Leaves in Representation Diagrams of Bipartite Distance-Regular Graphs

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Received May 22, 2001; Revised January 27, 2003; Accepted February 10, 2003

Abstract. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the eigenvalues of Γ and let q_{ij}^h ($0 \leq h, i, j \leq D$) denote the Krein parameters of Γ . Pick an integer h ($1 \leq h \leq D-1$). The *representation diagram* $\Delta = \Delta_h$ is an undirected graph with vertices $0, 1, \dots, D$. For $0 \leq i, j \leq D$, vertices i, j are adjacent in Δ whenever $i \neq j$ and $q_{ij}^h \neq 0$. It turns out that in Δ , the vertex 0 is adjacent to h and no other vertices. Similarly, the vertex D is adjacent to $D-h$ and no other vertices. We call $0, D$ the *trivial* vertices of Δ . Let l denote a vertex of Δ . It turns out that l is adjacent to at least one vertex of Δ . We say l is a *leaf* whenever l is adjacent to exactly one vertex of Δ . We show Δ has a nontrivial leaf if and only if Δ is the disjoint union of two paths.

Keywords: primitive idempotent, eigenvalue, association scheme, Q-polynomial, antipodal

1. Introduction

In recent research on distance-regular graphs, the following theme emerges. Let Γ denote a distance-regular graph and let E and F denote primitive idempotents of Γ . When is the entrywise product $E \circ F$ a linear combination of a “small” number of primitive idempotents of Γ ?

We refer the reader to the articles of MacLean [5–7], Pascasio [9–11], and the present author [4] for work on this theme. In this paper we consider the case where $E \circ F$ is a linear combination of F and one other primitive idempotent. To keep things simple, we assume Γ is bipartite. Before we state our main result, we recall a bit of notation.

Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. (Definitions are contained in the next section.) Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the eigenvalues of Γ . Recall that $\theta_0 = k$ and $\theta_D = -k$; we call θ_0 and θ_D the *trivial* eigenvalues of Γ . For $0 \leq i \leq D$, let E_i denote the primitive idempotent of Γ associated with θ_i . Let q_{ij}^h ($0 \leq h, i, j \leq D$) denote the Krein parameters of Γ . Recall that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h \quad (0 \leq i, j \leq D),$$

where $\circ$ denotes entrywise multiplication.

Pick an integer h ($1 \leq h \leq D-1$). We recall the *representation diagram* $\Delta = \Delta_h$ [12–14]. Δ is an undirected graph with vertices $0, 1, \dots, D$. For $0 \leq i, j \leq D$, vertices i and j are adjacent in Δ whenever $i \neq j$ and $q_{ij}^h \neq 0$.

It turns out that in Δ , the vertex 0 is adjacent to h and no other vertices. Similarly, the vertex D is adjacent to $D-h$ and no other vertices. We call 0 and D the *trivial* vertices of Δ .

Let l denote a vertex of Δ . As we see in the next section, l is adjacent to at least one vertex of Δ . We say l is a *leaf* whenever l is adjacent to exactly one vertex of Δ . Our main result is the following.

Theorem 1.1 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Pick an integer h ($1 \leq h \leq D-1$). The representation diagram Δ_h has a nontrivial leaf if and only if Δ_h is the disjoint union of two paths.*

Hypercubes and doubled Odd graphs have representation diagrams satisfying the conditions of Theorem 1.1. At diameters greater than 5, these are the only such graphs known.

