Spherical 7-Designs in 2^n -Dimensional Euclidean Space

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Abstract. We consider a finite subgroup Θ_n of the group $O(N)$ of orthogonal matrices, where $N = 2^n$, $n = 1, 2, \dots$. This group was defined in [7]. We use it in this paper to construct spherical designs in 2^n -dimensional Euclidean space R^N . We prove that representations of the group Θ_n on spaces of harmonic polynomials of degrees 1, 2 and 3 are irreducible. This and the earlier results [1–3] imply that the orbit $\Theta_{n,2}\mathbf{x}^t$ of any initial point $\mathbf{x}$ on the sphere S_{N-1} is a 7-design in the Euclidean space of dimension 2^n .

Keywords: spherical design, orthogonal matrix, Euclidean space

1. Introduction

A spherical t -design in N -dimensional Euclidean space $\mathbf{R}^N$ is a finite nonempty set X of points on the unit sphere $S_{N-1} = \{\mathbf{x} = (x_1, \dots, x_N) \in \mathbf{R}^N \mid x_1^2 + \dots + x_N^2 = 1\}$ such that

$$\frac{1}{|S_{N-1}|} \int_{S_{N-1}} f(\mathbf{x}) d\mathbf{x} = \frac{1}{|X|} \sum_{\mathbf{x} \in X} f(\mathbf{x})$$

for all polynomials $f(\mathbf{x})$ of degree at most t where $|S_{N-1}|$ denotes the surface area of the S_{N-1} . Account of basic properties of spherical t -designs may be found in [1].

Let $\text{Hom}(k)$ be the space of all homogeneous N -variable polynomials of degree k over $\mathbf{R}$ and let $\text{Harm}(k)$ be the space of all homogeneous harmonic polynomials of degree k , i.e. the space of all homogeneous polynomials $y = y(\mathbf{x})$ satisfying the potential equation $\frac{\partial^2 y}{\partial x_1^2} + \dots + \frac{\partial^2 y}{\partial x_N^2} = 0$. The dimension of $\text{Harm}(k)$ is $\binom{N+k-1}{k} - \binom{N+k-3}{k-2}$ [5, 6]. In what follows we assume that $N > 2$.

The space $\text{Harm}(k)$ is an irreducible invariant subspace of the representation of the orthogonal group $O(N)$ on $\text{Hom}(k)$. Speaking more precisely, the space $\text{Hom}(k)$ can be represented as the direct sum:

$$\text{Hom}(k) = \text{Harm}(k) + \text{Harm}(k-2)|\mathbf{x}|^2 + \dots + \text{Harm}(k-2l)|\mathbf{x}|^{2l}, \quad l = \left\lceil \frac{k}{2} \right\rceil.$$

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Each term in this sum is an invariant irreducible subspace of the representation of $O(N)$ on $\text{Hom}(k)$, where $|\mathbf{x}|$ denotes the norm of the vector $\mathbf{x}$ and $|\mathbf{x}|^2$ is used as a shorthand for the polynomial $x_1^2 + \dots + x_N^2$ [6, 7].

Since Θ_n is a subgroup of the group $O(N)$ the spaces $\text{Harm}(k - 2j)$ are also invariant spaces of the representation of Θ_n on $\text{Hom}(k)$ and they in turn can be decomposed into direct sum of some nontrivial invariant subspaces.

Next we state some basic results, that will be useful in what follows.

Theorem A [2, Th. 6.10 and Th. 3.1], [3, Th. 1] *Let G be a finite subgroup of the group $O(N)$ and let ρ_k be a representation of G on $\text{Harm}(k)$. If all representations ρ_i for $i = 1, \dots, t$ are irreducible then for any $\mathbf{x} \in S_{N-1}$ the set*

$$X = \{g\mathbf{x} \mid g \in G\} \subset S_{N-1}$$

is a spherical $2t$ -design.

If the set X in addition satisfies $\sum_{\mathbf{x} \in X} f(\mathbf{x}) = 0$ for all $f(\mathbf{x}) \in \text{Harm}(2t + 1)$, then X is a spherical $(2t + 1)$ -design.

