# Discrete Polymatroids

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**Abstract.**   The discrete polymatroid is a multiset analogue of the matroid. Based on the polyhedral theory on integral polymatroids developed in late 1960's and in early 1970's, in the present paper the combinatorics and algebra on discrete polymatroids will be studied.

**Keywords:**


## Introduction

Matroid theory is one of the most fascinating research areas in combinatorics. Let $[n] = \{1, 2, \ldots, n\}$ and $2^{[n]}$ the set of all subsets of $[n]$. For $A \subset [n]$ write $|A|$ for the cardinality of $A$. A *matroid* on the ground set $[n]$ is a nonempty subset $\mathcal{M} \subset 2^{[n]}$ satisfying

(M1) if $F_1 \in \mathcal{M}$ and $F_2 \subset F_1$, then $F_2 \in \mathcal{M}$;
(M2) if $F_1$ and $F_2$ belong to $\mathcal{M}$ and $|F_1| < |F_2|$, then there is $x \in F_2 \backslash F_1$ such that $F_1 \cup \{x\} \in \mathcal{M}$.

The members of $\mathcal{M}$ are the *independent sets* of $\mathcal{M}$. A *base* of $\mathcal{M}$ is a maximal independent set of $\mathcal{M}$. It follows from (M2) that if $B_1$ and $B_2$ are bases of $\mathcal{M}$, then $|B_1| = |B_2|$. The set of bases of $\mathcal{M}$ possesses the "exchange property" as follows:

(B) If $B_1$ and $B_2$ are bases of $\mathcal{M}$ and if $x \in B_1 \backslash B_2$, then there is $y \in B_2 \backslash B_1$ such that $(B_1 \backslash \{x\}) \cup \{y\}$ is a base of $\mathcal{M}$.

Moreover, the set of bases of $\mathcal{M}$ possesses the "symmetric exchange property" as follows:

(S) If $B_1$ and $B_2$ are bases of $\mathcal{M}$ and if $x \in B_1 \backslash B_2$, then there is $y \in B_2 \backslash B_1$ such that both $(B_1 \backslash \{x\}) \cup \{y\}$ and $(B_2 \backslash \{y\}) \cup \{x\}$ are bases of $\mathcal{M}$.

On the other hand, given a nonempty set $\mathcal{B} \subset 2^{[n]}$, there exists a matroid $\mathcal{M}$ on $[n]$ with $\mathcal{B}$ its set of bases if and only if $\mathcal{B}$ possesses the exchange property (B).

Let $\varepsilon_1, \ldots, \varepsilon_n$ denote the canonical basis vectors of $\mathbb{R}^n$. If we associate each $F \subset [n]$ with $\sum_{i \in F} \varepsilon_i$, then a matroid on $[n]$ may be regarded as a set of $(0, 1)$-vectors of $\mathbb{R}^n$. Considering nonnegative integer vectors instead of $(0, 1)$-vectors enables us to define the concept of discrete polymatroids. Let $\mathbb{R}_+^n$ denote the set of vectors $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ with each $u_i \geq 0$. If $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are two vectors belonging to $\mathbb{R}_+^n$, then we write $u \leq v$ if all components $v_i - u_i$ of $v - u$ are nonnegative and, moreover, write $u < v$ if $u \leq v$ and $u \neq v$. We say that $u$ is a *subvector* of $v$ if $u \leq v$. The *modulus* of $u = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ is $|u| = u_1 + \cdots + u_n$. Also, let $\mathbb{Z}_+^n = \mathbb{R}_+^n \cap \mathbb{Z}^n$.

A *discrete polymatroid* on the ground set $[n]$ is a nonempty finite set $P \subset \mathbb{Z}_+^n$ satisfying

(D1) if $u \in P$ and $v \in \mathbb{Z}_+^n$ with $v \leq u$, then $v \in P$;
(D2) if $u = (u_1, \ldots, u_n) \in P$ and $v = (v_1, \ldots, v_n) \in P$ with $|u| < |v|$, then there is $i \in [n]$ with $u_i < v_i$ such that $u + \varepsilon_i \in P$.

A *base* of $P$ is a vector $u \in P$ such that $u < v$ for no $v \in P$. It follows from (D2) that if $u_1$ and $u_2$ are bases of $P$, then $|u_1| = |u_2|$. Discrete polymatroids can be also characterized in terms of the exchange property of their bases. In fact, a nonempty finite set $B \subset \mathbb{Z}_+^n$ is the set of bases of a discrete polymatroid on $[n]$ if and only if $B$ satisfies

 (i) all $u \in B$ have the same modulus;
(ii) if $u = (u_1, \ldots, u_n) \in P$ and $v = (v_1, \ldots, v_n) \in P$ belong to $B$ with $u_i > v_i$, then there is $j \in [n]$ with $u_j < v_j$ such that $u - \varepsilon_i + \varepsilon_j \in B$.

Consult Theorem 2.3 for the details.

In, e.g., [17, Ch. 18] and [15, p. 382] we can find the concept of polymatroids, which originated in Edmonds [5]. A *polymatroid* on $[n]$ is a nonempty compact subset $\mathcal{P} \subset \mathbb{R}_+^n$ satisfying

(P1) if $u \in \mathcal{P}$ and $v \in \mathbb{R}_+^n$ with $v \leq u$, then $v \in \mathcal{P}$;
(P2) if $u = (u_1, \ldots, u_n) \in \mathcal{P}$ and $v = (v_1, \ldots, v_n) \in \mathcal{P}$ with $|u| < |v|$, then there is $i \in [n]$ with $u_i < v_i$ and $0 < N < v_i - u_i$ such that $u + N\varepsilon_i \in \mathcal{P}$.

A brief summary of fundamental materials on polymatroids is given in Section 1. It follows that a polymatroid $\mathcal{P}$ on $[n]$ is a convex polytope in $\mathbb{R}^n$. We say that a polymatroid $\mathcal{P} \subset \mathbb{R}_+^n$ is *integral* if all vertices of $\mathcal{P}$ are integer vectors of $\mathbb{R}^n$.

Now, a combinatorial aspect of the present paper is to understand the relation between discrete polymatroids and integral polymatroids.

Even though the following fundamental Theorems 3.4 and 4.1 are known in the areas of combinatorial optimization and discrete convex analysis ([6, 10–13]), we want to give a self-contained presentation for the convenience of the reader working on commutative algebra and algebraic combinatorics.

**Theorem 3.4**  *A nonempty finite set $P \subset \mathbb{Z}_+^n$ is a discrete polymatroid if and only if* $\mathrm{conv}(P) \subset \mathbb{R}_+^n$ *is an integral polymatroid with* $\mathrm{conv}(P) \cap \mathbb{Z}^n = P$. *Here* $\mathrm{conv}(P)$ *is the convex hull of $P$ in $\mathbb{R}^n$.*

**Theorem 4.1** *If $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are bases of a discrete polymatroid $P \subset \mathbb{Z}_+^n$, then for each $i \in [n]$ with $u_i > v_i$ there is $j \in [n]$ with $u_j < v_j$ such that both $u - \varepsilon_i + \varepsilon_j$ and $v - \varepsilon_j + \varepsilon_i$ are bases of $P$.*

To the reader who is eager to study these fundamental facts in detail and from a viewpoint of applied mathematics, we strongly recommend Murota's monograph [12] which offers an attractive introduction to discrete convex analysis. In [12, Section 4.4], first of all, starting from the symmetric exchange property (Theorem 4.1) it is proved that the symmetric exchange property is equivalent to the exchange property. Second, our Theorem 3.4 is [12, Theorem 4.12]. Finally, [12, Theorem 4.15] relates the exchange property and the submodular function definition.

On the other hand, an algebraic aspect of the present paper is to study Ehrhart rings and base rings together with toric ideals of discrete polymatroids. Let $K$ be a field and let $t_1, \ldots, t_n$ and $s$ indeterminates over $K$. If $u = (u_1, \ldots, u_n) \in \mathbb{Z}_+^n$, then $t^u = t_1^{u_1} \ldots t_n^{u_n}$. Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid and $B$ its set of bases. Let $K[P]$ denote the homogeneous semigroup ring $K[t^u s : u \in P]$. The *base ring* of $P$ is the subalgebra $K[B] = K[t^u s : u \in B]$ ($\cong K[t^u : u \in B]$) of $K[P]$. Let $K[x_u : u \in B]$ denote the polynomial ring in $|B|$ variables over $K$ and write $I_B \subset K[x_u : u \in B]$ for the *toric ideal* of $K[B]$, i.e., $I_B$ is the kernel of the surjective $K$-algebra homomorphism $\xi : K[x_u : u \in B] \to K[B]$ defined by $\xi(x_u) = t^u$ for all $u \in B$. Clearly, all symmetric relations belong to $I_B$. More precisely, if $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ belong to $B$ with $u_i > v_i$ and $u_j < v_j$ and if both $u - \varepsilon_i + \varepsilon_j$ and $v - \varepsilon_j + \varepsilon_i$ belong to $B$, then the quadratic binomial $x_u x_v - x_{u-\varepsilon_i+\varepsilon_j} x_{v-\varepsilon_j+\varepsilon_i}$ belong to $I_B$. White [18] conjectured that for a matroid all symmetric exchange relations generate $I_B$. It is natural to conjecture that this also holds for discrete polymatroids. Much stronger, commutative algebraists cannot escape from the temptation to study the following problems:

(a) Does the toric ideal $I_B$ of a discrete polymatroid possess a quadratic Gröbner basis?
(b) Is the base ring $K[B]$ of a discrete polymatroid Koszul?

These must be the most attractive research problems on base rings of discrete polymatroids. One of our results on toric ideals of discrete polymatroids is

**Theorem 5.3**    (a) *Suppose that each matroid has the property that the toric ideal of its base ring is generated by symmetric exchange relations, then this is also true for each discrete polymatroid.*

(b) *If $P \subset \mathbb{Z}_+^n$ is a discrete polymatroid whose set of base $B$ satisfies the strong exchange property (i.e., if $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in B$, then for all $i$ and $j$ with $u_i > v_i$ and $u_j < v_j$ one has $u - \varepsilon_i + \varepsilon_j \in B$), then*

(i) *$I_B$ has a quadratic Gröbner basis, and hence $K[B]$ is Koszul;*
(ii) *$I_B$ is generated by symmetric exchange relations.*

It is known that if $\mathcal{P}_1, \ldots, \mathcal{P}_k \subset \mathbb{R}_+^n$ are polymatroid then their polymatroid sum

$$\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k = \left\{ x = \sum_{i=1}^k x_i : x_i \in \mathcal{P}_i, 1 \leq i \leq k \right\}$$

is again a polymatroid. Moreover, if each $\mathcal{P}_i$ is integral, then $\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$ is also integral and for each integer vector $x \in \mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$ there exist integer vectors $x_i \in \mathcal{P}_i$, $1 \leq i \leq k$, with $x = \sum_{i=1}^{k} x_i$. An immediate consequence of this fact is the following important

**Theorem 6.1**   *If $P \subset \mathbb{Z}_+^N$ is a discrete polymatroid, then the homogeneous semigroup ring $K[P]$ is normal.*

**Corollary 6.2**   *The base ring $K[B]$ of a discrete polymatroid $P$ is normal.*

Thus in particular both $K[P]$ and $K[B]$ are Cohen–Macaulay. We will also compare algebraic properties of $K[P]$ with those of $K[B]$. See Theorem 6.3 for the details.

Once we know that both $K[P]$ and $K[B]$ are Cohen–Macaulay, it is natural to study the problem when $K[P]$ is Gorenstein and when $K[B]$ is Gorenstein. A combinatorial criterion for $K[P]$ to be Gorenstein is obtained in Theorem 7.3. To classify all Gorenstein base rings seems, however, quite difficult. We will introduce the concept of "generic" discrete polymatroids and find a combinatorial characterization of discrete polymatroids $P$ satisfying the condition that (i) $P$ is generic and (ii) the base ring of $P$ is Gorenstein. See Theorem 7.6.

Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid and $B$ the set of bases of $P$. The convex hull $\text{conv}(B) \subset \mathbb{R}_+^n$ of $B$ is said to be the *base polytope* of $P$. What can be said about base polytopes of discrete polymatroids? A beautiful characterization of base polytopes arising from matroids is obtained in [7].

Let $V_n^{(d)} \subset \mathbb{R}_+^n$ denote the set of vectors $u \in \mathbb{R}_+^n$ of modulus $d$. If $\mathcal{Q} \subset V_n^{(d)}$ is an integral convex polytope and if $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ belong to $\mathbb{Z}_+^n$, then we write $\mathcal{Q}_{a,b}$ for the integral convex polytope which is the convex hull of all vectors $w = (w_1, \ldots, w_n) \in \mathcal{Q} \cap \mathbb{Z}^n$ satisfying $\min\{a_i, b_i\} \leq w_i \leq \max\{a_i, b_i\}$ for $i = 1, \ldots, n$.

**Theorem 8.2**   *An integral convex polytope $\mathcal{Q} \subset V_n^{(d)}$ is the base polytope of a discrete polymatroid $P \subset \mathbb{Z}_+^n$ of rank $d$ if and only if, for any vectors $a$ and $b$ belonging to $\mathbb{Z}_+^n$, the integral convex polytope $\mathcal{Q}_{a,b}$ possesses the property that if $u$ and $v$ are vertices of $\mathcal{Q}_{a,b}$ and if the segment $\text{conv}\{u, v\}$ is an edge of $\mathcal{Q}_{a,b}$, then $v - u$ is of the form $\ell(\varepsilon_i - \varepsilon_j)$, where $i \neq j$ and $0 \neq \ell \in \mathbb{Z}_+$.*

We say that a convex polytope $\mathcal{P} \subset \mathbb{R}^n$ of dimension $e$ is *simple* if each vertex of $\mathcal{P}$ belongs to just $e$ edges of $\mathcal{P}$.

**Corollary 8.3**   *Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid with $B$ the set of its bases. Suppose that the base polytope $\text{conv}(B) \subset \mathbb{R}_+^n$ is simple. Then the base ring $K[B]$ is a $K$-algebra with isolated singularity.*

Finally, in Section 9, two simple techniques to construct discrete polymatroids will be studied.

