost is positive. We do not know the design of the optimal function beyond the deductible, since no hypothesis has been made about the variation of the marginal costs. We already know that a constant marginal cost—that is, linear costs—generates some full insurance beyond a straight deductible (Arrow [1963]). But if the second derivative of the cost function is not zero, the design of the optimal indemnity function beyond the deductible depends not only on the sign of $c''(\cdot)$ but also on the prudence behavior[11] of the agent as shown by Spaeter [1996].

3. Concluding remarks

Many of the results presented in this article are well known in the risk-sharing literature. However, they have been often obtained by using different mathematical tools or more restrictive initial conditions. Thus, the use of the angular norm topology constitutes an accurate and original approach of optimal insurance contracts designs in presence of administrative costs.

Looking for the optimal indemnity schedule is not an obvious optimization problem: the decision variable is actually a function depending on a random variable, the risk of damage, and it belongs to an infinite dimensional space. When this indemnity function must verify the insurance constraint, the angular norm nicely reflects this condition.

When there is no moral hazard, by considering any function defined on the vector space endowed with this norm, we have shown that the optimal contracts are on the frontier of the indemnity functions set. Hence, Proposition 2 brings up the nonoptimality of pure coinsurance. It is to be noted that, if costs equal zero, the insurance premium is actuarial and Mossin [1968] has shown that the agent chooses full insurance. This case can also be derived from our approach: the indemnity function belongs to the frontier. But it is useful to remember that Mossin's insurance contract is exogenously specified: the insured chooses a pure coinsurance coefficient α (equal to one in case of full insurance).

Thanks to the unique optimality condition, we have also proved that the solution, if it exists, always displays a deductible when costs are increasing with indemnities. But it can have, *a priori*, some very different designs and the analyses of Arrow [1963], Raviv [1979], and Huberman, Mayers, and Smith [1983] can be kept as special cases.

Finally, we have generalized the existing results and we have presented them in a unified way. We have also shown, in the framework of the angular norm topology, that an optimal contract comprises a deductible when the first derivative of the cost function is strictly positive. But in order to state precisely what happens beyond the deductible, the variation of marginal costs must be taken into consideration. This study is done by Spaeter [1996], where the same optimality condition is used, notably when costs are nonlinear.

Appendix

Proof of Lemma 1

First, let us note that the function $\sup_{x\neq 0}\frac{|f(x)|}{x}$ is well defined since the elements of $C^0\,[0,T]$ are differentiable at zero. The positiveness property is straightforward: $\|f\| = 0$ if and only if $f(x) = 0$ for any $x \in [0, T]$. The convexity inequality is verified for any function f and g belonging to $C^0\,[0,T]$, by writing

$$\begin{aligned}\|f+g\| &= \sup_{x\neq 0}\frac{|f(x)+g(x)|}{x} \leq \sup_{x\neq 0}\frac{|f(x)|+|g(x)|}{x}\\ &\leq \sup_{x\neq 0}\frac{|f(x)|}{x} + \sup_{x\neq 0}\frac{|g(x)|}{x} = \|f\| + \|g\|\end{aligned}$$

Finally, $\|cf\| = |c|\|f\|$ is trivially verified. □

Proof of Lemma 2

Let $(f_n, n \in \mathbb{N})$ be a sequence of elements of $\mathcal{I}$ that converges toward a limit f. According to the convergence definition, we have

$$\lim_n \|f_n - f\| = \lim_n \sup_{x\neq 0}\frac{|f_n(x) - f(x)|}{x} = 0.$$

For every $x \in\,]0, T]$, $\lim_n \frac{|f_n(x)-f(x)|}{x} = 0$ implies $0 \leq \frac{f(x)}{x} \leq 1$. Therefore, $\mathcal{I}$ is closed; the boundedness is deduced from the preceding mentioned inclusion $\mathcal{I} \subset B(\vec{O}, 1)$ and the convexity is straightforward.

Let us now consider f such that $0 < \|f\| < 1$ and note $2\alpha = \min(\min_{x\neq 0}|\frac{f(x)}{x}|; 1 - \|f\|)$. The ball $B(f;\alpha)$ is defined by

$$B(f,\alpha) = \{g \in C^0\,[0,T]/\|f-g\| \leq \alpha\}$$

with

$$\|f-g\| = \sup_{x\neq 0}\frac{|g(x)-f(x)|}{x}.$$

Let us suppose there exists x_0 such that $\frac{g(x_0)}{x_0} \geq 1$. Then we have $\|g\| \geq 1$ but $\frac{f(x_0)}{x_0} \leq \|f\|$, and this leads to $\sup_{x \neq 0} \frac{|g(x)-f(x)|}{x} \geq 2\alpha$. Finally, g does not belong to $B(f, \alpha)$.