In [7] we constructed a finite group $\Sigma_{n,p}$ which for p being an odd prime is a group of $p^n \times p^n$ -matrices over $\mathbf{C}$, and for $p = 2$ is a group of $2^n \times 2^n$ -matrices over $\mathbf{R}$.

The group $\Sigma_{n,p}$, $p > 2$, has an isomorphic image in a certain group $\tilde{\Sigma}_{n,p}$ of $2p^n \times 2p^n$ -matrices over $\mathbf{R}$ [7]. The group $\Sigma_{n,2}$ has a remarkable subgroup Θ_n of index 2 comprising all matrices from $\Sigma_{n,2}$ with rational entries.

The order of the group Θ_n is asymptotically $c2^{n(2n+1)}$, $c = 1.38 \dots$, $n \rightarrow \infty$.

We used the group Θ_n to construct orbit codes $\mathcal{K}(\mathbf{x}') = \Theta_n \mathbf{x}$ with $\mathbf{x} = \mathbf{o}_1$, where $\mathbf{o}_1 = (1, 0, \dots, 0)$ (see [7]). The cardinality of the code $\mathcal{K}(\mathbf{o}_1)$ is asymptotically $2.38 \dots$, $2^{n(n+1)/2}$, $n \rightarrow \infty$, and its Euclidean code distance is 1. The order of the stabilizer of the point $\mathbf{o}_1$ in the group Θ_n is $O(2^{n(3n+1)/2})$.

1. Definition and properties of the group $\Sigma_{1,p}$ Let $\mathbf{F}_p$ be a p -element Galois field, let $f(x) \in \mathbf{F}_p[x]$ denotes the polynomial of the second degree, let

$$E_{f,p} = \text{diag}(\exp(2\pi i f(0)/p), \dots, \exp(2\pi i f(p-1)/p))$$

be a diagonal matrix, where $i = \sqrt{-1}$, and let $A(s) = A_p(s) = \|w_{a,b}^s\|$ be a unitary symmetrical $p \times p$ matrix, where $a, b \in \mathbf{F}_p$, $s = 1, 2, \dots, p-1$, and $w_{a,b}^s = p^{-1/2} \exp(2\pi i abs/p)$. Note, that $(A(s))^{-1} = A(-s)$. Consider a group

$$\Sigma_{1,p} = \langle A(s), E_{f,p}; s = 1, 2, \dots, p-1, f(x) \in \mathbf{F}_p[x], \deg f(x) \leq 2 \rangle,$$

generated by $p-1$ unitary matrices $A(s)$ and p^3 diagonal matrices $E_{f,p}$.

Theorem B [7] *The group $\Sigma_{1,p}$ is a finite group of order $\sigma_{1,p}$, where*

1. $\sigma_{1,p} = 4p^4(p^2 - 1)$ whenever $p \equiv 3 \pmod{4}$,
2. $\sigma_{1,p} = 2p^4(p^2 - 1)$ whenever $p \equiv 1 \pmod{4}$,
3. $\sigma_{1,p} = 2^4$ whenever $p = 2$.

The entries of the matrices in $\Sigma_{1,p}$ and in matrices of the group $\Sigma_{n,p}$ to be defined later are complex numbers whenever $p > 2$. There exists an isomorphic mapping ϕ of the group $\Sigma_{n,p}$ to the group $\tilde{\Sigma}_{n,p}$ of orthogonal $2p^n \times 2p^n$ matrices over $\mathbf{R}$. There also exists a mapping ρ from U^{p^n-1} to the sphere S^{2p^n-1} such that for any matrix $P \in \Sigma_{n,p}$ we have $\lambda(\mathbf{x} - \mathbf{x}P) = \lambda'(\rho(\mathbf{x}) - \rho(\mathbf{x})\phi(P))$, where λ is the usual metric in $\mathbf{C}^{p^n}$ and λ' is the Euclidean metric in $\mathbf{R}^{2p^n}$. Thus, the maps ϕ and ρ are isomorphic and isometric transformations of codes on U^{p^n-1} into codes on S^{2p^n-1} .