## 1. Polymatroids

The present section is a brief summary, based on [17, Ch. 18], of fundamental materials on polymatroids. Fix an integer $n > 0$ and set $[n] = \{1, 2, \ldots, n\}$. The canonical basis vectors of $\mathbb{R}^n$ will be denoted by $\varepsilon_1, \ldots, \varepsilon_n$. Let $\mathbb{R}^n_+$ denote the set of those vectors $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ with each $u_i \geq 0$, and $\mathbb{Z}^n_+ = \mathbb{R}^n_+ \cap \mathbb{Z}^n$. For a vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n_+$ and for a subset $A \subset [n]$, we set

$$u(A) = \sum_{i \in A} u_i.$$

Thus in particular $u(\{i\})$, or simply $u(i)$, is the $i$th component $u_i$ of $u$. The *modulus* of $u$ is

$$|u| = u([n]) = \sum_{i=1}^{n} u_i.$$

Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ be two vectors belonging to $\mathbb{R}^n_+$. We write $u \leq v$ if all components $v_i - u_i$ of $v - u$ are nonnegative and, moreover, write $u < v$ if $u \leq v$ and $u \neq v$. We say that $u$ is a *subvector* of $v$ if $u \leq v$. In addition, we set

$$u \vee v = (\max\{u_1, v_1\}, \ldots, \max\{u_n, v_n\}),$$
$$u \wedge v = (\min\{u_1, v_1\}, \ldots, \min\{u_n, v_n\}).$$

Thus $u \wedge v \leq u \leq u \vee v$ and $u \wedge v \leq v \leq u \vee v$.

**Definition 1.1** A *polymatroid* on the *ground set* $[n]$ is a nonempty compact subset $\mathcal{P}$ in $\mathbb{R}^n_+$, the set of *independent vectors*, such that

(P1) every subvector of an independent vector is independent;
(P2) if $u, v \in \mathcal{P}$ with $|v| > |u|$, then there is a vector $w \in P$ such that

$$u < w \leq u \vee v.$$

A *base* of a polymatroid $\mathcal{P} \subset \mathbb{R}^n_+$ is a maximal independent vector of $\mathcal{P}$, i.e., an independent vector $u \in \mathcal{P}$ with $u < v$ for no $v \in \mathcal{P}$. Every base of $\mathcal{P}$ has the same modulus rank($\mathcal{P}$), the *rank* of $\mathcal{P}$. In fact, if $u$ and $v$ are bases of $\mathcal{P}$ with $|u| < |v|$, then by (P2) there exists $w \in P$ with $u < w \leq u \vee v$, contradicting the maximality of $u$.

Let $\mathcal{P} \subset \mathbb{R}^n_+$ be a polymatroid on the ground set $[n]$. Let $2^{[n]}$ denote the set of all subsets of $[n]$. The *ground set rank function* of $\mathcal{P}$ is a function $\rho : 2^{[n]} \to \mathbb{R}_+$ defined by setting

$$\rho(A) = \max\{v(A) : v \in \mathcal{P}\}$$

for all $\emptyset \neq A \subset [n]$ together with $\rho(\emptyset) = 0$.

**Proposition 1.2** (a) *Let* $\mathcal{P} \subset \mathbb{R}_+^n$ *be a polymatroid on the ground set* $[n]$ *and* $\rho$ *its ground set rank function. Then* $\rho$ *is nondecreasing, i.e., if* $A \subset B \subset [n]$, *then* $\rho(A) \leq \rho(B)$, *and is submodular, i.e.,*

$$\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B)$$

*for all* $A, B \subset [n]$. *Moreover,* $\mathcal{P}$ *coincides with the compact set*

$$\{x \in \mathbb{R}_+^n : x(A) \leq \rho(A), A \subset [n]\}. \tag{1}$$

(b) *Conversely, given a nondecreasing and submodular function* $\rho : 2^{[n]} \to \mathbb{R}_+$ *with* $\rho(\emptyset) = 0$, *the compact set* (1) *is a polymatroid on the ground set* $[n]$ *with* $\rho$ *its ground set rank function.*

It follows from Proposition 1.2 (a) that a polymatroid $\mathcal{P} \subset \mathbb{R}_+^n$ on the ground set $[n]$ is a convex polytope in $\mathbb{R}^n$. In addition, the set of bases of $\mathcal{P}$ is a face of $\mathcal{P}$ with supporting hyperplane

$$\left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = \mathrm{rank}(\mathcal{P}) \right\}.$$

We refer the reader to, e.g., [9] for basic terminologies on convex polytopes.

How can we find the vertices of a polymatroid? A complete answer was obtained by Edmonds [5]. We will associate any permutation $\pi = (i_1, \ldots, i_n)$ of $[n]$ with $A_\pi^1 = \{i_1\}$, $A_\pi^2 = \{i_1, i_2\}, \ldots, A_\pi^n = \{i_1, \ldots, i_n\}$.

**Proposition 1.3** *Let* $\mathcal{P} \subset \mathbb{R}_+^n$ *be a polymatroid on the ground set* $[n]$ *and* $\rho$ *its ground set rank function. Then the vertices of* $\mathcal{P}$ *are all points* $v = v(k, \pi) \in \mathbb{R}_+^n$, *where* $v = (v_1, \ldots, v_n)$ *and*

$$\begin{aligned}
v_{i_1} &= \rho\left(A_\pi^1\right), \\
v_{i_2} &= \rho\left(A_\pi^2\right) - \rho\left(A_\pi^1\right), \\
v_{i_3} &= \rho\left(A_\pi^3\right) - \rho\left(A_\pi^2\right), \\
&\cdots \\
v_{i_k} &= \rho\left(A_\pi^k\right) - \rho\left(A_\pi^{k-1}\right), \\
v_{i_{k+1}} &= v_{i_{k+2}} = \cdots = v_{i_n} = 0,
\end{aligned}$$

*and* $k$ *ranges over the integers belonging to* $[n]$, *and* $\pi = (i_1, \ldots, i_n)$ *ranges over all permutations of* $[n]$. *In particular the vertices of the face of* $\mathcal{P}$ *consisting of all bases of* $\mathcal{P}$ *are all points* $v = v(n, \pi) \in \mathbb{R}_+^n$, *where* $\pi$ *ranges over all permutations of* $[n]$.

We say that a polymatroid is *integral* if all of its vertices have integer coordinates; in other words, a polymatroid is integral if and only if its ground set rank function is integer valued.

Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be polymatroids on the ground set $[n]$. The *polymatroid sum* $\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$ of $\mathcal{P}_1, \ldots, \mathcal{P}_k$ is the compact subset in $\mathbb{R}^n_+$ consisting of all vectors $x \in \mathbb{R}^n_+$ of the form

$$x = \sum_{i=1}^{k} x_i, \quad x_i \in \mathcal{P}_i.$$

**Proposition 1.4** *Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be polymatroids on the ground set $[n]$ and $\rho_i$ the ground set rank function of $\mathcal{P}_i$, $1 \leq i \leq k$. Then the polymatroid sum $\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$ is a polymatroid on $[n]$ and $\sum_{i=1}^{k} \rho_i$ is its ground set rank function. Moreover, if each $\mathcal{P}_i$ is integral, then $\mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$ is integral, and for each integer vector $x \in \mathcal{P}_1 \vee \cdots \vee \mathcal{P}_k$ there exist integer vectors $x_i \in \mathcal{P}_i$, $1 \leq i \leq k$, with $x = \sum_{i=1}^{k} x_i$.*

## 2.  Discrete polymatroids

In this section we introduce discrete polymaroids. They may be viewed as generalizations of matroids.

**Definition 2.1**  Let $P$ be a nonempty finite set of integer vectors in $\mathbb{R}^n_+$ which contains with each $u \in P$ all its integral subvectors. The set $P$ is called a *discrete polymatroid* on the ground set $[n]$ if for all $u, v \in P$ with $|v| > |u|$, there is a vector $w \in P$ such that

$$u < w \leq u \vee v.$$

A *base* of $P$ is a vector $u \in P$ such that $u < v$ for no $v \in P$. We denote $B(P)$ the set of bases of a discrete polymatroid $P$. Just as for polymatroids one proves that all elements in $B(P)$ have the same modulus. This common number is called the *rank* of $P$.

The following simple lemma is useful for induction arguments.

**Lemma 2.2**  *Let $P$ be a discrete polymatroid.*
(a) *Let $d \leq \operatorname{rank} P$. Then the set $P' = \{u \in P \colon |u| \leq d\}$ is a discrete polymatroid of rank $d$ with the set of bases $\{u \in P \colon |u| = d\}$.*
(b) *Let $x \in P$. Then the set $P_x = \{v - x \colon v \geq x\}$ is a discrete polymatroid of rank $d - |x|$.*

**Proof:**  (a)  Let $u, v \in P$ with $d \geq |v| > |u|$. There exists $w \in P$ such that $u < w \leq u \wedge v$. Since $w > u$, and since $P$ contains all subvectors of $w$, there exists an integer $i$ such that $u + \varepsilon_i \leq w$. Then $u < u + \varepsilon_i \leq u \wedge v$, and since $u + \varepsilon_i \leq d$, it belongs to $P'$. This proves that $P'$ is a discrete polymatroid. It is clear that $\{u \in P : |u| = d\}$ is the set of bases of $P'$.

(b) Let $u', v' \in P_x$ with $|v'| > |u'|$, and set $u = u' + x$ and $v = v' + x$. Then $u, v \in P$ and $|v| > |u|$. Hence there exists $w \in P$ with $u < w \leq u \wedge v$. Set $w' = w - x$. Then $w' \in P_x$ and $u' < w' \leq u' \wedge v'$.                                                                                                                                     $\square$

Discrete polymatroids can be characterized in terms of their set of bases as follows:

**Theorem 2.3** *Let P be a nonempty finite set of integer vectors in $\mathbb{R}_+^n$ which contains with each $u \in P$ all its integral subvectors, and let $B(P)$ be the set of vectors $u \in P$ with $u < v$ for no $v \in P$. The following conditions are equivalent:*
(a) *P is a discrete polymatroid;*
(b) *if $u, v \in P$ with $|v| > |u|$, then there is an integer $i$ such that $u + \varepsilon_i \in P$ and*

$$u + \varepsilon_i \le u \vee v;$$

(c)  (i) *all $u \in B(P)$ have the same modulus,*
     (ii) *if $u, v \in B(P)$ with $u(i) > v(i)$, then there exists $j$ with $u(j) < v(j)$ such that $u - \varepsilon_i + \varepsilon_j \in B(P)$.*

**Proof:**    That (a) implies (b) was already shown in the proof of 2.2, and implication (b) $\Rightarrow$ (a) is trivial.

(b) $\Rightarrow$ (c): We noticed already that (c)(i) holds. Thus it remains to prove (c)(ii). Let $u, v \in B(P)$ with $u(i) > v(i)$ for some $i$. Then $u(i) - 1 \ge v(i)$, and hence $|u - \varepsilon_i| = |v| - 1 < |v|$. Thus by (b) there exists an integer $j$ such that $(u - \varepsilon_i) + \varepsilon_j \le (u - \varepsilon_i) \wedge v$. We have $j \ne i$, because otherwise $u(i) = (u - \varepsilon_i + \varepsilon_j)(i) \le \max\{u(i) - 1, v(i)\} = u(i) - 1$, a contradiction. Thus $u(j) + 1 = (u - \varepsilon_i + \varepsilon_j)(j) \le \max\{u(j), v(j)\} \le v(j)$, that is, $v(j) > u(j)$.

(c) $\Rightarrow$ (b): Let $v, u \in P$ with $|v| > |u|$, and let $w' \in B(P)$ with $u < w'$. Since all $w' \in B(P)$ have the same modulus, it follows that $|v| \le |w'|$. Thus we can choose a subvector $w$ of $w'$ in $P$ with $u \le w$ and $|w| = |v|$. Let $P' = \{x \in P : |x| \le |v|\}$. By 2.2(a), $P'$ is a discrete polymatroid, and hence by the implication (b) $\Rightarrow$ (c) which we have already shown, we know that $w$ and $v$ satisfy the exchange property (c)(ii).

Suppose that $w(j) \le \max\{u(j), v(j)\}$ for all $j$. Since $u(j) \le w(j)$ for all $j$, it follows that $w(j) \le v(j)$ for all $j$. However since $|w| = |v|$, this implies $w = v$. Then $u < v$, and the assertion is trivial.

Now assume that there exists an integer $j$ such that $w(j) > \max\{u(j), v(j)\}$. Then by the exchange property, there exists an integer $i$ with $v(i) > w(i)$ such that $w - \varepsilon_i + \varepsilon_j \in P$. Since $u + \varepsilon_i$ is a subvector of $w - \varepsilon_j + \varepsilon_i$, it follows that $u + \varepsilon_i \in P$. Furthermore we have $(u + \varepsilon_i)(i) = u(i) + 1 \le w(i) + 1 \le v(i)$, so that $u + \varepsilon_i \le u \wedge v$.    $\square$

The following corollary is immediate from 2.3 and the definition of a matroid.

**Corollary 2.4** *Let B be a nonempty finite set of integer vectors in $\mathbb{R}_+^n$. The following conditions are equivalent:*
(a) *B is the set of bases of a matroid;*
(b) *B is the set of bases of a discrete polymatroid, and for all $u \in B$ one has $u(i) \le 1$ for $i = 1, \ldots, n$.*

The exchange property 2.3(c)(ii) suggests the following

**Definition 2.5**    Let $B$ be a nonempty set of integer vectors which have the same modulus. Then $B$ satisfies:

(W) the *weak exchange property*, if for all $u$, $v \in B$, $u \neq v$, there exist $i$ and $j$ with $u(i) > v(i)$ and $u(j) < v(j)$ such that $u - \varepsilon_i + \varepsilon_j \in B$,

(S) the *strong exchange property*, if for all $u$, $v \in B$, $u \neq v$, and all $i$ and $j$ with $u(i) > v(i)$ and $u(j) < v(j)$ one has $u - \varepsilon_i + \varepsilon_j \in B$.

**Example 2.6**  (a) Let $B$ be the set of bases of a polymatroid on the ground set $[n]$ with $n \leq 3$. It is immediate that $B$ satisfies the strong exchange property.

(b) The set $\{(1, 1, 1, 1), (0, 2, 0, 2), (0, 1, 1, 2), (1, 2, 0, 1)\}$ is the set of bases of a discrete polymatroid. It does not satisfy the strong exchange property.

(c) Let $s_1, \ldots, s_n$ and $d$ be nonnegative integers. The set

$$V = \{u : u(i) \text{ is an integer with } 0 \leq u(i) \leq s_i \text{ and } |u| = d\}$$

is called of *Veronese type*. It satisfies the strong exchange property.