2. Preliminaries

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X , edge set R , path-length distance function ∂ , and diameter $D := \max\{\partial(x, y) : x, y \in X\}$. Let k denote a nonnegative integer. We say Γ is *regular* with *valency* k whenever for all $x \in X$, $|\{z \in X : \partial(x, z) = 1\}| = k$. We say Γ is *distance-regular* whenever for all integers h, i, j ($0 \leq h, i, j \leq D$) and all $x, y \in X$ with $\partial(x, y) = h$, the scalar $p_{ij}^h = |\{z \in X : \partial(x, z) = i, \partial(y, z) = j\}|$ is independent of x and y . For notational convenience, set $c_i := p_{i-1}^i$ ($1 \leq i \leq D$), $a_i := p_{ii}^i$ ($0 \leq i \leq D$), $b_i := p_{i+1}^i$ ($0 \leq i \leq D-1$), and $c_0 := 0$, $b_D := 0$. For the rest of this section, suppose Γ is distance-regular. To avoid trivialities, assume $D \geq 3$ and $k \geq 3$. We observe Γ is regular with valency $k = b_0$. Further, we observe $c_i + a_i + b_i = k$ for $0 \leq i \leq D$.

We say Γ is *bipartite* whenever there exists a partition $X = X^+ \cup X^-$ such that no edge joins two vertices in the same cell of the partition. Observe Γ is bipartite if and only if $a_i = 0$ ($0 \leq i \leq D$), and in this case,

$$c_i + b_i = k \quad (0 \leq i \leq D). \quad (1)$$

For the rest of this section, suppose Γ is bipartite.

Let $\sim$ denote the binary relation on X such that for any $x, y \in X$, we have $x \sim y$ whenever $x = y$ or $\partial(x, y) = D$. We say Γ is *antipodal* whenever $\sim$ is an equivalence relation.

Let $\mathbb{R}$ denote the field of real numbers. By $\text{Mat}_X(\mathbb{R})$ we mean the $\mathbb{R}$ -algebra consisting of all matrices whose entries are in $\mathbb{R}$ and whose rows and columns are indexed by X .

For each integer i ($0 \leq i \leq D$), let A_i denote the matrix in $\text{Mat}_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{otherwise} \end{cases} \quad (x, y \in X).$$

Abbreviate $A := A_1$. We call A the *adjacency matrix* of Γ . Let M denote the sub-algebra of $\text{Mat}_X(\mathbb{R})$ generated by A . We call M the *Bose-Mesner algebra* of Γ . By [1, Theorem 20.7], $A_0, A_1, \dots, A_D$ is a basis for M .

By [2, Theorem 2.6.1], M has a second basis $E_0, E_1, \dots, E_D$ such that $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq D$). We call $E_0, E_1, \dots, E_D$ the (*primitive*) *idempotents* of Γ .

Observe there exists a sequence of scalars $\theta_0, \theta_1, \dots, \theta_D$ taken from $\mathbb{R}$ such that

$$A = \sum_{i=0}^D \theta_i E_i.$$

We call θ_i the *eigenvalue* of Γ associated with E_i . Note $\theta_0, \theta_1, \dots, \theta_D$ are distinct since A generates M . Throughout this paper, we assume the eigenvalues are labeled so that $\theta_0 > \theta_1 > \dots > \theta_D$. By [2, p. 82], $\theta_0 = k$ and $\theta_{D-i} = -\theta_i$ for $0 \leq i \leq D$. We call θ_0 and θ_D the *trivial eigenvalues* of Γ .

Let θ_h denote an eigenvalue of Γ and let E_h denote the associated idempotent. Since $A_0, A_1, \dots, A_D$ is a basis for M , there exist real scalars $\sigma_0, \sigma_1, \dots, \sigma_D$ such that

$$E_h = m_h |X|^{-1} \sum_{i=0}^D \sigma_i A_i, \quad (2)$$

where $m_h = \text{rank } E_h$. We call $\sigma_0, \sigma_1, \dots, \sigma_D$ the *cosine sequence* associated with θ_h . By [2, p. 128],

$$c_i \sigma_{i-1} + b_i \sigma_{i+1} = \theta_h \sigma_i \quad (0 \leq i \leq D), \quad (3)$$

where σ_{-1} and σ_{D+1} denote indeterminates.