2. Definition of the group $\Sigma_{n,p}$, $n > 1$ Let $D = \|d_{i,j}\|$ be a $n \times n$ matrix over Galois field $\mathbf{F}_p$, i.e., $D \in M_n(\mathbf{F}_p)$.

1. By $\mathbf{Ker}(D)$ we denote the linear space of zeroes of matrix D over the field $\mathbf{F}_p$, i.e. the set of all vectors $\mathbf{x}$ in n -dimensional space $(\mathbf{F}_p)^n$ such that $D\mathbf{x}^t = \mathbf{0}$. By $\mathbf{Im}(D)$ we denote the space spanned by columns of the matrix D , i.e. $\mathbf{Im}(D) = \{D\mathbf{x}^t; \mathbf{x} \in (\mathbf{F}_p)^n\}$.
2. Let $D, R, T \in M_n(\mathbf{F}_p)$, $\alpha^t \in \mathbf{F}_p^n$ and $m = \dim \mathbf{Ker}(T)$. A $p^n \times p^n$ matrix $C(D, R, T, \alpha) = [v_{\mathbf{a}, \mathbf{b}}]$, $\mathbf{a}, \mathbf{b} \in (\mathbf{F}_p)^n$, over the complex numbers, i.e. matrix from $M_{p^n}(\mathbf{C})$, is defined by

$$v_{\mathbf{a}, \mathbf{b}} = 0,$$

whenever $R\mathbf{a} - T\mathbf{b} + \alpha \neq \mathbf{0}$, and by

$$v_{\mathbf{a}, \mathbf{b}} = p^{-m/2} \exp(2\pi i \mathbf{a} D \mathbf{b}^t / p),$$

whenever

$$R\mathbf{a} - T\mathbf{b} + \alpha = \mathbf{0}. \quad (1)$$

Let $\mathbf{U}_n = \{\alpha_1, \dots, \alpha_N\} = (\mathbf{F}_p)^n$, $N = p^n$, be the set of all elements of the space $(\mathbf{F}_p)^n$, listed in the lexicographical order. We use the set $\mathbf{U}_n$ for indexing rows and columns of the matrices $C(D, R, T, \alpha)$ in such a way that the entry in the intersection of the i th row and the j th column is equal to v_{α_i, α_j} .

Note that the identity matrix $\tilde{E}$ in $M_{2^n}(\mathbf{R})$ can be represented as $\tilde{E} = C(0, Q, Q, 0)$, where Q is an arbitrary nondegenerate matrix in $M_n(\mathbf{F}_p)$. The matrix $C(0, E, E, -\alpha) = \Gamma_\alpha$ is a substitution matrix corresponding to the translation $\sigma : \mathbf{x} \rightarrow \mathbf{x} + \alpha$ in the space $(\mathbf{F}_p)^n$.

3. Denote by $B(D, R, T, \alpha)$ a matrix $C(D, R, T, \alpha)$ such that the matrices D, R, T and the vector α satisfy the following two conditions
 - a. $\mathbf{Im}(R) = \mathbf{Im}(T)$ and $\alpha \in \mathbf{Im}(R)$.
 - b. The bilinear form $\mathbf{x} D \mathbf{y}^t$ has for $m > 0$ the following property: for any not identically zero vector $\mathbf{x}_0$ in $\mathbf{Ker}(R)$ the linear function $\mathbf{x}_0 D \mathbf{y}^t$ mapping $\mathbf{y}$ to $\mathbf{Ker}(T)$ is not identically zero, i.e. $\mathbf{x}_0 D \mathbf{y}_0^t \neq 0$ for some $\mathbf{y}_0$ in $\mathbf{Ker}(T)$.
4. Let $f(\mathbf{x}) \in \mathbf{F}_p[\mathbf{x}]$, $\mathbf{x} = (x_1, \dots, x_n)$, be a polynomial of the second degree. By E_f we denote the diagonal matrix $E_f = \text{diag}(\exp(2\pi i f(\alpha_1)/p), \dots, \exp(2\pi i f(\alpha_N)/p))$.