In fact, let $u$, $v \in V$ with $u(i) > v(i)$ and $u(j) < v(j)$ then $u(i) - 1 \geq 0$ and $u(j) + 1 \leq s_j$, so that $u - \varepsilon_i + \varepsilon_j \in V$.

(d) The set $\{(2, 1, 1), (2, 2, 0), (3, 0, 1), (3, 1, 0)\ (4, 0, 0)\}$ satisfies the strong exchange property, but is not of Veronese type.

(e) A set $S$ of integral vectors in $\mathbb{R}_+^n$ of modulus $d$ is called *strongly stable*, if for all $u \in S$, all $i$ with $u(i) > 0$ and all $j < i$, one has that $u - \varepsilon_i + \varepsilon_j \in S$. The strongly stable set $\{(3, 0, 1), (1, 3, 0), (3, 1, 0), (2, 2, 0), (4, 0, 0)\}$ does not satisfies the weak exchange property. But any strongly stable set $S$ satisfies the following somewhat weaker exchange property: for all $u$, $v \in S$ there exist integers $i$ and $j$ such that $u(i) > v(i)$ and $u(j) < v(j)$, and either $u - \varepsilon_i + \varepsilon_j \in S$, or $v - \varepsilon_j + \varepsilon_i \in S$.

Indeed, let $u$, $v \in S$, $u \neq v$. Then there is an integer $i$ such that $u(i) > v(i)$. Let $i$ be the largest such number. If $u(j) = v(j)$ for all $j > i$, then, since $u$ and $v$ have the same modulus, there exists $j < i$ with $u(j) < v(j)$. Since $S$ is strongly stable it follows that $u - \varepsilon_i + \varepsilon_j \in S$. On the other hand, if $u(j) < v(j)$ for some $j > i$, then $v - \varepsilon_j + \varepsilon_i \in S$, since $S$ is strongly stable.

(f) A strongly stable set $S$ is called *principal*, if there exists an element $u \in S$ such that the smallest strongly stable set containing $u$ coincides with $S$. In that case $u$ is called the Borel generator of $S$. The set in example (d) is a strongly stable principal set with Borel generator $(2, 1, 1)$.

A strongly stable principal set satisfies the exchange property of 2.3(c)(ii), i.e. it is the set of bases of a polymatroid. This will follow from Example 9.4 in Section 9. On the other hand, the set

$$\{(0, 1, 0, 1), (0, 1, 1, 0), (0, 2, 0, 0), (1, 0, 0, 1), (1, 0, 1, 0), (1, 1, 0, 0), (2, 0, 0, 0)\}$$

is strongly stable principal with Borel generator $(0, 1, 0, 1)$, but does not satisfy the strong exchange property.

We close this section with the following

**Theorem 2.7**  *Let $P$ be a nonempty finite set of integer vectors in $\mathbb{R}_+^n$ which contains with each $u \in P$ all its integral subvectors. The following conditions are equivalent:*
(a)  *$P$ is a discrete polymatroid of rank $d$ on the ground set $[n]$;*
(b)  *$B = \{(u, d - |u|) : u \in P\}$ is the set of bases of a discrete polymatroid of rank $d$ on the ground set $[n + 1]$.*

**Proof:**  (a) $\Rightarrow$ (b): We will show that $B$ satisfies condition (c) of 2.3. Let $u, v \in P$, $i = d - |u|$, $j = d - |v|$ and set $u' = (u, i)$ and $v' = (v, j)$. We may suppose that $|v| \geq |u|$, so that $j \leq i$. If $i = j$, then $u$ and $v$ are bases of $P' = \{w \in P : |w| \leq d - i\}$, see 2.2. Thus $u'$ and $v'$ satisfy the exchange property 2.3(c)(ii), and hence we may assume that $j < i$. We consider two cases.

In the first case assume that $v'(k) > u'(k)$ for some $k$. Then $k \leq n$, and $v - \varepsilon_k \in P$, since $v - \varepsilon_k$ is a subvector of $v$, and so $v' - \varepsilon_k + \varepsilon_{n+1} \in B$.

In the second case assume that $u'(k) > v'(k)$ for some $k$. Since $|v| > |u|$, 2.3(b) implies that there exists an integer $l$ such that $u + \varepsilon_l \in P$ and $u + \varepsilon_l \leq u \wedge v$. It follows that $u(l) < u(l) + 1 \leq v(l)$. If $k \leq n$, then $u - \varepsilon_k + \varepsilon_l \in P$, because it is a subvector of $u - \varepsilon_l$. Hence $u' - \varepsilon_k + \varepsilon_l = (u - \varepsilon_k + \varepsilon_l, i) \in B$. On the other hand, if $k = n + 1$, that is, $u'(k) = i$, then $u' - \varepsilon_k + \varepsilon_l = (u + \varepsilon_l, i - 1) \in B$, because $u + \varepsilon_l \in P$.

(b) $\Rightarrow$ (a): Let $u, v \in P$ with $|v| > |u|$. Then $d - |v| < d - |u|$, and so by the exchange property of $B$ there exists an integer $i$ with $v(i) > u(i)$ and such that $(u + \varepsilon_i, d - |u|) \in B$. This implies that $u + \varepsilon_i \in P$, and since $v(i) > u(i)$ we also have that $u + \varepsilon_i \leq u \wedge v$.  $\square$

Theorem 2.7 may be regarded as a version of [6, Theorem 3.58].

## 3.   Integral polymatroids and discrete polymatroids

The purpose of the present section is to show our first fundamental Theorem 3.4 which says that a nonempty finite set $P \subset \mathbb{Z}_+^n$ is a discrete polymatroid if and only if $\mathrm{conv}(P) \subset \mathbb{R}_+^n$ is an integral polymatroid with $\mathrm{conv}(P) \cap \mathbb{Z}^n = P$. Here $\mathrm{conv}(P)$ is the convex hull of $P$ in $\mathbb{R}^n$.

First of all, we collect a few basic lemmata on integral polymatroids and discrete polymatroids which will be required to prove Theorem 3.4.

**Lemma 3.1**  *If $\mathcal{P} \subset \mathbb{R}_+^n$ is an integral polymatroid and if $u, v \in \mathcal{P} \cap \mathbb{Z}^n$ with $|v| > |u|$, then there is $w \in \mathcal{P} \cap \mathbb{Z}^n$ such that $u < w \leq u \vee v$.*

Lemma 3.1 appears (not explicitly) in [17, Lemma 5, p. 340]. Its proof is also valid to show Lemma 3.1 by noting that the ground set rank function $\rho$ of an integral polymatroid $\mathcal{P}$ is integer valued.

Let $\mathcal{P} \subset \mathbb{Z}_+^n$ be a discrete polymatroid and $B(P)$ the set of bases of $P$. We define the nondecreasing function $\rho_P : 2^{[n]} \to \mathbb{R}_+$ associated with $P$ by setting

$$\rho_P(X) = \max\{u(X) : u \in B(P)\}$$

for all $\emptyset \neq X \subset [n]$ together with $\rho_P(\emptyset) = 0$.

In general, for $u, v \in B(P)$ we define the *distance* between $u$ and $v$ by

$$\text{dis}(u, v) = \frac{1}{2} \sum_{i=1}^{n} |u(i) - v(i)|.$$

A crucial property of $\text{dis}(u, v)$ is that if $u(i) > v(i)$ and $u(j) < v(j)$ together with $u' = u - \varepsilon_i + \varepsilon_j \in B(P)$, then $\text{dis}(u, v) > \text{dis}(u', v)$.

**Lemma 3.2**  *If $X_1 \subset X_2 \subset \cdots \subset X_s \subset [n]$ is a sequence of subsets of $[n]$, then there is $u \in B(P)$ such that $u(X_k) = \rho_P(X_k)$ for all $1 \le k \le s$.*

**Proof:**  We work with induction on $s$ and suppose that there is $u \in B(P)$ such that $u(X_k) = \rho_P(X_k)$ for all $1 \le k < s$. Choose $v \in B(P)$ with $v(X_s) = \rho_P(X_s)$. If $u(X_s) < v(X_s)$, then there is $i \in [n]$ with $i \notin X_s$ such that $u(\{i\}) > v(\{i\})$. The exchange property 2.3(c)(ii) says that there is $j \in [n]$ with $u(j) < v(j)$ such that $u_1 = u - \varepsilon_i + \varepsilon_j \in B(P)$. Since $u(X_{s-1}) = \rho_P(X_{s-1})$, it follows that $j \notin X_{s-1}$. Hence $u_1(X_k) = \rho_P(X_k)$ for all $1 \le k < s$. Moreover, $u_1(X_s) \ge u(X_s)$ and $\text{dis}(u, v) > \text{dis}(u_1, v)$.

If $u_1(X_s) = v(X_s)$, then $u_1$ is a desired base of $P$. If $u_1(X_s) < v(X_s)$, then the above technique will yield a base $u_2$ of $P$ such that $u_2(X_k) = \rho_P(X_k)$ for all $1 \le k < s$, $u_2(X_s) \ge u_1(X_s)$ and $\text{dis}(u_1, v) > \text{dis}(u_2, v)$. It is now clear that repeated applications of this argument guarantee the existence of a base $u_q$ of $P$ such that $u_q(X_k) = \rho_P(X_k)$ for all $1 \le k \le s$.    □

**Corollary 3.3**  *The function $\rho_P : 2^{[n]} \to \mathbb{R}_+$ is submodular.*

**Proof:**  Let $A, B \subset [n]$. By Lemma 3.2 there is $u \in B(P)$ such that $u(A \cap B) = \rho_P(A \cap B)$ and $u(A \cup B) = \rho_P(A \cup B)$. Hence

$$\begin{aligned}
\rho_P(A) + \rho_P(B) &\ge u(A) + u(B) \\
&= u(A \cup B) + u(A \cap B) \\
&= \rho_P(A \cup B) + \rho_P(A \cap B),
\end{aligned}$$

as desired.    □

We now come to our first fundamental

**Theorem 3.4**  *A nonempty finite set $P \subset \mathbb{Z}_+^n$ is a discrete polymatroid if and only if $\text{conv}(P) \subset \mathbb{R}_+^n$ is an integral polymatroid with $\text{conv}(P) \cap \mathbb{Z}^n = P$.*

**Proof:**  The "if" part follows from Lemma 3.1. To see why the "only if" part is true, let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid and $\rho_P : 2^{[n]} \to \mathbb{R}_+$ the nondecreasing and submodular function associated with $P$. Write $\mathcal{P} \subset \mathbb{R}_+^n$ for the integral polymatroid with $\rho_P$ its ground set rank function, i.e.,

$$\mathcal{P} = \{u \in \mathbb{R}_+^n : u(X) \le \rho_P(X), \, X \subset [n]\}.$$

Since each base $u$ of $P$ satisfies $u(X) \leq \rho_P(X)$ for all $X \subset [n]$, it follows that $P \subset \mathcal{P}$. Moreover, since $\mathcal{P}$ is convex, one has $\mathrm{conv}(P) \subset \mathcal{P}$. Now, Lemma 3.2 together with Proposition 1.3 guarantees that all vertices of $\mathcal{P}$ belong to $P$. Thus $\mathcal{P} = \mathrm{conv}(P)$.

To complete our proof we must show $\mathcal{P} \cap \mathbb{Z}^n = P$. For each $i \in [n]$, write $P_i \subset \mathbb{Z}_+^n$ for the discrete polymatroid $P_{\varepsilon_i}$ in the notation of Lemma 2.2(b) and $B_i = B(P_i)$, the set of bases of $P_i$. We compute the nondecreasing and submodular function $\rho_{P_i} : 2^{[n]} \to \mathbb{R}_+$ associated with $P_i$. We distinguish three cases:

(a) Let $i \notin X \subset [n]$ with $\rho_P(X \cap \{i\}) > \rho_P(X)$. Choose $u \in B(P)$ with $u(X) = \rho_P(X)$ and with $u(X \cup \{i\}) = \rho_P(X \cup \{i\})$. Then $u(i) = u(X \cup \{i\}) - u(X) \geq 1$ and $u \in B(P)$. Since $i \notin X$, one has $(u - \varepsilon_i)(X) = u(X)$. Since $(u - \varepsilon_i)(X) \leq \rho_{P_i}(X) \leq \rho_P(X) = u(X)$, it follows that $\rho_{P_i}(X) = \rho_P(X)$.

(b) Let $i \notin X \subset [n]$ with $\rho_P(X \cup \{i\}) = \rho_P(X)$. If $u \in B(P)$ with $u(i) \geq 1$, then

$$
\begin{aligned}
(u - \varepsilon_i)(X) &\leq (u - \varepsilon_i)(X \cup \{i\}) \\
&= u(X \cup \{i\}) - 1 \\
&\leq \rho_P(X \cup \{i\}) - 1 = \rho_P(X) - 1.
\end{aligned}
$$

Thus $\rho_{P_i}(X) \leq \rho_P(X) - 1$. Choose $v \in B(P)$ with $v(X) = \rho_P(X)$. Then $v(i) = 0$. (Otherwise, since $i \notin X$, $\rho_P(X \cup \{i\}) \geq v(X \cup \{i\}) = v(X) + v(i) > v(X) = \rho_P(X)$, a contradiction.) Let $u_0 \in B(P)$ with $u_0(i) \geq 1$. Then the exchange property says that there is $j \in [n]$ with $u_0(j) < v(j)$ such that $u_0 - \varepsilon_i + \varepsilon_j \in B(P)$. Thus we assume $u_0(i) = 1$. If $u_0(X) < v(X) - 1$, then $u_0(X \cup \{i\}) = u_0(X) + 1 < v(X) = v(X \cup \{i\})$. Thus there is $j \notin X \cup \{i\}$ with $u_0(j) > v(j)$. Hence there is $i \neq k \in [n]$ with $u_0(k) < v(k)$ such that $u_1 = u_0 = \varepsilon_j + \varepsilon_k \in B(P)$. Then $u_1(i) = 1$, $u_1(X) \geq u_0(X)$ and $\mathrm{dis}(u_0, v) > \mathrm{dis}(u_1, v)$. Thus, as in the proof of Lemma 3.2, we can find $u \in B(P)$ with $u(i) = 1$ such that $u(X) = (u - \varepsilon_i)(X) = v(X) - 1$. Hence $\rho_{P_i}(X) = \rho_P(X) - 1$.