Let $\circ$ denote entrywise multiplication in $\text{Mat}_X(\mathbb{R})$ and observe

$$A_i \circ A_j = \delta_{ij} A_i \quad (0 \leq i, j \leq D). \quad (4)$$

This implies M is closed under $\circ$. Since $E_0, E_1, \dots, E_D$ is a basis for M , there exist scalars $q_{ij}^h \in \mathbb{R}$ ($0 \leq h, i, j \leq D$) such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h. \quad (5)$$

We call the q_{ij}^h the *Krein parameters* of Γ .

In the next two lemmas, we recall a few basic facts about the product $\circ$ and the Krein parameters.

Lemma 2.1 [9, Lemma 3.3, Theorem 3.6] *Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 3$.*

- (i) $E_0 \circ E_i = |X|^{-1} E_i$ for $0 \leq i \leq D$.
- (ii) $E_D \circ E_i = |X|^{-1} E_{D-i}$ for $0 \leq i \leq D$.
- (iii) For $1 \leq i, j \leq D-1$, $E_i \circ E_j$ is not a scalar multiple of a single idempotent of Γ .

Lemma 2.2 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$.*

- (i) $q_{ij}^h = q_{ji}^h$ ($0 \leq h, i, j \leq D$).
- (ii) $m_h q_{ij}^h = m_i q_{jh}^i = m_j q_{hi}^j$ ($0 \leq h, i, j \leq D$).
- (iii) $q_{0j}^h = \delta_{hj}$ ($0 \leq h, j \leq D$).
- (iv) $q_{Dj}^h = \delta_{h, D-j}$ ($0 \leq h, j \leq D$).
- (iii) $q_{D-i, j}^{D-h} = q_{ij}^h$ ($0 \leq h, i, j \leq D$).

Proof: (i) Immediate from (5). (ii) [2, Lemma 2.3.1(iv)] (iii) Immediate from Lemma 2.1(i). (iv) Immediate from Lemma 2.1(ii). (v) Taking the entrywise product of both sides of (5) with E_D and applying Lemma 2.1(ii), we get the result. $\square$

Definition 2.3 [12] *Let Γ denote a distance-regular graph with diameter D . Pick an integer h ($0 \leq h \leq D$). We define the *representation diagram* $\Delta = \Delta_h$. Δ is an undirected graph with vertices $0, 1, \dots, D$. For $0 \leq i, j \leq D$, vertices i and j are adjacent in Δ whenever $i \neq j$ and $q_{ij}^h \neq 0$. We sometimes say Δ is the representation diagram associated with the eigenvalue θ_h .*

Let C denote a connected component of Δ . We say C is a *path* whenever there exists an ordering $v_0, v_1, \dots, v_l$ of the vertices of C such that for $0 \leq i, j \leq l$, vertices v_i, v_j are adjacent in Δ if and only if $|i - j| = 1$.

Lemma 2.4 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$. With reference to Definition 2.3, the following hold.*

- (i) For $0 \leq h, i, j \leq D$, vertices i and j are adjacent in Δ_h if and only if $D - i$ and $D - j$ are adjacent in Δ_h .
- (ii) Δ_0 has no edges.
- (iii) In Δ_D , vertex i ($0 \leq i \leq D$) is adjacent to $D - i$ and no other vertices.
(If D is even then vertex $D/2$ is not adjacent to any vertices.)
- (iv) Suppose $h \neq 0$. In Δ_h , vertex 0 is adjacent to h and no other vertices. Moreover, vertex D is adjacent to $D - h$ and no other vertices.
- (v) Suppose $1 \leq h \leq D - 1$. Each vertex of Δ_h is adjacent to at least one other vertex.

Proof: (i)–(iv) Immediate from Lemma 2.2. (v) Let i denote a vertex of Δ_h and suppose i is not adjacent to any vertices of Δ_h . By (iv) above, we find $1 \leq i \leq D - 1$. By Definition 2.3,

$q_{ij}^h = 0$ for $j \neq i$. Applying Lemma 2.2(ii), we find $q_{hi}^j = 0$ for $j \neq i$, which implies $E_h \circ E_i$ is a scalar multiple of E_i . This contradicts Lemma 2.1(iii). $\square$

We call 0 and D the *trivial* vertices of a representation diagram.