Notice that

$$B(D, R, T, \alpha) = E_l B(D, R, T, 0) \Gamma_\beta,$$

where Γ_β is the substitution matrix corresponding to the translation $\sigma : \mathbf{x} \rightarrow \mathbf{x} + \beta$ in the space $\mathbf{F}_p^n$, β is an arbitrary vector in $\mathbf{F}_p^n$ satisfying $\beta T = -\alpha$ and $l = l(\mathbf{x}) = \mathbf{x} D \beta^t$ is the linear function.

5. Let $p = 2$, or $p = 1 \pmod{4}$. We denote by $\Sigma_{n,p}$ the set of all unitary $p^n \times p^n$ matrices of the form

$$P = \pm E_{f_1} B(D, R, T, \alpha) E_{f_2}, \quad (2)$$

where $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ range over all pairs n -variate polynomials of degree at most 2, and matrices D, R, T and element α are chosen from the set of all quadruples $\{D, R, T, \alpha\}$, satisfying the properties 3.a. and 3.b. If $p = 3 \pmod{4}$, then by $\Sigma_{n,p}$ we denote the set of all matrices of the form $i^\varepsilon P$, where ε is chosen in the set of numbers $\{1, 2, 3, 4\}$, and the matrix P is chosen in the set of all matrices of the form (2). The matrices in $\Sigma_{n,p}$ are all unitary as follows from Lemma A.

Lemma A [7] *The matrix $C(D, R, T, \alpha)$ is unitary iff conditions 3.a., 3.b hold, i.e., the matrix $C(D, R, T, \alpha)$ is unitary matrix iff it coincides with the matrix $B(D, R, T, \alpha)$.*

Theorem C [7] *The set $\Sigma_{n,p}$ of unitary matrices is closed under multiplication, i.e. $\Sigma_{n,p}$ is the finite group. The order of the group $\Sigma_{n,p}$ is*

$$\sigma_{n,p} = \vartheta(p)(p^n - 1) \cdot \dots \cdot (p^n - p^{n-1}) \left(\sum_{m=0}^n p^{n-m} \begin{bmatrix} n \\ n-m \end{bmatrix}_p \tau_{m,p} \right) \tau_{n,p}/p \quad (3)$$

where $\vartheta(2) = 1$, $\vartheta(p) = 2$ whenever $p = 1 \pmod{4}$, $\vartheta(p) = 4$, whenever $p = -1 \pmod{4}$, $\begin{bmatrix} n \\ m \end{bmatrix}_p = \begin{bmatrix} n-m \\ m \end{bmatrix}_p = (p^n - 1) \cdot \dots \cdot (p^n - p^{m-1}) / (p^m - 1) \cdot \dots \cdot (p^m - p^{m-1})$, $m \geq 1$, is a Gaussian coefficient, and $\tau_{n,2} = 2^{n(n+1)/2+1}$, $\tau_{n,p} = p^{n(n+3)/2+1}$, $p > 2$, is the number of n -variable polynomials of the second degree in $\mathbf{F}_p[\mathbf{x}]$, $\mathbf{x} = (x_1, \dots, x_N)$, $N = p^n$.

In particular, $\sigma_{2,2} = 2304 = 2^8 \cdot 3^2$, $\sigma_{3,2} = 2^{14} \cdot 3^2 \cdot 5 \cdot 7$, $\sigma_{4,2} = 2^{22} \cdot 3^5 \cdot 5^2 \cdot 7$, $\sigma_{5,2} = 2^{32} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$, $\sigma_{6,2} = 2^{44} \cdot 3^8 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 17 \cdot 31$.

Corollary A [7] *The order $\sigma_{n,p}$ of the group $\Sigma_{n,p}$ is $\sigma_{n,p} = O(p^{2n^2+3n+1})$ whenever $p > 2$, and $\sigma_{n,2} \sim c 2^{n(2n+1)+1}$, $n \rightarrow \infty$. In this asymptotic $c = 1.77 \dots$.*

For $p = 2$ the set Θ_n is defined as a set of matrices P of the form (2) with $\dim \mathbf{Ker}(R)$ even, i.e the set $\Theta_n \subset \Sigma_{n,2}$ is the set of all matrices over the rationals.