(c) Let $i \in X \subset [n]$. Then $\rho_{P_i}(X) = \rho_P(X) - 1$. In fact, since $i \in X$, by Lemma 3.2 there is $u \in B(P)$ with $u(i) = \rho_P(\{i\}) \geq 1$ and with $u(X) = \rho_P(X)$. Then $(u - \varepsilon_i)(X) = \rho_P(X) - 1$.

Let $\mathcal{P}_i \in \mathbb{R}_+^n$ denote the integral polymatroid with $\rho_{P_i}$ its ground set rank function. Then $\mathcal{P}_i = \mathrm{conv}(P_i)$ and, working with induction on the rank of $P$ enables us to assume that $\mathcal{P}_i \cap \mathbb{Z}^n = P_i$. If $x \in \mathcal{P} \cap \mathbb{Z}^n$ with $x(i) \geq 1$, then $y = x - \varepsilon_i$ belongs to $\mathcal{P}_i$. (In fact, if $i \notin X \subset [n]$ with $\rho_{P_i}(X) = \rho_P(X) - 1$, then $\rho_P(X \cup \{i\}) = \rho_P(X)$. Thus replacing $u$ with $x$ in the inequalities $(u - \varepsilon_i)(X) \leq \cdots = \rho_P(X) - 1$ appearing in the discussion (b) shows that $y(X) \leq \rho_{P_i}(X)$.) Thus $y \in \mathcal{P}_i \cap \mathbb{Z}^n = P_i$. Hence $y \leq u - \varepsilon_i$ for some $u \in B(P)$ with $u(i) \geq 1$. Thus $x \leq u \in B(P)$ and $x \in P$. Hence $\mathcal{P} \cap \mathbb{Z}^n = P$, as desired.                                $\square$

## 4.   The symmetric exchange theorem for polymatroids

We now establish our second fundamental theorem on discrete polymatroids; the symmetric exchange theorem.

**Theorem 4.1**  *If $u = (a_1, \ldots, a_n)$ and $v = (b_1, \ldots, b_n)$ are bases of a discrete polymatroid $P \subset \mathbb{Z}_+^n$, then for each $i \in [n]$ with $a_i > b_i$ there is $j \in [n]$ with $a_j < b_j$ such that both $u - \varepsilon_i + \varepsilon_j$ and $v - \varepsilon_j + \varepsilon_i$ are bases of $P$.*

**Proof:**  Let $B'$ denote the set of those bases $w$ of $P$ with $u \wedge v \leq w \leq u \vee v$. It then turns out that $B'$ satisfies the exchange property 2.3(c)(ii) for polymatroids. Thus $B'$ is the set of bases of a discrete polymatroid $P' \subset \mathbb{Z}_+^n$. Considering $u' = u - u \wedge v$ and $v' = v - u \wedge v$ instead of $u$ and $v$, we will assume that $P' \subset \mathbb{Z}_+^s$ is a discrete polymatroid, where $s \leq n$, and

$$u = (a_1, \ldots, a_r, 0, \ldots, 0) \in \mathbb{Z}_+^s, \quad v = (0, \ldots, 0, b_{r+1}, \ldots, b_s) \in \mathbb{Z}_+^s,$$

where each $0 < a_i$ and each $0 < b_j$ and where $|u| = |v| = \operatorname{rank}(P')$. Our work is to show that for each $1 \leq i \leq r$ there is $r + 1 \leq j \leq s$ such that both $u - \varepsilon_i + \varepsilon_j$ and $v - \varepsilon_j + \varepsilon_i$ are bases of $P'$. Let, say, $i = 1$.

*First case*: Suppose that $u - \varepsilon_1 + \varepsilon_j$ are bases of $P'$ for all $r + 1 \leq j \leq s$. It follows from the exchange property that, given arbitrary $r$ integers $a_1', \ldots, a_r'$ with each $0 \leq a_i' \leq a_i$, there is a base $w'$ of $P'$ of the form

$$w' = (a_1', \ldots, a_r', b_{r+1}', \ldots, b_s'),$$

where each $b_j' \in \mathbb{Z}$ with $0 \leq b_j' \leq b_j$. Thus in particular there is $r + 1 \leq j_0 \leq s$ such that $v - \varepsilon_{j_0} + \varepsilon_1$ is a base of $P'$. Since $u - \varepsilon_1 + \varepsilon_j$ is a base of $P'$ for each $r + 1 \leq j \leq s$, both $u - \varepsilon_1 + \varepsilon_{j_0}$ and $v - \varepsilon_{j_0} + \varepsilon_1$ are bases of $P'$, as desired.

*Second case*: Let $r \geq 2$ and $r + 2 \leq s$. Suppose that there is $r + 1 \leq j \leq s$ with $u - \varepsilon_1 + \varepsilon_j \notin P'$. Let $X \subset \{r+1, \ldots, s\}$ denote the set of those $r + 1 \leq j \leq s$ with $u - \varepsilon_1 + \varepsilon_j \notin P'$. Recall that Theorem 3.4 guarantees that $\operatorname{conv}(P') \subset \mathbb{R}_+^s$ is an integral polymatroid on the ground set $[s]$ with $\operatorname{conv}(P') \cap \mathbb{Z}^s = P'$. Let $\rho = \rho_{P'}$ denote the ground set rank function of the integral polymatroid $\operatorname{conv}(P') \subset \mathbb{R}_+^s$. Thus $\rho(Y) = \max\{w(Y) : w \in B'\}$ for $\emptyset \neq Y \subset [s]$ together with $\rho(\emptyset) = 0$. In particular $\rho(Y) = u(Y)$ if $Y \subset \{1, \ldots, r\}$ and $\rho(Y) = v(Y)$ if $Y \subset \{r + 1, \ldots, s\}$.

For each $j \in X$, since $u - \varepsilon_1 + \varepsilon_j \notin \operatorname{conv}(P')$, there is a subset $A_j \subset \{2, 3, \ldots, r\}$ with

$$\rho(A_j \cup \{j\}) \leq u(A_j).$$

Thus

$$\begin{aligned}
\rho(\{2, 3, \ldots, r\} \cup \{j\}) &\leq \rho(A_j \cup \{j\}) + \rho(\{2, 3, \ldots, r\} \setminus A_j) \\
&\leq u(A_j) + u(\{2, 3, \ldots, r\} \setminus A_j) \\
&= u(\{2, 3, \ldots, r\}) = \rho(\{2, 3, \ldots, r\}).
\end{aligned}$$

Hence, for all $j \in X$,

$$\rho(\{2, 3, \ldots, r\} \cup \{j\}) = u(\{2, 3, \ldots, r\}).$$

By [15, Lemma 1.3.3] it follows that

$$\rho(\{2, 3, \ldots, r\} \cup X) = u(\{2, 3, \ldots, r\}).$$

Now, since $\rho$ is submodular,

$$\rho(\{2, 3, \ldots, r\} \cup X) + \rho(\{1\} \cup X) \geq \rho(X) + \rho(\{1, 2, \ldots, r\} \cup X)$$
$$= v(X) + \text{rank}(P').$$

Thus

$$u(\{2, 3, \ldots, r\}) + \rho(\{1\} \cup X) \geq v(X) + \text{rank}(P').$$

Since

$$\text{rank}(P') - u(\{2, 3, \ldots, r\}) = a_1, \text{ and}$$
$$\rho(\{1\} \cup X) \leq \rho(\{1\} + \rho(X) = a_1 + v(X),$$

it follows that

$$\rho(\{1\} \cup X) = a_1 + v(X).$$

Hence, for all $X' \subset X$,

$$\begin{aligned}
a_1 + v(X) &= a_1 + v(X') + v(X \backslash X') \\
&= \rho(\{1\} + \rho(X') + \rho(X \backslash X') \\
&\geq \rho(\{1\} \cup X') + \rho(X \backslash X') \\
&\geq \rho(\{1\} \cup X) \\
&= a_1 + v(X).
\end{aligned}$$

Thus, for all $X' \subset X$,

$$\rho(\{1\} \cup X') = a_1 + v(X').$$

By virtue of Lemma 3.2 there is a base $w$ of $P'$ with $w(1) = a_1$ and with $w(j) = v(j)$ $(= \rho(\{j\}))$ for all $j \in X$. Again the exchange property (for $w$ and $v$) guarantees that for each $1 \leq i \leq r$ with $w(i) > 0$ there is $j \in \{r + 1, \ldots, s\} \backslash X$ such that $w - \varepsilon_i + \varepsilon_j$, is a base of $P'$. Hence repeated applications of the exchange property yield a base of $w'$ of $P'$ of the form $w' = v - \varepsilon_{j_0} + \varepsilon_1$, where $j_0 \in \{r + 1, \ldots, s\} \backslash X$. Hence both $u - \varepsilon_1 + \varepsilon_{j_0}$ and $v - \varepsilon_{j_0} + \varepsilon_1$ are bases of $P'$, as required.                                        □

## 5. Base rings of discrete polymatroids and their relations

Let $K$ be a field, and $P$ a discrete polymatroid with the set of bases $B$. The toric ring $K[B]$ generated over $K$ by the monomials $t^u = \prod_i t_i^{u(i)}$, $u = (u(1), \dots, u(n)) \in B$, is called the *base ring of P*. Let $S = K[x_u : u \in B]$ be the polynomial ring in the indeterminates $x_u$ with $u \in B$. Let $I_B$ be the kernel of the $K$-algebra homomorpism $\xi : S \to K[B]$ with $\xi(x_u) = t^u$. There are some obvious elements in $I_B$, namely, those arising from symmetric exchange: Let $u, v \in B$ with $u(i) > v(i)$ and $u(j) < v(j)$, such that $u - \varepsilon_i + \varepsilon_j$ and $v - \varepsilon_j + \varepsilon_i$ belong to $B$. Then clearly $x_u x_v - x_{u-\varepsilon_i+\varepsilon_j} x_{v-\varepsilon_j+\varepsilon_i} \in I_B$. We call such a relation a *symmetric exchange relation*. White conjectured (cf. [18]) that for a matroid the symmetric exchange relations generate $I_B$. It is natural to conjecture that this also holds for discrete polymatroids.

In order to formulate the next result we have to recall the notion of sortability, introduced by Sturmfels [16]:

Let $u, v \in B$, and write $t^u t^v = t_{i_1} \dots t_{i_{2d}}$ with $i_1 \leq i_2 \leq \cdots \leq i_{2d}$. Then we set $t^{u'} = \prod_j^d t_{2j-1}$ and $t^{v'} = \prod_j^d t_{2j}$. This defines a map

$$\text{sort}: B \times B \to M_d \times M_d, \quad (u, v) \to (u', v'),$$

where $M_d$ denotes the set of all integer vectors of modulus $d$.

The map 'sort' is called the *sorting operator*, and $B$ is called *sortable*, if $\text{Im(sort)} \subseteq B \times B$.

The pair $(u, v)$ is called *sorted* if $(u, v) \in \text{Im(sort)}$, equivalently, if $\text{sort}(u, v) = (u, v)$. It is clear from the definition that $(u, v)$ is sorted if and only if

$$u(1) + \cdots + u(i) - 1 \leq v(1) + \cdots + v(i) \leq u(1) + \cdots + u(i) \quad \text{for } i = 1, \dots, n.$$

This implies that $u - v$ is vector with entries $\pm 1$ and 0. To such a vector we attach sequence of $+$ and $-$ signs: reading from the left to right we put the sign $+$ or the sign $-$ if we reach the entry $+1$ or the entry $-1$. For example to $(0, 1, 1, 0, -1, -1)$ belongs the sequence $+ + - -$.

For the above characterization of sorted pairs one easily deduces

**Lemma 5.1** *The following conditions are equivalent*:
(a) $(u, v)$ *or* $(v, u)$ *is sorted*;
(b) $u - v$ *is a vector with entries $\pm 1$ and 0 and with $\pm$-sequence $+ - + - \cdots + -$.*

In [3], De Negri noticed the following fact which easily follows from a theorem of Sturmfels [16].

**Lemma 5.2** *Suppose $B \subset M_d$ is sortable. Then $I_B$ has a Gröbner basis consisting of the sorting relations $x_u x_v - x_{u'} x_{v'}$ with $u, v \in B$ and $(u', v') = \text{sort}(u, v)$.*

The main result of this section is

**Theorem 5.3** (a) *Suppose that each matroid has the property that the toric ideal of its base ring is generated by symmetric exchange relations, then this is also true for each discrete polymatroid.*

(b) *If P is a discrete polymatroid whose set of bases B satisfies the strong exchange property, then*

(i) *B is sortable, and hence $I_B$ has a quadratic Gröbner basis and $K[B]$ is Koszul,*

(ii) *$I_B$ is generated by symmetric exchange relations.*

For the proof of 5.3(a) we shall need.

**Lemma 5.4** *Let $u_1, \ldots, u_d \in B$. Then, modulo exchange relations, $\prod_j^d x_{u_j}$ can be rewritten as $\prod_j^d x_{v_j}$ with $|v_j(i) - v_k(i)| \leq 1$ for all $i$, $j$, $k$.*

**Proof:** Suppose that for some $i, k$ and $l$ we have $|u_k(i) - u_l(i)| > 1$. Without loss of generality we may assume that $k = 1, l = 2$ and $u_1(i) > u_2(i)$. By the symmetric exchange property there exists $j$ with $u_2(j) > u_1(j)$ such that $u_1 - \varepsilon_i + \varepsilon_j, u_2 - \varepsilon_j + \varepsilon_i \in B$.

We set

$$
u'_k = \begin{cases} u_k, & \text{for } k \neq 1, 2, \\ u_1 - \varepsilon_i + \varepsilon_j, & \text{for } k = 1, \\ u_2 - \varepsilon_j + \varepsilon_i, & \text{for } k = 2. \end{cases}
$$

It is clear that $u'_1(i) - u'_2(i) = u_1(i) - u_2(i) - 2 \geq 0$.

We introduce the number

$$
c_i(u_1, \ldots, u_d) = \sum_{1 \leq k < l \leq d} |u_k(i) - u_l(i)|.
$$

Then $c_k(u_1, \ldots, u_d) = c_k(u'_1, \ldots, u'_d)$ for $k \neq i, j$.