Lemma 2.5 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. The following are equivalent for $1 \leq h \leq D - 1$.*

- (i) Δ_h is not connected.
- (ii) Γ is antipodal and h is even.

Suppose (i)–(ii) hold. Then Δ_h has two connected components, one consisting of the even vertices and one consisting of the odd vertices.

Proof: Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote the cosine sequence associated with θ_h .

(i) $\rightarrow$ (ii) Since Δ_h is not connected and by [2, Proposition 2.11.1], $\sigma_i = 1$ for some i ($1 \leq i \leq D$). By [2, Proposition 4.4.7], Γ is antipodal and $\sigma_D = 1$. Now h is even by [2, p. 142].

(ii) $\rightarrow$ (i) By [2, p. 142], $\sigma_D = 1$. Now Δ_h is not connected by [2, Proposition 2.11.1].

Suppose (i)–(ii) hold. We already mentioned $\sigma_D = 1$. By [2, Proposition 4.4.7], $\sigma_i \neq 1$ for $1 \leq i \leq D - 1$. Now by [2, Proposition 2.11.1], Δ_h has two components. By [2, p. 413], $q_{ij}^h = 0$ if one of i and j is even and the other is odd. The result follows. $\square$

Example 2.6 Let Γ denote a bipartite distance-regular graph with diameter 3 and valency $k \geq 3$. With reference to Definition 2.3, the following hold.

- (i) Δ_1 is the path 0, 1, 2, 3.
- (ii) Suppose Γ is not antipodal. Then Δ_2 is the path 0, 2, 1, 3.
- (iii) Suppose Γ is antipodal. Then Δ_2 is the disjoint union of the paths 0, 2 and 1, 3.

Proof: (i) By Lemma 2.4(iv), vertex 0 is adjacent to 1 and no other vertices. Also, vertex 3 is adjacent to 2 and no other vertices. By Lemma 2.5, Δ_1 is connected, so 1 is adjacent to 2 and we are done.

(ii), (iii) Similar to the proof of (i). $\square$

3. Leaves

Definition 3.1 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Fix h ($1 \leq h \leq D - 1$) and let $\Delta = \Delta_h$ denote a representation diagram of Γ . Let l denote a vertex of Δ . By Lemma 2.4(v), l is adjacent to at least one vertex of Δ . We say l is a *leaf* whenever l is adjacent to exactly one vertex of Δ . Observe l is a leaf if and only if there exists an idempotent F of Γ with $F \neq E_l$ such that

$$E_h \circ E_l \in \text{Span} \{E_l, F\}. \quad (6)$$

By Lemma 2.4, l is a leaf if and only if $D - l$ is a leaf. Also, the trivial vertices 0 and D of Δ are leaves.

Theorem 3.2 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Let Δ_h ($1 \leq h \leq D-1$) denote a representation diagram of Γ . Suppose Δ_h has at least one nontrivial leaf. Then the following hold.*

- (i) Δ_h is the disjoint union of two paths, one consisting of the even vertices and one consisting of the odd vertices.
- (ii) Γ is antipodal and h is even.

Proof: We abbreviate $\Delta := \Delta_h$.

(i) If $D = 3$ the result follows from Example 2.6, so suppose $D \geq 4$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote the cosine sequence associated with θ_h . We show there exists $\beta \in \mathbb{R}$ such that $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1}$ is independent of i for $1 \leq i \leq D-1$.