The structure of the group $\Sigma_{n,p}$ was studied by Lev Kazarin [9].

Theorem D [7] *The set Θ_n is a subgroup of index 2 of the group $\Sigma_{n,2}$.*

2. 7-Designs

We also use the set $\mathbf{U}_n$ defined above for indexing unknown quantity x_α of 2^n -variable polynomials $f(\mathbf{x})$ in $\mathbf{R}[\mathbf{x}]$.

Lemma B *For any $n \geq 3$, the group Θ_n has a triplewise transitive subgroup Υ of substitution matrices, i.e. for any two monomials $x_{\alpha_i}x_{\alpha_j}x_{\alpha_k}$ and $x_{\beta_i}x_{\beta_j}x_{\beta_k}$, where $\{\alpha_i, \alpha_j, \alpha_k\}$ and $\{\beta_i, \beta_j, \beta_k\}$ are both three-element subsets of the set $\mathbf{F}_2^n$, there is a substitution matrix Γ in the group Υ such that $\Gamma x_{\alpha_i}x_{\alpha_j}x_{\alpha_k} = x_{\beta_i}x_{\beta_j}x_{\beta_k}$.*

Proof: A substitution matrix is a matrix with rational entries. Therefore to prove the lemma it suffices to show that $\Upsilon \subset \Sigma_{n,2}$.

We shall show that the subgroup of $\Sigma_{n,2}$ of all substitution matrices $\Gamma(\sigma)$ corresponding to affine maps $\sigma : \mathbf{x} \rightarrow \mathbf{x}Q + \alpha$, where Q is a nondegenerate matrix over $\mathbf{F}_2$ and $\alpha \in \mathbf{F}_2^n$, has all stated properties. Denote this subgroup by Υ .

Direct calculations show that

$$\Gamma(\sigma) = B(0, E, Q, -\alpha),$$

hence the group Υ is a subgroup of $\Sigma_{n,2}$.

To show triplewise transitivity of the subgroup Υ , we prove that it contains substitution matrix corresponding to the affine map $\sigma : \mathbf{x} \rightarrow Q\mathbf{x} + \beta$, which transforms three-element set $\beta = \{\beta_1, \beta_2, \beta_3\}$ into the set $\{0, \mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_i = (\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0)$, $i = 1, \dots, n$.

Indeed, we first use the substitution matrix Γ_{β_3} to transform the set β into the set $\gamma = \{0, \gamma_1, \gamma_2\}$, $\gamma_i = \beta_i + \beta_3$, $i = 1, 2$. Then use a linear map $\sigma' : \mathbf{x} \rightarrow Q\mathbf{x}$ with nondegenerate matrix Q such that $Q\gamma_i = \mathbf{e}_i$, $i = 1, 2$, to transform the set γ into the set $\{0, \mathbf{e}_1, \mathbf{e}_2\}$. The required substitution matrix is $\Gamma = \Gamma(\sigma')\Gamma_{\beta_3}$. The proof is complete. $\square$

Quadratic homogeneous harmonic polynomial in $N = 2^n$ variables looks as follows

$$F(\mathbf{x}) = \sum_{i < j} a_{\alpha_i, \alpha_j} x_{\alpha_i} x_{\alpha_j} + \sum_{i=1}^N a_{\alpha_i} x_{\alpha_i}^2, \quad \text{where} \quad \sum_{i=1}^N a_{\alpha_i} = 0$$

and a cubic one has the following form

$$F(\mathbf{x}) = \sum_{i < j < k} a_{\alpha_i, \alpha_j, \alpha_k} x_{\alpha_i} x_{\alpha_j} x_{\alpha_k} + \sum_{i < j} a_{\alpha_i, \alpha_j} x_{\alpha_i} x_{\alpha_j}^2 + \sum_{i > j} b_{\alpha_i, \alpha_j} x_{\alpha_i} x_{\alpha_j}^2 + \sum_{i=1}^N a_{\alpha_i} x_{\alpha_i}^3,$$

where

$$\sum_{j=1}^{i-1} a_{\alpha_i, \alpha_j} + \sum_{j=i+1}^N b_{\alpha_j, \alpha_i} + 3a_{\alpha_i} = 0 \quad \text{for all } \alpha_i.$$

This implies that the dimension of Harm(3) is $\binom{N}{3} + 2\binom{N}{2} + \binom{N}{1} - N = \binom{N+2}{3} - \binom{N}{1}$ which agrees with the above-mentioned relation.