With similar arguments, if there exists x_1 such that $\frac{g(x_1)}{x_1} \leq 0$, we have $\min_{x \neq 0} \frac{|f(x)-g(x)|}{x} \geq 2\alpha$ and a fortiori $\|f - g\| \geq 2\alpha$. Consequently, every element of $B(f, \alpha)$ is in $\mathcal{I}$ and f belongs to $\mathcal{I}^0$.

Now, let us consider a function f with a norm equal to 1. As $[0, T]$ is a compact set, there exists x_0 such that $\frac{f(x_0)}{x_0} = 1$. For every $\epsilon > 0$, the ball $B(f, \epsilon), \epsilon > 0$, contains an element g of $C^0[0, T]$, which is not in $\mathcal{I}$. To prove this point, let $\gamma > 0$ and define

$$
\begin{aligned}
&g(x) = f(x) \quad \text{if } x \notin\ ]x_0 - \gamma, x_0 + \gamma[\\
&g(x) = f(x) + \frac{\epsilon}{2}\left(\frac{x - x_0 + \gamma}{\gamma}\right) \quad \text{if } x \in [x_0 - \gamma, x_0[\\
&g(x) = f(x) - \frac{\epsilon}{2}\left(\frac{x - x_0 + \gamma}{\gamma}\right) \quad \text{if } x \in [x_0; x_0 + \gamma[.
\end{aligned}
$$

It is obvious that $d(f, g) = \|f - g\| = \frac{\epsilon}{2}$ and g is continuous. Consequently, any function the norm of which equals one is on the frontier of $\mathcal{I}$.

Suppose now that $\|f\| < 1$ and that there exists $x_1 \neq 0$ such that $f(x_1) = 0$. One must show, as above, that for every $\epsilon > 0$, the ball $B(f, \epsilon)$ contains a function g that is not in $\mathcal{I}$.

Using the same construction, we can define for $\gamma > 0$

$$
\begin{aligned}
&g(x) = f(x) \quad \text{if } x \notin\ ]x_1 - \gamma, x_1 + \gamma[\\
&g(x) = f(x) - \frac{\epsilon}{2}\left(\frac{x - x_1 + \gamma}{\gamma}\right) \quad \text{if } x \in [x_1 - \gamma, x_1[\\
&g(x) = f(x) + \frac{\epsilon}{2}\left(\frac{x - x_1 - \gamma}{\gamma}\right) \quad \text{if } x \in [x_1; x_1 + \gamma[.
\end{aligned}
$$

Still here $d(f, g) = \frac{\epsilon}{2}$ and $g(x_1) < 0$, then $g \notin \mathcal{I}$ and f belongs to the frontier of $\mathcal{I}$. □

Proof of Lemma 4

Let us state $\mathcal{K}(I^*, X) = \frac{U'(\theta(I^*,X))}{E_P[U'(\theta(I^*,X))]}$. Given $I^* \in \mathcal{I}^0$ and $c(\cdot)$ a continuous cost function such that $c'(\cdot) > 0$, Eq. (3) becomes

$$
\begin{aligned}
&E_P[\phi(X)(1 + c'(I^*(X)) - \mathcal{K}(I^*, X))] = 0,\\
&\quad \forall \phi \in C^0[0, T] \text{ such that } F'(I^*, \phi) \text{ exists.}
\end{aligned}
\tag{9}
$$

Let us prove that this equality is verified only if $1 + c'(I^*(x)) - \mathcal{K}(I^*, x)$ equals 0, $\forall x > 0$. It will lead to a contradiction because $c'() > 0$ and $\mathcal{K}(I^*, X)$ is a random variable verifying $E_P(\mathcal{K}(I^*, X)) = 1$.

For $x = 0, \phi(x)$ equals 0 and the product in brackets (Eq. (9)) equals zero whatever $1 + c'(I^*(x)) - \mathcal{K}(I^*, x)$ is worth.

The proof is made in two steps: first, we consider the discrete random variable case; then we study the continuous one by taking the limit.

Let X be a positive discrete random variable the realizations of which x_i, $i = 1, 2, \ldots, n$, belong to $\mathbb{R}^+$. Let us note A_i, $i = 1, \ldots, n$, the set of states $\omega \in \Omega$ such that $X(\omega) = x_i (P(A_i) > 0)$. The A_i's form a finite partition of Ω and X can be written as $X = \sum_{i=1}^n x_i \cdot \mathbf{1}_{A_i}$, where $\mathbf{1}_{A_i}$ is the indicator function of A_i.