By assumption, Δ has a nontrivial leaf. Let us denote this leaf by l . Let t denote the vertex of Δ to which l is adjacent. Apparently, $t \neq l$ and there exist $\epsilon, \zeta \in \mathbb{R}$ with $\zeta \neq 0$ such that

$$E_h \circ E_l = \epsilon E_l + \zeta E_t. \quad (7)$$

Let $\rho_0, \rho_1, \dots, \rho_D$ and $\tau_0, \tau_1, \dots, \tau_D$ denote the cosine sequences associated with θ_l and θ_t , respectively. We use (2) to eliminate E_h, E_l and E_t from (7) and then apply (4). In the result, we equate coefficients of A_i and simplify to find that for $0 \leq i \leq D$,

$$\sigma_i \rho_i = x \rho_i + y \tau_i, \quad (8)$$

where $x = |X| m_h^{-1} \epsilon$ and $y = |X| m_t m_h^{-1} m_l^{-1} \zeta$. Note $y \neq 0$ because $\zeta \neq 0$.

We use (8) for $0 \leq i \leq 4$. Repeatedly applying (3) and (1), we find for $0 \leq i \leq 4$ that $\sigma_i = f_i(\theta_h)$, $\rho_i = f_i(\theta_l)$ and $\tau_i = f_i(\theta_t)$, where the functions f_i are given by

$$f_0(\lambda) = 1, \quad f_1(\lambda) = \frac{\lambda}{k}, \quad f_2(\lambda) = \frac{\lambda^2 - k}{kb_1}, \quad (9)$$

$$f_3(\lambda) = \frac{\lambda^3 - (k + c_2 b_1) \lambda}{kb_1 b_2}, \quad f_4(\lambda) = \frac{\lambda^4 - (k + c_2 b_1 + c_3 b_2) \lambda^2 + c_3 k b_2}{kb_1 b_2 b_3}. \quad (10)$$

We set $i = 0, 1$ in (8) to obtain two linear equations in x and y . To solve this system, we first verify the coefficient matrix is nonsingular. This coefficient matrix is

$$\begin{pmatrix} \rho_0 & \tau_0 \\ \rho_1 & \tau_1 \end{pmatrix}. \quad (11)$$

Evaluating the determinant of (11) using (9), we find this determinant equals $(\theta_t - \theta_l)k^{-1}$. This is nonzero, so the coefficient matrix is nonsingular. We now solve the system of equations to find in view of (9) that

$$x = \frac{\theta_h \theta_l - \theta_t k}{k(\theta_l - \theta_t)}, \quad y = \frac{\theta_l(k - \theta_h)}{k(\theta_l - \theta_t)}. \quad (12)$$

Note θ_l is a factor of y and so cannot be zero. We set $i = 2$ in (8), apply (9) and (12) and solve for θ_l to get

$$\theta_l = \frac{\theta_l^2 \theta_h + \theta_l^2 - \theta_h k - k^2}{b_1 \theta_l}. \quad (13)$$

We set $i = 3$ in (8), apply (9)–(10), (12) and (13), and simplify to find

$$\frac{(k^2 - \theta_h^2)(k^2 - \theta_l^2)}{\theta_l k^2 b_1^3 b_2^2} ((b_2 - (c_2 - 1)\theta_h)\theta_l^2 - b_2(k + \theta_h)) = 0. \quad (14)$$

The fraction is nonzero and $b_2(k + \theta_h) \neq 0$, so $b_2 - (c_2 - 1)\theta_h \neq 0$. We solve (14) for θ_l^2 to get

$$\theta_l^2 = \frac{(k + \theta_h)b_2}{b_2 - (c_2 - 1)\theta_h}. \quad (15)$$

We set $i = 4$ in (8) and apply (9)–(10), (12), (13) and (15) to find the left side minus the right is

$$\frac{(k^2 - \theta_h^2)(k + \theta_h)(k^2 - \theta_l^2)c_2}{\theta_l^2(b_2 - (c_2 - 1)\theta_h)^2 k^2 b_1^2 b_2 b_3^2} \quad (16)$$

times

$$(b_2 - b_3)\theta_h^3 + (b_2 - b_3 c_2)\theta_h^2 + (2b_3 b_2 - b_3 c_2 b_2 - b_2^2)\theta_h + b_2^2(b_3 - 1). \quad (17)$$

Since (16) is nonzero, (17) must be zero.