It should be noted that

$$\begin{aligned} E_f F(\mathbf{x}) &= F(E_f \mathbf{x}^t) = \sum_{i < j < k} (-1)^{f(\alpha_i) + f(\alpha_j) + f(\alpha_k)} a_{\alpha_i, \alpha_j, \alpha_k} x_{\alpha_i} x_{\alpha_j} x_{\alpha_k} \\ &+ \sum_{i < j} (-1)^{f(\alpha_i)} a_{\alpha_i, \alpha_j} x_{\alpha_i} x_{\alpha_j}^2 + \sum_{1 > j} (-1)^{f(\alpha_i)} b_{\alpha_i, \alpha_j} x_{\alpha_i} x_{\alpha_j}^2 \\ &+ \sum_{i=1}^N (-1)^{f(\alpha_i)} a_{\alpha_i} x_{\alpha_i}^3. \end{aligned}$$

Lemma C For any $n > 1$

$$\sum_{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2} E_f x_{\beta_1} x_{\beta_2} x_{\beta_3} = \begin{cases} 2^{n(n+1)/2} x_{\mathbf{0}} x_{\mathbf{e}_1} x_{\mathbf{e}_2}, & \text{if } \{\beta_1, \beta_2, \beta_3\} = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2\}; \\ 0, & \text{if } \{\beta_1, \beta_2, \beta_3\} \neq \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2\}, \end{cases}$$

where the sum in $\sum_{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2}$ ranges over the set of all diagonal matrices E_f with Boolean function $f(\mathbf{x}) = f_0 + \sum_{i=1}^N f_{\alpha_i} x_{\alpha_i} + \sum_{i < j} f_{\alpha_i, \alpha_j} x_{\alpha_i} x_{\alpha_j}$ satisfying

$$f(\mathbf{0}) + f(\mathbf{e}_1) + f(\mathbf{e}_2) = 0. \quad (4)$$

Proof: Let $B_{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2}$ be the linear space of all Boolean functions $f(\mathbf{x})$, $\deg f(\mathbf{x}) \leq 2$, such that (4) holds. The dimension of $B_{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2}$ is one less than the dimension of the space of all Boolean functions of second degree, i.e. it is equal to $n(n+1)/2$.

Consider a function $l(f) = l_{\beta_1, \beta_2, \beta_3}(f) = f(\beta_1) + f(\beta_2) + f(\beta_3)$ on the space $B_{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2}$. It is obvious, $l(f)$ is a linear function and in particular $l_{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2}(f)$ is identically zero. We have to prove that the function $l(f)$ is nondegenerate whenever $\{\beta_1, \beta_2, \beta_3\} \neq \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2\}$. For this it suffices to show that in the space $B_{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2}$ there is a function $f(\mathbf{x})$ such that $l(f) = 1$.

Let $L_{\mathbf{e}_1, \mathbf{e}_2}$ be the two-dimensional linear subspace of the space $\mathbb{F}_2^n$ spanned by $\mathbf{e}_1, \mathbf{e}_2$. First we consider the case when the vectors $\{\beta_1, \beta_2, \beta_3\}$ are pairwise distinct and $\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2\} \neq \{\beta_1, \beta_2, \beta_3\}$.

We consider two subcases:

- a) $\{\beta_1, \beta_2, \beta_3\} \subset L_{\mathbf{e}_1, \mathbf{e}_2}$, e.g., $\beta_1 = \mathbf{e}_1, \beta_2 = \mathbf{e}_2, \beta_3 = \mathbf{e}_1 + \mathbf{e}_2$;
- b) the vector $\beta_1 = (\beta_{1,1}, \dots, \beta_{1,n})$ does not belong to $L_{\mathbf{e}_1, \mathbf{e}_2}$.