It is easy to show that

$$
c_i(u'_1, \ldots, u'_d) < c_i(u_1, \ldots, u_d), \tag{2}
$$

and

$$
c_j(u'_1, \ldots, u'_d) \leq c_j(u_1, \ldots, u_d). \tag{3}
$$

Thus, induction completes the proof.                                         $\square$

**Proof:** [Proof of 5.3] (a) Let $\prod_j x_{u_j} - \prod_j x_{v_j} \in I_B$. Applying Lemma 5.4 we may assume that for all $j$ and $k$, all components of $u_j$ and $u_k$, and of $v_j$ and $v_k$ differ at most by 1. Let $w = \sum_j^d u_j = \sum_j^d v_j$, and for $i = 1, \ldots, n$ let $a_i - \min\{u_j(i) : j = 1, \ldots, d\}$, and similarly $b_i = \min\{v_j(i) : j = 1, \ldots, d\}$. Then $u_j(i) = a_i$ or $u_j(i) = a_i + 1$. Let $k_i$ be the number of $j$ with $u_j(i) = a_i + 1$. Then $w(i) = \sum_{j=1}^d u_j(i) = da_i + k_i$. Since $0 \leq k_i < d$, we conclude that $a_i = [w(i)/d]$, where $\lfloor c \rfloor$ denotes the integer part of a number $c$. In particular it follows that $a_i = b_i$ for $i = 1, \ldots, n$.

Now we consider the following subset

$$B' = \{u \in B : a_i \le u(i) \le a_i + 1\}$$

of $B$. The vectors $u_1, \ldots, u_d$ and $v_1, \ldots, v_d$ belong to $B'$, and $B'$ is closed under the exchange relations.

We may identify $B'$ with the set of bases of a matroid $B''$, via the assignment $u \mapsto u^* = u - (a_1, \ldots, a_n)$. Then the relation $\prod_j x_{u_j} - \prod_j x_{v_j} \in I_B$ corresponds to the relation $\prod_j x_{u_j^*} - \prod_j x_{v_j^*} \in I_{B''}$. By our hypothesis, $\prod_j x_{u_j^*} - \prod_j x_{v_j^*}$ reduces to 0 modulo the symmetric exchange relations of $B''$. Thus $\prod_j x_{u_j} - \prod_j x_{v_j}$ reduces to 0 modulo the symmetric exchange relations of $B'$, and hence of $B$.

(b) (i) Let $(u, v) \in B \times B$. In the proof of (a) we have shown that by a finite number of exchanges we can transform the pair $(u, v)$ into a pair whose difference vector has entries $\pm 1$ and 0. Thus we may assume that $(u, v)$ itself has this property. Since $u$ and $v$ have the same modulus, it follows that the $\pm 1$-sequence of $u - v$ has as many $+$ signs as $-$ signs. If the $\pm$-sequence does not have the pattern $+ - + - \cdots + -$, then obviously we can obtain such a sequence by a finite number of symmetric exchanges. Thus 5.1 shows that $B$ is sortable. The other statements of (i) follow from 5.2.

(ii) We have seen in the proof of (i) that by a finite number of symmetric exchanges we can sort any pair $(u, v) \in B$. It follows that the sorting relation $x_u x_v - x_{u'} x_{v'}$ is a linear combination of the corresponding symmetric exchange relations. Since by (i), the sorting relations generate $I_B$, the assertion follows. $\qquad\square$

## 6. The Ehrhart ring and the base ring of a discrete polymatroid

Let $K$ be a field, and let $P$ be a discrete polymatroid of rank $d$ on the ground set $[n]$ with set of bases $B$. Since $P$ is the set of integer vectors of an integral polymatroid $\mathcal{P}$ (see Theorem 3.4), we may study the *Ehrhart ring of $\mathcal{P}$*. To this end one considers the cone $\mathcal{C} \subset \mathbb{R}^{n+1}$ with $\mathcal{C} = \mathbb{R}_+\{(p, 1): p \in \mathcal{P}\}$ Then $Q = \mathcal{C} \cap \mathbb{Z}^{n+1}$ is subsemigroup of $\mathbb{Z}^{n+1}$, and the Ehrhart ring of $\mathcal{P}$ is defined to be the toric ring $K[\mathcal{P}] \subset K[t_1, \ldots, t_n, s]$ generated over $K$ by the monomials $t^u s^i, (u, i) \in Q$. By Gordan's lemma (cf. [1, Proposition 6.1.2]), $K[\mathcal{P}]$ is normal.

Notice that $K[\mathcal{P}]$ is naturally graded if we assign to $t^u s^i$ the degree $i$. We denote by $K[P]$ the $K$-subalgebra of $K[\mathcal{P}]$ which is generated over $K$ by the elements of degree 1 in $K[\mathcal{P}]$. Since $P = \mathcal{P} \cap \mathbb{Z}^n$ it follows that $K[P] = K[t^u s: u \in P]$. Observe that $K[B]$ may be identified with the subalgebra $K[t^u s: u \in B]$ of $K[P]$. After this identification $K[B]$ is an algebra retract of $K[P]$ with retraction map $\pi: K[P] \to K[B]$, where $\pi$ is the residue class map modulo the prime ideal $\wp$ generated by the elements $t^u s, u \in P, |u| < d$.

As an immediate consequence of 1.4 we obtain the following important.

**Theorem 6.1** $K[P] = K[\mathcal{P}]$. *In particular, $K[P]$ is normal.*

**Corollary 6.2** $K[B]$ *is normal.*

**Proof:** By [1, Theorem 6.1.4], $K[P]$ is normal if and only if the semigroup $S$ generated by $\hat{P} = \{(u, 1) : u \in P\}$ is normal. This means that if $G$ is the smallest subgroup of $\mathbb{Z}^{n+1}$ containing $S$ and if $ma \in S$ for some $m \in \mathbb{N}$ and $a \in G$, then $a \in S$. We apply the same criterion to show that $K[B]$ is normal. Let $H$ be the subgroup of $\mathbb{Z}^{n+1}$ consisting of all $(u, i)$ with $|u| = id$, and let $T$ be the semigroup generated by $\hat{B} = \{(u, 1) : u \in B\}$. Then $H \cap G$ is the smallest group containing the semigroup $T$. Hence if $ma \in T$ for some $m \in \mathbb{N}$ and $a \in H \cap G$, then $a \in S \cap H = T$, as desired. $\qquad\square$

Our next aim is to compare algebraic properties of $K[P]$ with those of $K[B]$. We say that a $K$-algebra $A$ has a quadratic Gröbner basis if the defining ideal of $A$ has this property for some term order.

**Theorem 6.3** (a) *Suppose $K[P]$ has quadratic relations, or a quadratic Gröbner basis or is Koszul, then $K[B]$ has these properties, too.*

(b) *Given a property $\mathcal{E}$. Suppose that $K[B(P)]$ satisfies $\mathcal{E}$ for all discrete polymatroids $P$. Then also $K[P]$ satisfies $\mathcal{E}$ for all discrete polymatroids $P$.*

In particular it follows from the theorem that the properties that the Ehrhart ring of all discrete polytopes is Koszul, if and only if this is the case for all base rings of all discrete polytopes. Not all properties behave this way. For example, there exist discrete polytopes (see 7.1) for which $K[P]$ is Gorenstein but $K[B]$ is not, and vice versa.

**Proof:** [Proof of 6.3] (a) We noticed already that $K[B]$ may be viewed as an algebra retract of $K[P]$. Hence the statements concerning quadratic relations and Koszulness follow from [14, Proposition 1.4, Corollary 2.5].

Now let $A \subset B$ be an arbitrary retract of standard graded toric $K$-algebras. Then we may assume that $A = K[x_1, \ldots, x_n]/I$, and $B = K[x_1, \ldots, x_n, y_1, \ldots, y_m]/J$ with toric ideals $I$ and $J$ contained in the square of the corresponding graded maximal ideals, and that the retraction map $B \to A$ is induced by $\pi(y_j) = 0$ for $j = 1, \ldots, m$.

We use the following well-known fact: suppose $J$ has a Gröbner basis $G$ with respect to the term order $<$, and suppose that for all $f \in G$ with $\text{in}_<(f) \in K[x_1, \ldots, x_n]$ one has that $f \in k[x_1, \ldots, x_n]$. Then $G' = G \cap K[x_1, \ldots, x_n]$ is a Gröbner basis of $I$ (with respect to the restricted term order).

Since $B$ is toric, we may assume that the elements of $G$ are binomials. Suppose now that $f \in G$ with $f = m_1 - m_2$ and $\text{in}(f) = m_1 \in K[x_1, \ldots, x_n]$, and suppose that $m_2 \notin K[x_1, \ldots, x_n]$. Then $m_1 = \pi(f) \in I$, a contradiction since $I$ is a prime ideal.

(b) We notice that $K[P]$ is isomorphic to the toric ring $K[t^u s^{d-|u|} : u \in P]$ which, as we know from 2.7, is the base ring of a polymatroid. Thus the conclusion follows. $\qquad\square$

**Remark 6.4** A monomial ideal generated by monomials corresponding to the base of a discrete polymatroid is called a *polymatroidal ideal*. It has been shown in [8] that polymatroidal ideals have linear quotients, so that, in particular, they have a linear resolution. As another immediate application of 1.4 one has that the product of polymatroidal ideals is again polymatroidal. This fact has been shown directly in [2].

## 7. Gorenstein polymatroids

Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid and $B = B(P)$ the set of bases of $P$. We know that both the Ehrhart ring $K[P]$ and the base ring $K[B]$ are normal; thus Cohen–Macaulay. It is then natural to ask when these rings are Gorenstein. We begin with

**Example 7.1**  (a) Let $P \subset \mathbb{Z}_+^3$ be the discrete polymatroid consisting of all integer vectors $u \in \mathbb{Z}_+^3$ with $|u| \leq 3$. Then the base ring $K[B]$ is the Veronese subring $K[x, y, z]^{(3)}$, the subring of $K[x, y, z]$ generated by all monomials of degree 3. Thus $K[B]$ is Gorenstein. On the other hand, since the Hilbert series of the Ehrhart ring $K[P]$ is $(1 + 16\lambda + 10\lambda^2)/(1 - \lambda)^4$, it follows that $K[P]$ is not Gorenstein.
 (b) Let $P \subset \mathbb{Z}_+^3$ be the discrete polymatroid consisting of all integer vectors $u \in \mathbb{Z}_+^3$ with $|u| \leq 4$. Then $K[B] = K[x, y, z]^{(4)}$ is not Gorenstein. On the other hand, the Hilbert series of the Ehrhart ring $K[P]$ is $(1 + 31\lambda + 31\lambda^2 + \lambda^3)/(1 - \lambda)^4$; thus $K[P]$ is Gorenstein.
 (c) Let $P \subset \mathbb{Z}_+^2$ be the discrete polymatroid with $B = \{(1, 2), (2, 1)\}$ its set of bases. Then both $K[P]$ and $K[B]$ are Gorenstein.

Let, in general, $\mathcal{P} \subset \mathbb{R}^n$ be an integral convex polytope of dimension $n$ and $K[\mathcal{P}]$ the Ehrhart ring of $\mathcal{P}$. A combinatorial criterion [4, Corollary (1. 2)] for $K[\mathcal{P}]$ to be Gorenstein is stated below. Let $\delta > 0$ denote the smallest integer for which $\delta(\mathcal{P} \backslash \partial \mathcal{P}) \cap \mathbb{Z}^n \neq \emptyset$. Here $\delta \mathcal{P} = \{\delta \alpha : \alpha \in \mathcal{P}\}$, $\partial \mathcal{P}$ is the boundary of $\mathcal{P}$ and $\mathcal{P} \backslash \partial \mathcal{P}$ is the interior of $\mathcal{P}$. Then $K[\mathcal{P}]$ cannot be Gorenstein unless $\delta(\mathcal{P} \backslash \partial \mathcal{P})$ possesses a unique integer vector. In case that $\delta(\mathcal{P} \backslash \partial \mathcal{P})$ possesses a unique integer vector, say $\alpha_0 \in \mathbb{Z}^n$, let $\mathcal{Q} = \delta \mathcal{P} - \alpha_0 = \{\alpha - \alpha_0 : \alpha \in \delta \mathcal{P}\}$. Thus $\mathcal{Q} \subset \mathbb{R}^n$ is an integral convex polytope of dimension $n$ and the origin of $\mathbb{R}^n$ is a unique integer vector belonging to the interior $\mathcal{Q} \backslash \partial \mathcal{Q}$ of $\mathcal{Q}$. Now, [4, Corollary (1.2)] says that the Ehrhart ring $K[\mathcal{P}]$ is Gorenstein if and only if the following condition is satisfied: If the hyperplane $\mathcal{H} \subset \mathbb{R}^n$ determined by a linear equation $\sum_{i=1}^n a_1 x_i = b$, where each $a_i$ and $b$ are integers and where the greatest common divisor of $a_1, \ldots, a_n, b$ is equal to 1, is a supporting hyperplane of $\mathcal{Q}$ such that $\mathcal{H} \cap \mathcal{Q}$ is a facet of $\mathcal{Q}$, then $b$ is either 1 or −1.

First of all, we discuss the problem for which discrete polymatroids $P \subset \mathbb{Z}_+^n$ the Ehrhart ring $K[P]$ is Gorenstein. In what follows, we will assume that the canonical basis vectors $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathbb{R}^n$ belong to $P$. In order to apply the above criterion to the Ehrhart ring $K[P]$, it is required to know linear equations of supporting hyperplanes which define facets of the integral polymatroid $\mathrm{conv}(P) \subset \mathbb{R}_+^n$. Note that, since each $\varepsilon_i \in P$, the dimension of $\mathrm{conv}(P)$ is equal to $n$. Let $\rho$ denote the ground set rank function of $\mathrm{conv}(P)$. We say that $\emptyset \neq A \subset [n]$ is $\rho$-*closed* if any subset $B \subset [n]$ properly containing $A$ satisfies $\rho(A) < \rho(B)$, and that $\emptyset \neq A \subset [n]$ is $\rho$-*separable* if there exist two nonempty subsets $A_1$ and $A_2$ of $A$ with $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$ such that $\rho(A) = \rho(A_1) + \rho(A_2)$. For $\emptyset \neq A \subset [n]$ define the hyperplane $\mathcal{H}_A \subset \mathbb{R}^n$ by

$$\mathcal{H}_A = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i \in A} x_i = \rho(A)\right\}.$$

In addition, for each $i \in [n]$ define the hyperplane $\mathcal{H}^{(i)} \subset \mathbb{R}^n$ by

$$\mathcal{H}^{(i)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = 0\}.$$

Edmonds [5] says

**Proposition 7.2** *The facets of* $\operatorname{conv}(P) \subset \mathbb{R}^n_+$ *are all* $\mathcal{H}^{(i)} \cap \operatorname{conv}(P)$, *where i ranges over all integers belonging to* $[n]$, *and all* $\mathcal{H}_A \cap \operatorname{conv}(P)$, *where A ranges over all* $\rho$-*closed and* $\rho$-*inseparable subsets of* $[n]$.