By [3, Lemma 9.3] and since (17) is zero, there exists $\beta \in \mathbb{R}$ such that $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1}$ is independent of i for $1 \leq i \leq D - 1$. Now, by [4, Theorem 5.4], either Δ is a path or Δ is as in (i). Since Δ has a nontrivial leaf, Δ cannot be a path. So Δ is as in (i), as desired.

(ii) By (i), Δ is not connected. Now the result follows by Lemma 2.5. $\square$

Example 3.3 Let Γ denote a bipartite antipodal distance-regular graph with diameter 4 and valency $k \geq 3$. With reference to Definition 2.3, the following hold.

- (i) Δ_2 is the disjoint union of the paths 0, 2, 4 and 1, 3.
- (ii) Suppose $h = 1$ or $h = 3$. Then Δ_h has no nontrivial leaves.

Proof: (i) Immediate from Lemma 2.4(iv) and Lemma 2.5.

(ii) Since h is odd, Δ_h has no nontrivial leaves by Theorem 3.2. $\square$

Example 3.4 Let Γ denote a bipartite antipodal distance-regular graph with diameter 5 and valency $k \geq 3$. With reference to Definition 2.3, the following hold.

- (i) Δ_2 is the disjoint union of the paths 0, 2, 4 and 5, 3, 1.
- (ii) Δ_4 is the disjoint union of the paths 0, 4, 2 and 5, 1, 3.
- (iii) Suppose $h = 1$ or $h = 3$. Then Δ_h has no nontrivial leaves.

Proof: (i) By Lemma 2.5, the even vertices of Δ_2 comprise a connected component. This component consist of the path 0, 2, 4 since vertex 0 is adjacent to 2 but not 4 by Lemma 2.4(iv). Now the odd vertices form the path 5, 3, 1 by Lemma 2.4(i).

(ii) Similar to the proof of (i).

(iii) Since h is odd, Δ_h has no nontrivial leaves by Theorem 3.2. $\square$

In the next lemma and the following two examples, we recall some information about the D -cube.

Lemma 3.5 [2, Section 9.2] *Let Γ denote the D -cube.*

- (i) Γ is antipodal.
- (ii) For $0 \leq h, i, j \leq D$, $q_{ij}^h \neq 0$ if $|i - j| = h$ and $q_{ij}^h = 0$ if $|i - j| > h$.

Example 3.6 Let D denote an odd integer with $D \geq 3$ and let Γ denote the D -cube. With reference to Definition 2.3, the following hold.

- (i) Δ_2 is the disjoint union of the paths 0, 2, 4, $\dots$, $D - 1$ and 1, 3, 5, $\dots$, D .
- (ii) Δ_{D-1} is the disjoint union of the paths 0, $D - 1$, 2, $D - 3$, $\dots$ and D , 1, $D - 2$, 3, $\dots$
- (iii) Suppose $h \neq 2$ and $h \neq D - 1$. Then Δ_h has no nontrivial leaves.

Proof: (i) Γ is antipodal by Lemma 3.5(i), so by Lemma 2.5, Δ_2 has two connected components, one consisting of the even vertices and one consisting of the odd vertices. For $0 \leq i, j \leq D$, we see by Lemma 3.5(ii) that $q_{ij}^2 \neq 0$ if $|i - j| = 2$ and $q_{ij}^2 = 0$ if $|i - j| > 2$. The result follows.

(ii) We mentioned Γ is antipodal, so by Lemma 2.5, Δ_{D-1} has two connected components, one consisting of the even vertices and one consisting of the odd vertices. For $0 \leq i, j \leq D$, we see by Lemma 3.5(ii) that $q_{ij}^1 \neq 0$ if $|i - j| = 1$ and $q_{ij}^1 = 0$ if $|i - j| > 1$. Applying Lemma 2.2(v), we then get $q_{D-i,j}^{D-1} \neq 0$ if $|i - j| = 1$ and $q_{D-i,j}^{D-1} = 0$ if $|i - j| > 1$. The result follows.