In the subcase a) the function $f(\mathbf{x}) = x_1 x_2$ will do since $f(\mathbf{0}) + f(\mathbf{e}_1) + f(\mathbf{e}_2) = 0$ and $f(\mathbf{e}_1 + \mathbf{e}_2) + f(\mathbf{e}_1) + f(\mathbf{e}_2) = 1$. If $\beta_1 = \mathbf{0}, \beta_2 = \mathbf{e}_2, \beta_3 = \mathbf{e}_1 + \mathbf{e}_2$ or $\beta_1 = \mathbf{0}, \beta_2 = \mathbf{e}_1, \beta_3 = \mathbf{e}_1 + \mathbf{e}_2$ then the function $f(\mathbf{x}) = x_1 + x_2$ has the required properties.

Now we pass to the subcase b). Consider a set $M = \{\beta'_1, \beta'_2, \beta'_3\}$, where $\beta'_i = (0, 0, \beta_{i,3}, \dots, \beta_{i,n})$, $i = 1, 2, 3$. Under the hypothesis of this subcase β'_1 is not identically zero vector. If either the elements of M are linearly independent or $\beta'_1 = \beta'_2 = \beta'_3$, then obviously there is a vector $\mathbf{l} = (0, 0, l_3, \dots, l_n)$ such that $\langle \mathbf{l}, \beta'_1 \rangle = \langle \mathbf{l}, \beta'_2 \rangle = \langle \mathbf{l}, \beta'_3 \rangle = 1$. In this case the function $f(\mathbf{x}) = \langle \mathbf{l}, \mathbf{x} \rangle$ satisfies both $f(\mathbf{0}) + f(\mathbf{e}_1) + f(\mathbf{e}_2) = 0$ and $f(\beta_1) + f(\beta_2) + f(\beta_3) = 1$.

If $\beta'_1 = \beta'_2 + \beta'_3$ and $\beta'_1 \neq 0, \beta'_2 \neq 0$, then there are two vectors $\mathbf{l}_i = (0, 0, l_{i,3}, \dots, l_{i,n})$, $i = 1, 2$, such, that $\langle \mathbf{l}_1, \beta'_1 \rangle = \langle \mathbf{l}_1, \beta'_2 \rangle = 1, \langle \mathbf{l}_1, \beta'_3 \rangle = 0$ and $\langle \mathbf{l}_2, \beta'_1 \rangle = \langle \mathbf{l}_2, \beta'_3 \rangle = 1, \langle \mathbf{l}_2, \beta'_2 \rangle = 0$, since the vectors β'_1 and β'_2 are linearly independent. In this case the function $f(\mathbf{x}) = \langle \mathbf{l}_1, \mathbf{x} \rangle \langle \mathbf{l}_2, \mathbf{x} \rangle$ satisfies both $f(\mathbf{0}) + f(\mathbf{e}_1) + f(\mathbf{e}_2) = 0$ and $f(\beta_1) + f(\beta_2) + f(\beta_3) = 1$.

If $\beta'_1 = \beta'_2 + \beta'_3$ and $\beta'_1 = \beta'_2 \neq 0$, i.e. $\beta'_3 = 0$, then the vectors $\beta''_i = (\beta_{i,1}, \beta_{i,2}, 0, \dots, 0)$, $i = 1, 2$, are distinct. For example, let $\beta''_1 \neq 0$. There exist two vectors $\mathbf{l}' = (l_1, l_2, 0, \dots, 0)$ and $\mathbf{l} = (0, 0, l_3, \dots, l_n)$ such, that $\langle \beta''_1, \mathbf{l}' \rangle = 1, \langle \beta''_2, \mathbf{l}' \rangle = 0$, and $\langle \mathbf{l}, \beta'_1 \rangle = 1$. The function $f(\mathbf{x}) = \langle \mathbf{l}', \mathbf{x} \rangle \langle \mathbf{l}, \mathbf{x} \rangle$ satisfies both $f(\mathbf{0}) + f(\mathbf{e}_1) + f(\mathbf{e}_2) = 0$ and $f(\beta_1) + f(\beta_2) + f(\beta_3) = 1$.