We are now in the position to state a characterization for the Ehrhart ring $K[P]$ of a discrete polymatroid $P \subset \mathbb{Z}^n_+$ to be Gorenstein. For $A \subset [n]$ write $|A|$ for the cardinality of $A$.

**Theorem 7.3** *Let* $P \subset \mathbb{Z}^n_+$ *be a discrete polymatroid and suppose that the canonical basis vectors* $\varepsilon_1, \ldots, \varepsilon_n$ *of* $\mathbb{R}^n$ *belong to P. Let* $\rho$ *denote the ground set rank function of the integral polymatroid* $\mathcal{P} = \operatorname{conv}(P) \subset \mathbb{R}^n$. *Then the Ehrhart ring* $K[P]$ *of P is Gorenstein if and only if there exists an integer* $\delta \geq 1$ *such that*

$$\rho(A) = \frac{1}{\delta}(|A| + 1)$$

*for all* $\rho$-*closed and* $\rho$-*inseparable subsets A of* $[n]$.

**Proof:** Let $\delta \geq 1$ denote the smallest integer for which $\delta(\mathcal{P} \backslash \partial \mathcal{P}) \cap \mathbb{Z}^n \neq \emptyset$. Since $\varepsilon_1, \ldots, \varepsilon_n$ belong to $P$, it follows that $\delta(\mathcal{P} \backslash \partial \mathcal{P}) \cap \mathbb{Z}^n \neq \emptyset$ if and only if $(1, \ldots, 1) \in \delta(\mathcal{P} \backslash \partial \mathcal{P})$. By virtue of Proposition 7.2 the equations of the supporting hyperplanes which define the facets of $\delta \mathcal{P} - (1, \ldots, 1)$ are all $x_i = -1, 1 \leq i \leq n$, and all $\sum_{i \in A} x_i = \delta\rho(A) - |A|$, where $A$ ranges over all $\rho$-closed and $\rho$-inseparable subsets of $[n]$. Since $(1, \ldots, 1) \in \delta\mathcal{P}$, one has $|A| \leq \delta\rho(A)$. Thus if $K[P]$ is Gorenstein, then $\delta\rho(A) - |A| = 1$ for all $\rho$-closed and $\rho$-inseparable subsets of $[n]$. Conversely, if there is an integer $\delta \geq 1$ such that $\delta\rho(A) - |A| = 1$ for all $\rho$-closed and $\rho$-inseparable subsets of $[n]$, then $(1, \ldots, 1)$ is a unique integer vector belonging to $\delta(\mathcal{P} \backslash \partial \mathcal{P})$ and $K[P]$ is Gorenstein. $\square$

**Example 7.4** (a) Let $P^{(d)}_n \subset \mathbb{Z}^n_+$ be the discrete polymatroid consisting of all integer vectors $u \in \mathbb{Z}^n_+$ with $|u| \leq d$ and $B^{(d)}_n$ the set of bases of $P^{(d)}_n$. Let $\rho$ denote the ground set rank function of the integral polymatroid $\operatorname{conv}(P) \subset \mathbb{R}^n_+$. Then $\rho(A) = d$ for all $\emptyset \neq A \subset [n]$. Thus $[n]$ is the only $\rho$-closed and $\rho$-inseparable subset of $[n]$. Hence the Ehrhart ring $K[P^{(d)}_n]$ is Gorenstein if and only if $d$ divides $n + 1$. On the other hand, the base ring $K[B^{(d)}_n]$ is the Veronese subring $K[t_1, \ldots, t_n]^{(d)}$. Thus $K[B^{(d)}_n]$ is Gorenstein if and only if $d$ divides $n$. (Note, in fact, that $K[P^{(d)}_n]$ is just the Veronese subring $K[x_1, \ldots, x_n, s]^{(d)}$.)

(b) Fix $\delta \geq 1$ which divides $n + 1$. Let $\rho : 2^{[n]} \to \mathbb{R}_+$ denote the nondecreasing function defined by

$$\rho(A) = \lceil (\max(A) + 1)/\delta \rceil$$

for all $\emptyset \neq A \subset [n]$ together with $\rho(\emptyset) = 0$, where $\max(A)$ is the biggest integer belonging to $A$ and where $\lceil x \rceil$ is the smallest integer $\geq x$. Then $\rho$ is submodular. (In fact, if $A$, $B \subset [n]$ are nonempty with $\max(B) \leq \max(A)$, then $\max(A \cup B) = \max(A)$ and $\max(A \cap B) \leq \max(B)$. Thus $\rho(A \cup B) = \rho(A)$ and $\rho(A \cap B) \leq \rho(B)$.) If $\emptyset \neq A \subset [n]$ is $\rho$-closed, then $A = [r]$ for some $1 \leq r \leq n$ such that $\delta$ divides $r + 1$. Moreover, if $\delta$ divides $r + 1$, then $A = [r]$ satisfies $\rho(A) = (|A| + 1)/\delta$. Let $d = (n + 1)/\delta$. Let $P \subset \mathbb{Z}_+^n$ be the discrete polymatroid of rank $d$ arising from $\rho$, i.e., $P$ consists of all integer vectors $u \in \mathbb{Z}_+^n$ such that $u(A) \leq \rho(A)$ for all $\emptyset \neq A \subset [n]$. More precisely, $u = (u_1, \ldots, u_n) \in \mathbb{Z}_+^n$ belongs to $P$ if and only if

$$u_1 + u_2 + \cdots + u_{i\delta - 1} \leq i, \quad i = 1, 2, \ldots, d.$$

It follows from Theorem 7.3 that the Ehrhart ring $K[P]$ is Gorenstein.

We also discuss the base ring of this discrete polymatroid. Let $\delta \geq 2$. Then $u \in \mathbb{Z}_+^n$ is a base of $P$ if and only if $u$ is of the form

$$u = \varepsilon_{i_1} + \varepsilon_{i_2} + \cdots + \varepsilon_{i_d},$$

where $1 \leq i_1 \leq \delta - 1$ and where $\delta(j - 1) \leq i_j \leq \delta j - 1$ for $j = 2, \ldots, d$. Thus the base ring $K[B]$ of $P$ is the Segre product

$$K[t_1, \ldots, t_{\delta - 1}] \sharp K[t_\delta, \ldots, t_{2\delta - 1}] \sharp \cdots \sharp K\big[t_{(d-1)\delta}, \ldots, t_{d\delta - 1}\big]$$

of the polynomial ring in $\delta - 1$ variables and $d - 1$ copies of the polynomial ring in $\delta$ variables. Thus $K[B]$ is not Gorenstein unless $\delta = n + 1$. If $\delta = 1$, then $(2, 1, \ldots, 1)$ is a unique base of $P$ and the base ring $K[B] = K[t_1^2 t_2 \ldots t_n]$ is Gorenstein.

(c) Let $\rho: 2^{[n]} \to \mathbb{R}_+$ denote the nondecreasing function defined by $\rho(A) = |A| + 1$ for all $\emptyset \neq A \subset [n]$ together with $\rho(\emptyset) = 0$. Then $\rho$ is submodular and all nonempty subsets of $[n]$ are $\rho$-closed and $\rho$-inseparable. Let $P \subset \mathbb{Z}_+^n$ be the discrete polymatroid of rank $n + 1$ arising from $\rho$. Then the Ehrhart ring $K[P]$ is Gorenstein. Moreover, since the set of bases of $P$ is $B = \{(2, 1, \ldots, 1), \ldots, (1, \ldots, 1, 2)\}$, the base ring $K[B]$ isomorphic to the polynomial ring in $n$ variables; thus $K[B]$ is Gorenstein.

We now turn to the problem when the base ring of a discrete polymatroid is Gorenstein. To obtain a perfect answer to this problem seems, however, quite difficult. For example, the discussions appearing in [4] for classifying Gorenstein rings belonging to the class of algebras of Veronese type, a distinguished class of discrete polymatroids (Example 2.6 (c)), is enough complicated. In what follows we introduce the concept of "generic" discrete polymatroids and find a characterization for the base ring of a generic discrete polymatroid to be Gorenstein.

Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid of rank $d$ and $B = B(P)$ the set of bases of $P$. We will, as before, assume that the canonical basis vectors $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathbb{R}^n$ belong to $P$. Let $\mathcal{F} = \text{conv}(B)$, the set of bases of the integral polymatroid $\mathcal{P} = \text{conv}(P) \subset \mathbb{R}_+^n$. Recall that $\mathcal{F}$ is a face of $\mathcal{P}$ with the supporting hyperplane $\mathcal{H}_{[n]} \subset \mathbb{R}^n$, i.e., $\mathcal{F} = \mathcal{H}_{[n]} \cap \mathcal{P}$.

Let $\rho: 2^{[n]} \to \mathbb{R}_+$ denote the ground set rank function of $\mathcal{P}$. Then

$$\mathcal{F} = \left\{ u \in \mathcal{H}_{[n]} \cap \mathbb{Z}_+^n : u(A) \leq \rho(A), \emptyset \neq A \subset [n], A \neq [n] \right\}.$$

Let $\varphi : \mathcal{H}_{[n]} \to \mathbb{R}^{n-1}$ denote the affine transformation defined by

$$\varphi(u_1, \ldots, u_n) = (u_1, \ldots, u_{n-1}).$$

Thus $\varphi$ is injective and $\varphi(\mathcal{H}_{[n]} \cap \mathbb{Z}^n) = \mathbb{Z}^{n-1}$. Since for all $A \subset [n]$ with $n \in A$ and $A \neq [n]$ the hyperplane $\varphi(\mathcal{H}_A \cap \mathcal{H}_{[n]}) \subset \mathbb{R}^{n-1}$ is determined by the linear equation $\sum_{i \in [n] \setminus A} x_i = d - \rho(A)$, it follows that

$$\varphi(\mathcal{F}) = \{ u \in \mathbb{R}_+^{n-1} : d - \rho([n] \setminus A) \leq u(A) \leq \rho(A), \emptyset \neq A \subset [n-1] \}.$$

We say that $P$ is *generic* if

(G1) each base $u$ of $P$ satisfies $u(i) > 0$ for all $1 \leq i \leq n$;
(G2) $\mathcal{F} = \text{conv}(B)$ is a facet of $\mathcal{P} = \text{conv}(P)$;
(G3) $\mathcal{F} \cap \mathcal{H}_A$ is a facet of $\mathcal{F}$ for all $\emptyset \neq A \subset [n]$ with $A \neq [n]$.

It follows that $P$ is generic if and only if (i) $\rho$ is strictly increasing, (ii) $\dim \varphi(\mathcal{F}) = n - 1$, and (iii) the facets of $\varphi(\mathcal{F})$ are all $\{ u \in \varphi(\mathcal{F}) : u(A) = \rho(A) \}$ and all $\{ u \in \varphi(\mathcal{F}) : u(A) = d - \rho([n] \setminus A) \}$, where $A$ ranges over all nonempty subsets of $[n-1]$.

**Example 7.5** (a) Let $n = 2$ and let $a_1, a_2 > 0$ be integers. Let $P \subset \mathbb{Z}_+^2$ denote the discrete polymatroid of rank $d$ consisting of those $u = (u_1, u_2) \in \mathbb{Z}_+^2$ such that $u_1 \leq a_1, u_2 \leq a_2$ and $u_1 + u_2 \leq d$. Then $P$ is generic if and only if $a_1 < d, a_2 < d$, and $d < a_1 + a_2$. If $P$ is generic, then the bases of $P$ are $(a_1, d - a_1), (a_1 - 1, d - a_1 + 1), \ldots, (d - a_2, a_2)$. Thus the base ring $K[B]$ of $P$ is Gorenstein if and only if either $a_1 + a_2 = d + 1$ or $a_1 + a_2 = d + 2$.

(b) Let $n = 3$. Let $P \subset \mathbb{Z}_+^3$ be a discrete polymatroid of rank $d$ with $B$ its set of bases, and $\rho$ the ground set rank function of the integral polymatroid $\text{conv}(P) \subset \mathbb{R}_+^3$. Then $\varphi(\mathcal{F}) \subset \mathbb{Z}_+^2$, where $\mathcal{F} = \text{conv}(B)$, consists of those $u = (u_1, u_2) \in \mathbb{R}_+^2$ such that

$$d - \rho(\{2, 3\}) \leq u_1 \leq \rho(\{1\}),$$
$$d - \rho(\{1, 3\}) \leq u_2 \leq \rho(\{2\}),$$
$$d - \rho(\{3\}) \leq u_1 + u_2 \leq \rho(\{1, 2\}).$$

Hence $P$ is generic if and only if

$$0 < \rho(\{i\}) < \rho(\{i, j\}) < d, \quad 1 \leq i \neq j \leq 3,$$
$$\rho(\{i\}) + \rho(\{j\}) > \rho(\{i, j\}), \quad 1 \leq i < j \leq 3,$$
$$\rho(\{i, j\}) + \rho(\{j, k\}) > d + \rho(\{j\}), \quad \{i, j, k\} = [3].$$

Moreover, if $P$ is generic, then the base ring $K[B]$ is Gorenstein if and only if

$$\rho(\{i\}) + \rho(\{j, k\}) = d + 2, \quad \{i, j, k\} = [3].$$

**Theorem 7.6** (a) *Let $n \geq 3$. Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid of rank $d$ and suppose that the canonical basis vectors $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathbb{R}^n$ belong to $P$. Let $\rho: 2^{[n]} \to \mathbb{R}_+$ denote the ground set rank function of the integral polymatroid $\mathrm{conv}(P) \subset \mathbb{R}_+^n$. If $P$ is generic and if the base ring $K[B]$ of $P$ is Gorenstein, then there is a vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}$ with each $\alpha_i > 1$ and with $d > |\alpha| + 1$ such that*

$$\rho(A) = \begin{cases} \alpha(A) + 1, & \text{if } \emptyset \neq A \subset [n-1], \\ d - \alpha([n] \setminus A) + 1, & \text{if } n \in A \neq [n]. \end{cases}$$

(b) *Conversely, given $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}$, where $n \geq 3$, with each $\alpha_i > 1$ and $d \in \mathbb{Z}$ with $d > |\alpha| + 1$, define the function $\rho: 2^{[n]} \to \mathbb{R}_+$ by (a) together with $\rho(\emptyset) = 0$ and $\rho([n]) = d$. Then*

 (i) *$\rho$ is strictly increasing and submodular;*
 (ii) *the discrete polymatroid $P = \{u \in \mathbb{Z}_+^n : u(A) \leq \rho(A), \emptyset \neq A \subset [n]\} \subset \mathbb{Z}_+^n$ arising from $\rho$ is generic;*
(iii) *the base ring $K[B]$ of $P$ is Gorenstein.*

**Proof:** (a) Suppose that a discrete polymatroid $P \subset \mathbb{Z}_+^n$ of rank $d$ is generic and that the base ring $K[B]$ of $P$ is Gorenstein. Let $\mathcal{F} = \mathrm{conv}(B)$. Since $K[B]$ is Gorenstein, there is an integer $\delta \geq 1$ such that

$$\delta(\rho(A) - (d - \rho([n] \setminus A))) = 2$$

for all $\emptyset \neq A \subset [n-1]$. Hence either $\delta = 1$ or $\delta = 2$.