Let us define the function $g(X)$ as $g(X) = 1 + c'(I^*(X)) - \mathcal{K}(I^*, X)$.

Given $\alpha_\phi^i = \phi(x_i)$ and $\beta_g^i = g(x_i)$, we have $\phi(X) = \sum_{i=1}^n \alpha_\phi^i \cdot \mathbf{1}_{A_i}$ and $g(X) = \sum_{i=1}^n \beta_g^i \cdot \mathbf{1}_{A_i}$. And $E_P[\phi(X)(1 + c'(I^*(X)) - \mathcal{K}(I^*, X))]$ becomes

$$E_P[\phi(X)g(X)] = E_P\left[\sum_{i=1}^n \alpha_\phi^i \cdot \mathbf{1}_{A_i} \sum_{i=1}^n \beta_g^i, \mathbf{1}_{A_i}\right] = \sum_{i=1}^n \alpha_\phi^i \beta_g^i P(A_i),$$

where $P(A_i)$ is the probability of the event A_i. The second equality holds because the A_i's form a partition of Ω.

Let $\alpha_\phi = (\alpha_\phi^i, i = 1, \ldots, n)$, $\beta_g = (\beta_g^i, i = 1, \ldots, n)$ and let M be the matrix with the $P(A_i)$ on its diagonal and zero elsewhere, (9) reduces to

$$\alpha_\phi' M \beta_g = \langle \alpha_\phi, \beta_g \rangle_M = 0,$$

where $\langle \cdot, \cdot \rangle_M$ is the inner product defined by the matrix M. Finally, we have to prove the following equivalence:

$$\langle \alpha_\phi, \beta_g \rangle_M = 0 \quad \text{if and only if } \beta_g = \vec{O}. \tag{10}$$

This can be done by classical geometrical arguments and it is not presented here[12]. In order to study the continuous case, we use the following lemma translated from Neveu ([1970], p. 34):

Lemma 5: *Any positive random variable X defined on (Ω, A) is the limit of at least one increasing sequence of positive real simple random variables.*

For the continuous case, let us note X the limit of a sequence of positive real simple random variables[13] $(X_n, n \in \mathbb{N}^*)$. We know that

$$1 + c'(I^*(X_i)) - \mathcal{K}(I^*, X_i) = 0, \quad \forall_i = 1, \ldots, n \tag{11}$$

for every strictly positive realization of X_i. Since $\lim_{n\to\infty} X_n = X$ and $c'(I^*(X))$ and $\mathcal{K}(I^*, X)$ are continuous by assumption, we finally have, according to (11),

$$\lim_{n\to\infty} 1 + c'(I^*(X_n)) - \mathcal{K}(I^*, X_n) = 1 + c'(I^*(X)) - \mathcal{K}(I^*, X)$$
$$\Leftrightarrow 1 + c'(I^*(x)) - \mathcal{K}(I^*, x) = 0, \quad \forall x \in]0, T].$$

This proves Lemma 4. □

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Notes

1. For a review, see Gollier [1992] and Lemaire [1991].
2. In addition to the deductible, a proportion of the damage is also borne by the insured.
3. If the amount of damage is greater than the deductible, indemnities are paid and the insured bears only a fraction of the deductible; this explains the formulation "partially disappearness".
4. We thank G. Dionne for this remark.
5. Doherty and Schlesinger [1983a, 1983b], among others, study the case of several risks.
6. For some basic topological notions, one can refer to Choquet [1973] and Schwartz [1970].
7. See Minoux [1983] or Duffie [1988].
8. It is not useful to consider a higher order since α tends toward zero in the differential.
9. o is the Landau's notation (see Roger [1991], p. 50).
10. Schlesinger [1981] shows that a more risk-averse agent prefers a lower deductible. In the same way, if the insured becomes less rich, in the way that his initial wealth w is lower, he prefers more coverage—that is, a lower deductible—if his absolute risk aversion is decreasing with respect to w. However, let us note that Schlesinger obtains these results of comparative statics by assuming that the insurance contract is exogenously defined: the indemnity function is equal to $\max_{x\geq 0}(0, x - D)$ and the insured can only choose the value of D that maximizes his expected utility.
11. The concept of prudence has been introduced by Leland [1968] and Sandmo [1970], and it has been developed and formalized by Kimball [1990].
12. See Spaeter-Roger [1995] for the details.
13. A simple random variable is one with finite support.

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