(iii) Follows from Theorem 3.2, [4, Theorem 5.4] and [3, Example 17.1] $\square$

Example 3.7 Let D denote an even integer with $D \geq 4$ and let Γ denote the D -cube. With reference to Definition 2.3, the following hold.

- (i) Δ_2 is the disjoint union of the paths 0, 2, 4, $\dots$, D and 1, 3, 5, $\dots$, $D - 1$.
- (ii) Suppose $h \neq 2$. Then Δ_h has no nontrivial leaves.

Proof: (i) By Lemma 2.5, Δ_2 has two connected components, one consisting of the even vertices and one consisting of the odd vertices. For $0 \leq i, j \leq D$, we see by Lemma 3.5(ii) that $q_{ij}^2 \neq 0$ if $|i - j| = 2$ and $q_{ij}^2 = 0$ if $|i - j| > 2$. The result follows.

(ii) Follows from Theorem 3.2, [4, Theorem 5.4] and [3, Example 17.1]. $\square$

Theorem 3.8 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Let $\Delta = \Delta_h$ ($1 \leq h \leq D-1$) denote a representation diagram of Γ . Suppose Δ has a nontrivial leaf. Suppose Γ is not one of Examples 2.6, 3.3, 3.4, 3.6, 3.7. Then either (a) D is odd and $h = D-1$ or (b) $D \equiv 1 \pmod{4}$ and $h = (D-1)/2$. In case (a) the nontrivial leaves are $(D-1)/2$ and $(D+1)/2$. In case (b) the nontrivial leaves are 1 and $D-1$.

Proof: Let l denote a nontrivial leaf of Δ . Then $D-l$ is also a leaf. Replacing l by $D-l$ if necessary, we assume $l \leq D/2$. Recall $E_h \circ E_l$ is a linear combination of E_l and one other idempotent of Γ . By [7, Lemma 4.4], Γ is either 2-homogeneous in the sense of Nomura [8] or taut in the sense of MacLean [7].

First suppose Γ is 2-homogeneous. By [8, Theorem 1.2], either Γ is antipodal with $D \leq 5$ or Γ is the D -cube. But this implies Γ is as in one of the examples we excluded.

Next suppose Γ is taut and D is even. Set $d := D/2$. By [7, Theorem 4.3], l is either 1 or d . By [7, Corollary 6.6], $l = d-1$. Combining these facts, we find $l = 1$ and $d = 2$. Now $D = 2d = 4$. Recall Γ is antipodal by Theorem 3.2(ii). Now Γ is as in Example 3.3(ii), which is a contradiction.

Finally, suppose Γ is taut and D is odd. Set $d := (D-1)/2$. By Theorem 3.3(ii), h is even. By [7, Theorem 4.3], the ordered pair (h, l) is one of $(D-1, d)$, $(d, 1)$, and $(D-d, 1)$. First suppose $(h, l) = (D-1, d)$. Then we have (a). Next suppose $(h, l) = (d, 1)$. Since $D = 2d+1$ and since $d = h$ is even, we have (b). Finally, suppose $(h, l) = (D-d, 1)$. By [7, Theorem 6.2, Corollary 6.3], $E_{D-d} \circ E_1 \in \text{Span}\{E_R, E_S\}$ for some R, S such that $1 < S < R$. But this contradicts (6). $\square$

Note 3.9 The doubled Odd graph $2.O_k$ is the only known graph for which (a) holds in Theorem 3.8. There is no known graph for which (b) holds.

Acknowledgment

This paper is part of the author's dissertation, written at the University of Wisconsin–Madison under the direction of Paul Terwilliger.

The author would like to thank the anonymous referees for their helpful comments.

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