If $\delta = 2$, then $(2\rho(\{1\}) - 1, \ldots, 2\rho(\{n-1\}) - 1) \in \mathbb{Z}_+^{n-1}$ must be a unique integer vector belonging to the interior of $2\varphi(\mathcal{F})$. Since $K[B]$ is Gorenstein it follows that

$$\sum_{i \in A}(2\rho(\{i\}) - 1) = 2\rho(A) - 1, \quad \emptyset \neq A \subset [n-1].$$

Thus

$$\rho(A) = \sum_{i \in A} \rho(\{i\}) - \frac{1}{2}(|A| - 1), \quad \emptyset \neq A \subset [n-1].$$

Since $n \geq 3$, it follows that $\rho(\{1, 2\}) \notin \mathbb{Z}$, a contradiction.

Now let $\delta = 1$ and set

$$\alpha_i = \rho(\{i\}) - 1 = d - \rho([n] \setminus \{i\}) + 1 > 1, \quad 1 \leq i \leq n - 1.$$

Then $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}$ is a unique integer vector belonging to the interior of $\varphi(\mathcal{F}) \subset \mathbb{R}_+^{n-1}$ and $\varphi(\mathcal{F}) - \alpha$ consists of those $u = (u_1, \ldots, u_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$d - \rho([n] \setminus A) - \alpha(A) \leq \sum_{i \in A} u_i \leq \rho(A) - \alpha(A), \quad \emptyset \neq A \subset [n-1].$$

Since $P$ is generic, the desired equality on $\rho$ follows immediately. Moreover, since $\rho([n-1]) = |\alpha| + 1 < \rho([n]) = d$, one has $d > |\alpha| + 1$, as required.

(b) Since each $\alpha_i > 1$ and since $d > |\alpha| + 1$, it follows that $0 < \rho(A) < \rho([n]) = d$ for all $\emptyset \neq A \subset [n]$ with $A \neq [n]$. Moreover, $\rho(\{i\}) > 2$ for all $1 \leq i \leq n$. If $\emptyset \neq A \subset B \subset [n]$ with $n \notin A$ and $n \in B$, then $\rho(B) - \rho(A) = d - (\alpha([n] \setminus B) + \alpha(A)) > d - |\alpha| > 1$. Hence $\rho$ is strictly increasing.

To see why $\rho$ is submodular, we distinguish three cases as follows. First, if $A, B \subset [n-1]$ with $A \neq \emptyset$ and $B \neq \emptyset$, then

$$\begin{aligned}
\rho(A) + \rho(B) &= \alpha(A) + \alpha(B) + 2 \\
&= \alpha(A \cup B) + \alpha(A \cap B) + 2 \\
&= \rho(A \cup B) + \rho(A \cap B)
\end{aligned}$$

unless $A \cap B \neq \emptyset$. Second, if $A, B \subset [n]$ with $n \in A \neq [n]$ and $n \in B \neq [n]$, then

$$\begin{aligned}
\rho(A) + \rho(B) &= 2d - (\alpha([n] \setminus A) + \alpha([n] \setminus B)) + 2 \\
&= 2d - (\alpha([n] \setminus (A \cap B)) + \alpha([n] \setminus (A \cup B))) + 2 \\
&= \rho(A \cup B) + \rho(A \cap B)
\end{aligned}$$

unless $A \cup B \neq [n]$. Third, if $n \in A$ and $B \subset [n-1]$, then assuming $A \cap B \neq \emptyset$ and $A \cup B \neq [n]$ one has

$$\begin{aligned}
\rho(A) + \rho(B) &= d - \alpha([n] \setminus A) + 1 + \alpha(B) + 1 \\
&= d - \alpha(([n] \setminus A) \setminus B) + 1 + \alpha(B \setminus ([n] \setminus A)) + 1 \\
&= d - \alpha([n] \setminus (A \cup B)) + 1 + \alpha(A \cap B) + 1 \\
&= \rho(A \cup B) + \rho(A \cap B),
\end{aligned}$$

as desired.

Let $\mathcal{F} = \text{conv}(B)$. Since

$$\varphi(\mathcal{F}) = \{u \in \mathbb{R}_+^{n-1} : \alpha(A) - 1 \leq u(A) \leq \alpha(A) + 1, \emptyset \neq A \subset [n-1]\},$$

it follows that $\varphi(\mathcal{F}) - \alpha \subset \mathbb{R}^{n-1}$ consists of those $u = (u_1, \ldots, u_{n-1}) \in \mathbb{R}^{n-1}$ such that

$$-1 \leq u_{i_1} + \cdots + u_{i_k} \leq 1, \quad 1 \leq i_1 < \cdots < i_k \leq n-1.$$

Hence $(\varphi(\mathcal{F}) - \alpha) \cap \mathbb{Z}^{n-1}$ consists of those $v = (v_1, \ldots, v_{n-1}) \in \mathbb{Z}^{n-1}$ such that $-1 \leq v_i \leq 1$ for all $1 \leq i \leq n-1$, $|\{i : v_i = 1\}| \leq 1$, and $|\{i : v_i = -1\}| \leq 1$. In particular

the canonical basis vectors $\varepsilon_1, \ldots, \varepsilon_{n-1}$ of $\mathbb{R}^{n-1}$ belong to $\varphi(\mathcal{F}) - \alpha$. Thus $\mathcal{F}$ is a facet of $\mathcal{P} = \mathrm{conv}(P)$. For $1 \le i_1 < \cdots < i_k \le n-1$ write $\mathcal{H}_{i_1\ldots i_k} \subset \mathbb{R}^{n-1}$ (resp. $\mathcal{H}'_{i_1\ldots i_k} \subset \mathbb{R}^{n-1}$) for the supporting hyperplane of $\mathcal{F}$ determined by the linear equation $x_{i_1} + \cdots + x_{i_k} = 1$ (resp. $x_{i_1} + \cdots + x_{i_k} = -1$). Then the vectors $\varepsilon_{i_1}, \ldots, \varepsilon_{i_k}$ (resp. $-\varepsilon_{i_1}, \ldots, -\varepsilon_{i_k}$) and $\varepsilon_{i_1} - \varepsilon_j$ (resp. $-\varepsilon_{i_1} + \varepsilon_j$), $j \in [n-1]\setminus\{i_1, \ldots, i_k\}$, belong to the face $(\varphi(\mathcal{F}) - \alpha) \cap \mathcal{H}_{i_1\ldots i_k}$ (resp. $(\varphi(\mathcal{F}) - \alpha) \cap \mathcal{H}'_{i_1\ldots i_k}$) of $\varphi(\mathcal{F}) - \alpha$. Thus $(\varphi(\mathcal{F}) - \alpha) \cap \mathcal{H}_{i_1\ldots i_k}$ (resp. $(\varphi(\mathcal{F}) - \alpha) \cap \mathcal{H}'_{i_1\ldots i_k}$) is a facet of $\varphi(\mathcal{F}) - \alpha$. Hence $P$ is generic. Moreover, since the Ehrhart ring $K[\varphi(\mathcal{F}) - \alpha]$ is Gorenstein, the base ring $K[B]$ $(\cong K[\varphi(\mathcal{F}) - \alpha])$ is Gorenstein, as desired. $\qquad\square$

## 8. Base polytopes of discrete polymatroids

Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid and $B = B(P)$ the set of bases of $P$. The convex hull $\mathrm{conv}(B) \subset \mathbb{R}_+^n$ of $B$ is said to be the *base polytope* of $P$. Since $\mathrm{conv}(P) \cap \mathbb{Z}^n = P$, one has $\mathrm{conv}(B) \cap \mathbb{Z}^n = B$. In other words, the base polytope of $P$ is just the set of bases of the integral polymatroid $\mathcal{P} = \mathrm{conv}(P) \subset \mathbb{R}_+^n$.

Write $V_n^{(d)} \subset \mathbb{R}_+^n$ for the set of all vectors $u = (u_1, \ldots, u_n) \in \mathbb{R}_+^n$ with $|u| = \sum_{i=1}^n u_i = d$. It is natural to ask which integral convex polytope $\mathcal{Q} \subset V_n^{(d)}$ can be the base polytope of a discrete polymatroid $P \subset \mathbb{Z}_+^n$ of rank $d$.

If $\mathcal{Q} \subset V_n^{(d)}$ is an integral convex polytope and if $a, b \in \mathbb{Z}_+^n$, then we write $\mathcal{Q}_{a,b}$ for the integral convex polytope which is the convex hull of the finite set

$$\{w \in \mathcal{Q} \cap \mathbb{Z}^n : a \wedge b \le w \le a \vee b\}$$

of integer vectors.

**Lemma 8.1** *If $\mathcal{Q} \subset V_n^{(d)}$ is the base polytope of a discrete polymatroid, then $\mathcal{Q}_{a,b}$ is again the base polytope of a discrete polymatroid.*

**Proof:** If $\mathcal{Q} \subset V_n^{(d)}$ is the base polytope of a discrete polymatroid, then $\mathcal{Q} \cap \mathbb{Z}^n$ is the set of bases of $P$. Since the finite set $\{w \in \mathcal{Q} \cap \mathbb{Z}^n : a \wedge b \le w \le a \vee b\}$ of integer vectors satisfies the exchange property 2.3 (c) for discrete polymatroids, it follows that $\mathcal{Q}_{a,b}$ is the base polytope of a discrete polymatroid. $\qquad\square$

We now come to a generalization of [7, Theorem 4.1] to discrete polymatroids. Let, as before, $\varepsilon_1, \ldots, \varepsilon_n$ denote the canonical basis vectors of $\mathbb{R}^n$.

**Theorem 8.2** *An integral convex polytope $\mathcal{Q} \subset V_n^{(d)}$ is the base polytope of a discrete polymatroid $P \subset \mathbb{Z}_+^n$ of rank $d$ if and only if, for any vectors $a$ and $b$ belonging to $\mathbb{Z}_+^n$, the integral convex polytope $\mathcal{Q}_{a,b}$ possesses the property that if $u$ and $v$ are vertices of $\mathcal{Q}_{a,b}$ and if the segment $\mathrm{conv}\{u, v\}$ is an edges of $\mathcal{Q}_{a,b}$, then $v - u$ is of the form $\ell(\varepsilon_i - \varepsilon_j)$, where $i \ne j$ and $0 \ne \ell \in \mathbb{Z}_+$.*

**Proof:** (**"only if"**) Because of Lemma 8.1, it suffices to show that the base polytope $\mathcal{Q}$ of a discrete polymatroid possesses the required property.

Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ be vertices of $\mathcal{Q}$ such that the segment conv($\{u, v\}$) is an edge of $\mathcal{Q}$, and suppose that $u_i > v_i$ for $1 \leq i \leq r$, $u_j < v_j$ for $r < j \leq s$, and $u_k = v_k$ for $s < k \leq n$. We claim that $r = 1$ and $s = 2$. Let, say, $r > 1$. Since $u_1 > v_1$, Theorem 4.1 guarantees that there is $r < j \leq s$ such that $u' = u - \varepsilon_1 + \varepsilon_j$ and $v' = v - \varepsilon_j + \varepsilon_1$ belong to $\mathcal{Q}$. Since $r > 1$, neither $u' = (u_1 - 1, u_2, \ldots)$ nor $v' = (v_1 + 1, v_2, \ldots)$ can lie on the edge conv($\{u, v\}$) of $\mathcal{Q}$. Thus $(u + v)/2 = (u' + v')/2$ yields a contradiction. Hence $r = 1$. Similarly, $s = 2$.

Let $r = 1$ and $s = 2$, and let $u = (u_1, u_2, w_3, \ldots, w_n)$ and $v = (v_1, v_2, w_3, \ldots, w_n)$ with $u_1 > v_1$ and $u_2 < v_2$. Since $u_1 - v_1 = v_2 - u_2$, it follows that $v - u = (u_1 - v_1)$ $(-1, 1, 0, \ldots, 0)$, as desired.

(**"If"**) Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ be integer vectors belonging to $\mathcal{Q}$, and suppose that $u_i > v_i$ for $1 \leq i \leq r$, $u_j < v_j$ for $r < j \leq s$, and $u_k = v_k$ for $s < k \leq n$. Then $u$ is a vertex of $\mathcal{Q}_{u,v}$. In fact, if $\mathcal{H}$ is the hyperplane of $\mathbb{R}^n$ defined by the linear equation $\sum_{i=1}^r x_i = \sum_{i=1}^r u_i$, then $\mathcal{H}$ is a supporting hyperplane of $\mathcal{Q}_{u,v}$ with $\mathcal{H} \cap \mathcal{Q}_{u,v} = \{u\}$.

If $w$ is a vertex of $\mathcal{Q}_{u,v}$ such that the segment conv$\{u, w\}$ is an edge of $\mathcal{Q}_{u,v}$ and if $w - u = \ell(\varepsilon_j - \varepsilon_i)$ with $i \neq j$ and with $0 \neq \ell \in \mathbb{Z}_+$, then we write $E_w$ for the vector $(w - u)/\ell = \varepsilon_j - \varepsilon_i$. Since $u + E_w$ belongs to $\mathcal{Q}_{u,v}$, it follows that $1 \leq i \leq r$ and $r < j \leq s$.

Let $\Omega$ denote the set of vertices $w$ of $\mathcal{Q}_{u,v}$ such that the segment conv$\{u, w\}$ is an edge of $\mathcal{Q}_{u,v}$, and write

$$v - u = \sum_{w \in \Omega} c_w E_w$$

with each $0 \leq c_w \in \mathbb{Q}$. What we must prove is that, for each $1 \leq i_0 \leq r$, there is $r < j_0 \leq s$ such that $u - \varepsilon_{i_0} + \varepsilon_{j_0}$ belongs to $\mathcal{Q}_{u,v}$.

Now, for each $1 \leq i_0 \leq r$, there is $w_0 \in \Omega$ with $c_{w_0} > 0$ such that $E_{w_0} = \varepsilon_{j_0} - \varepsilon_{i_0}$. Since $r < j_0 \leq s$ and since $u + E_{w_0} = u - \varepsilon_{i_0} + \varepsilon_{j_0} \in \mathcal{Q}_{u,v}$, the vector $u + E_{w_0}$ is what we want to find.                                                                                      $\square$

We conclude the present section with one of the fundamental properties of the base ring of a discrete polymatroid. We say that a convex polytope $\mathcal{P} \subset \mathbb{R}^n$ of dimension $e$ is *simple* if each vertex of $\mathcal{P}$ belongs to just $e$ edges of $\mathcal{P}$.

**Corollary 8.3** *Let $P \subset \mathbb{Z}_+^n$ be a discrete polymatroid with $B = B(P)$ the set of its bases. Suppose that the base polytope* conv($B$) $\subset \mathbb{R}_+^n$ *is simple. Then the base ring $K[B]$ is a $K$-algebra with isolated singularity.*

**Proof:** Let dim conv($B$) $= e$. Fix a vertex $v$ of conv($B$) and let $u_1, \ldots, u_e$ denote the vertices of conv($B$) such that the segment conv$\{v, u_i\}$, $1 \leq i \leq e$, is an edge of conv($B$). Write $Y_u$ for the $d \times n$ $\mathbb{Z}$-matrix whose rows are $u_1 - v, \ldots, u_e - v$. It follows from Theorem 8.2 that each row of $Y_u$ is of the form $\varepsilon_i - \varepsilon_j$ with $i \neq j$; thus $Y_u$ is totally unimodular, i.e., each subdeterminant of $Y_u$ belongs to $\{0, \pm 1\}$. Hence, for all $w \in B$, the vector $w - v$ can be expressed uniquely as $w - v = \sum_{i=1}^e q_i(u_i - v)$ with each $0 \leq q_i \in \mathbb{Z}$.

The base ring $K[B] \subset K[t_1, \ldots, t_n]$ is generated by those monomials $t^w = t_1^{w_1} \ldots t_n^{w_n}$ with $w = (w_1, \ldots, w_n) \in B$. Thus if $v$ is a vertex of $\text{conv}(B)$, then the localization

$$K[B]_{t^v} = K\left[t^v, \frac{1}{t^v}\right]\left[\left\{\frac{t^w}{t^v}\right\}_{w \in B}\right] = K\left[t^v, \frac{1}{t^v}\right][\{t^{w-v}\}_{w \in B}]$$

coincides with the polynomial ring $K[t^v, \frac{1}{t^v}][t^{w_1-v}, \ldots, t^{w_e-v}]$ over the regular ring $K[t^u, \frac{1}{t^u}]$. Hence the localization $K[B]_{t^v}$ is regular. Since $K[B]$ is a $K$-algebra with isolated singularity if and only if the localization $K[B]_{t^v}$ is regular for all vertices $v$ of $\text{conv}(B)$, it follows that $K[B]$ is a $K$-algebra with isolated singularity. $\qquad\square$

## 9. Constructions of discrete polymatroids

In this section two simple techniques to construct discrete polymatroids will be studied. The first one shows that a nondecreasing and submodular function defined in a sublattice of $2^{[n]}$ produces a discrete polymatroid. The second one yields the concept of transversal polymatroids.

A *sublattice* of $2^{[n]}$ is a collection $\mathcal{L}$ of subsets of $[n]$ with $\emptyset \in \mathcal{L}$ and $[n] \in \mathcal{L}$ such that for all $A$ and $B$ belonging to $\mathcal{L}$ both $A \cap B$ and $A \cup B$ belong to $\mathcal{L}$. A function $\mu: \mathcal{L} \to \mathbb{R}_+$ is called *submodular* if

$$\mu(A) + \mu(B) \geq \mu(A \cup B) + \mu(A \cap B), \quad A, B \in \mathcal{L}.$$

Even though Theorem 9.1 below is a direct consequence of [17, Theorem 1, p. 342] together with Theorem 3.4, for the sake of completeness we will give an easy proof based on [17, Theorem 2, p. 345].

**Theorem 9.1** *Let $\mathcal{L}$ be a sublattice of $2^{[n]}$ and $\mu: \mathcal{L} \to \mathbb{R}_+$ an integer valued nondecreasing and submodular function with $\mu(\emptyset) = 0$. Then*

$$P_{(\mathcal{L},\mu)} = \{u \in \mathbb{Z}_+^n : u(A) \leq \mu(A), \quad A \in \mathcal{L}\}.$$

*is a discrete polymatroid.*

**Proof:** Let $\rho: 2^{[n]} \to \mathbb{R}_+$ be the nondecreasing function defined by

$$\rho(X) = \min\{\mu(A): A \supset X, \quad A \in \mathcal{L}\}$$

together with $\rho(\emptyset) = 0$. Then $\rho$ is submodular. Let $P_\rho \subset \mathbb{Z}_+^n$ denote the discrete polymatroid arising from $\rho$. Since $\emptyset \neq X \subset [n]$ with $X \notin \mathcal{L}$ cannot be $\rho$-closed, it follows from Proposition 7.2 that

$$P_\rho = \{u \in \mathbb{Z}_+^n : u(A) \leq \rho(A), \quad A \in \mathcal{L}\}.$$

Since $\rho(A) = \mu(A)$ for all $A \in \mathcal{L}$, one has $P_\rho = P_{(\mathcal{L},\mu)}$, thus $P_{(\mathcal{L},\mu)}$ is a discrete polymatroid, as desired. $\qquad\square$

**Example 9.2**  Let $\mathcal{L}$ be a chain of length $n$ of $2^{[n]}$, say

$$\mathcal{L} = \{\emptyset, \{n\}, \{n-1, n\}, \ldots, \{1, \ldots, n\}\} \subset 2^{[n]}.$$

Given nonnegative integers $a_1, \ldots, a_n$, define $\mu: \mathcal{L} \to \mathbb{R}_+$ by

$$\mu(\{i, i+1, \ldots, n\}) = a_i + a_{i+1} + \cdots + a_n, \quad 1 \leq i \leq n$$

together with $\mu(\emptyset) = 0$. Then the discrete polymatroid $P_{(\mathcal{L},\mu)} \subset \mathbb{Z}_+^n$ is

$$P_{(\mathcal{L},\mu)} = \left\{ u = (u_1, \ldots, u_n) \in \mathbb{Z}_+^n : \sum_{j=i}^{n} u_j \leq \sum_{j=i}^{n} a_j, \quad 1 \leq i \leq n \right\}.$$

This example will be discussed again in Example 9.4.

Let $\mathcal{A} = (A_1, \ldots, A_d)$ be a family of nonempty subsets of $[n]$. It is *not* required that $A_i \neq A_j$ if $i \neq j$. Let

$$B_{\mathcal{A}} = \left\{ \varepsilon_{i_1} + \cdots + \varepsilon_{i_d} : i_k \in A_k, \quad 1 \leq k \leq d \right\} \subset \mathbb{Z}_+^n$$

and define the integer valued nondecreasing function $\rho_{\mathcal{A}}: 2^{[n]} \to \mathbb{R}_+$ by setting

$$\rho_{\mathcal{A}}(X) = |\{k : A_k \cap X \neq \emptyset\}|, \quad X \subset [n].$$

Again, the following Theorem 9.3 is well known. However, we give its proof for the sake of completeness.

**Theorem 9.3**  *The function $\rho_{\mathcal{A}}$ is submodular and $B_{\mathcal{A}}$ is the set of bases of the discrete polymatroid $P_{\mathcal{A}} \subset \mathbb{Z}_+^n$ arising from $\rho_{\mathcal{A}}$.*

**Proof:**  For each $1 \leq k \leq d$ define the function $\rho_k: 2^{[n]} \to \mathbb{R}_+$ by $\rho_k(X) = 1$ if $A_k \cap X \neq \emptyset$ and $\rho_k(X) = 0$ if $A_k \cap X = \emptyset$. Then $\rho_k$ is nondecreasing and submodular. Write $P_k \subset \mathbb{Z}_+^n$ for the discrete polymatroid arising from $\rho_k$. Then Proposition 1.4 guarantees that

$$P = \left\{ x \in \mathbb{Z}_+^n : x = \sum_{k=1}^{d} x_k, \quad x_k \in P_k \right\}$$

is a discrete polymatroid of rank $d$ and the ground set rank function of the integral polymatroid $\mathrm{conv}(P) \subset \mathbb{R}_+^n$ is $\rho = \sum_{k=1}^{d} \rho_k$. It is clear that

$$\rho(X) = |\{k : A_k \cap X \neq \emptyset\}|, \quad X \subset [n]$$

and the set of bases of $P$ coincides with $B_{\mathcal{A}}$, as desired. $\qquad\square$

We say that the discrete polymatroid $P_{\mathcal{A}} \subset \mathbb{Z}_+^n$ is the *transversal polymatroid* presented by $\mathcal{A}$.

**Example 9.4** Let $r_1, \ldots, r_d \in [n]$ and set $A_k = [r_k] = \{1, \ldots, r_k\}, 1 \leq k \leq d$. Let $\min(X)$ denote the smallest integer belonging to $X$, where $\emptyset \neq X \subset [n]$. If $\mathcal{A} = (A_1, \ldots, A_d)$, then

$$\rho_{\mathcal{A}}(X) = \rho_{\mathcal{A}}(\{\min(X)\}) = |\{k\colon \min(X) \leq r_k\}|.$$

If $\emptyset \neq X \subset [n]$ is $\rho_{\mathcal{A}}$-closed, then $X = \{\min(X), \min(X)+1, \ldots, n\}$. Let

$$a_i = |\{k\colon r_k = i\}|, \quad 1 \leq i \leq n.$$

Thus

$$\rho_{\mathcal{A}}(\{i, i+1, \ldots, n\}) = a_i + a_{i+1} + \cdots + a_n, \quad 1 \leq i \leq n.$$

The transversal polymatroid $P_{\mathcal{A}} \subset \mathbb{Z}_+^n$ presented by $\mathcal{A}$ is

$$P_{\mathcal{A}} = \left\{ u = (u_1, \ldots, u_n) \in \mathbb{Z}_+^n \colon \sum_{j=i}^n u_j \leq \sum_{j=i}^n a_j, \quad 1 \leq i \leq n \right\}.$$

Thus $P_{\mathcal{A}}$ coincides with the discrete polymatroid $P_{(\mathcal{L}, \mu)}$ in Example 9.2. If $P_i \subset \mathbb{Z}_+^n$ is the discrete polymatroid with $B_i = \{\varepsilon_1, \ldots, \varepsilon_i\}$ its set of bases, then $P_{\mathcal{A}} = a_1 P_1 \vee \cdots \vee a_n P_n$. Moreover, $B_{\mathcal{A}}$ is equal to the strongly stable principal set with $(a_1, \ldots, a_n)$ its Borel generator.

**Example 9.5** Let $P \subset \mathbb{Z}_+^4$ denote the discrete polymatroid of rank 3 consisting of those $u = (u_1, u_2, u_3, u_4) \in \mathbb{Z}_+^4$ with $u_i \leq 2$ for $1 \leq i \leq 4$ and with $|u| \leq 3$. Then $P$ is *not* transversal. Suppose, on the contrary, that $P$ is a transversal polymatroid presented by $\mathcal{A} = (A_1, A_2, A_3)$ with each $A_k \subset [4]$. Since $(2, 1, 0, 0), (2, 0, 1, 0), (2, 0, 0, 1) \in P$ and $(3, 0, 0, 0) \notin P$, we assume that $1 \in A_1, 1 \in A_2$ and $A_3 = \{2, 3, 4\}$. Since $(1, 2, 0, 0), (0, 2, 1, 0), (0, 2, 0, 1) \in P$ and $(0, 3, 0, 0) \notin P$, we assume that $2 \in A_1$ and $A_2 = \{1, 3, 4\}$. Since $(0, 0, 2, 1) \in P$ and $(0, 0, 3, 0) \notin P$, one has $4 \in A_1$. Hence $(0, 0, 0, 3) \in P$, a contradiction.

The discussions in Example 9.4 enables us to classify all Gorenstein strongly stable principal sets.

**Proposition 9.6** *Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}_+^n$ with $a_n \geq 1$ and write $B_a$ for the strongly stable principal set with $(a_1, \ldots, a_n)$ its Borel generator. Then the base ring $K[B_a]$ is Gorenstein if and only if $a_2 + \cdots + a_n$ divides $n$ and*

$$\frac{n-i+2}{a_i + a_{i+1} + \cdots + a_n} = \frac{n}{a_2 + \cdots + a_n}$$

*for all $3 \leq i \leq n$ with $a_{i-1} \neq 0$.*

**Proof:** The base ring $K[B_a]$ is isomorphic to the Ehrhart ring of the integral convex polytope $\mathcal{P} \in \mathbb{R}^{n-1}$ consisting of those $u = (u_2, u_3, \ldots, u_n)$ such that $u_i \geq 0, 2 \leq i \leq n$, and $\sum_{j=i}^{n} u_j \leq \sum_{j=i}^{n} a_j$ for $i = 2$ and for all $3 \leq i \leq n$ with $a_{i-1} \neq 0$. Since all of these inequalities define the facets of $\mathcal{P}$, the desired result follows immediately. Note that $\mathcal{P}$ is an integral polymatroid whose ground set rank function $\rho$ is given by $\rho(X) = \sum_{j=\min(X)}^{n} a_j$ for $\emptyset \neq X \subset \{2, \ldots, n\}$. $\qquad\square$

**Example 9.7**   (a) If $a = (0, 1, \ldots, 1, 2) \in \mathbb{Z}_+^n$ then $K[B_a]$ is Gorenstein.
   (b) If $a = (0, 1, 0, 2, 0, 3)$, then $K[B_a]$ is Gorenstein.
   (c) If $a = (0, \ldots, 0, a_n) \in \mathbb{Z}_+^n$ with $a_n \geq 1$, then $K[B_a]$ is Gorenstein if and only if $a_n$ divides $n$.

It would, of course, be of interest to classify all transversal polymatroids with Gorenstein base rings.

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