Now consider the case when there are at least two identical vectors in the set $\{\beta_1, \beta_2, \beta_3\}$. W.l.o.g. suppose that $\beta_2 = \beta_3$. Then $f(\beta_1) + f(\beta_2) + f(\beta_3) = f(\beta_1)$. Therefore we may consider a one-element set $\{\beta_1\}$ instead of a three-element one. In this case the proof that there is a function $f(\mathbf{x})$ in $B_{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2}$ such that $l(f) = f(\beta_1) = 1$, and $f(\mathbf{0}) + f(\mathbf{e}_1) + f(\mathbf{e}_2) = 0$ goes along the same lines but is a little bit easier. We leave this proof to reader.

Thus, we have proved that in all cases the function $l(f) = l_{\beta_1, \beta_2, \beta_3}(f)$ is not identically zero function provided $\{\beta_1, \beta_2, \beta_3\} \neq \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2\}$.

The proof of the lemma follows from the identity

$$\sum_{B_{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2}} (-1)^{l(f)} = 0, \quad (5)$$

which holds for any not identically zero function $l(f)$. This identity (5) follows from the fact that each nondegenerate linear function takes the value 0 in exactly one half of the points. The proof is complete. $\square$

Theorem E For any $n > 1$ the representations ρ_1, ρ_2 , and ρ_3 of the group $\Theta_{n,2}$ on $\text{Harm}(1)$, $\text{Harm}(2)$, and $\text{Harm}(3)$ respectively are irreducible.

Proof: First, consider the most complicated case, namely a representation ρ_3 on $\text{Harm}(3)$. The main idea of the proof is as follows. Suppose, contrary to our claim, that $\text{Harm}(3)$ is not irreducible. Then the Maschke theorem implies that $\text{Harm}(3)$ is a direct sum of two nontrivial invariant subspaces, say H and H' . If we could prove, that each of H and H' contains a monomial $x_{\mathbf{0}} x_{\mathbf{e}_1} x_{\mathbf{e}_2}$, then we would get $H \cap H' \neq \{0\}$, a contradiction.

Thus, the proof of the theorem is reduced to the proof of the following statement. Any not identically zero invariant subspace H of $\text{Harm}(3)$ contains the monomial $x_{\mathbf{0}} x_{\mathbf{e}_1} x_{\mathbf{e}_2}$. We proceed to prove this statement.

It should be noted that any harmonic polynomial which does not contain monomials of the form $x_{\alpha_i} x_{\alpha_j} x_{\alpha_k}$ with three-element set of indices $\alpha_i, \alpha_j, \alpha_k$, contains at least one monomial of the form $x_{\beta_i} x_{\beta_j}^2$ with not identically zero coefficient. Therefore no harmonic polynomial can be composed entirely by monomials of the form x_{β}^3 .

First we shall prove that any not identically zero invariant subspace H has polynomial, which contains some monomial $x_{\alpha_i} x_{\alpha_j} x_{\alpha_k}$ with not identically zero coefficient and three-element set $\{\alpha_i, \alpha_j, \alpha_k\}$ of indices.

Suppose the contrary, i.e. that any polynomial $F(\mathbf{x})$ in H has not identically zero coefficients only for monomials of either form $x_{\beta_1}x_{\beta_j}^2$ or $x_{\beta_j}^3$. Since the group Υ is triplewise transitive (see Lemma C) we can assume that monomial $x_0x_{\mathbf{e}_1}^2$ is one of them. In this case we prove, that there exist a polynomial $G(\mathbf{x}) \in H$ and a matrix $B \in \Theta_n$ such that the polynomial $G(B\mathbf{x})$ has some monomial $x_{\alpha_i}x_{\alpha_j}x_{\alpha_k}$ with not identically zero coefficient and a three-element set $\{\alpha_i, \alpha_j, \alpha_k\}$ of indices.

For this we consider the polynomial

$$m(\mathbf{x}) = m_F(\mathbf{x}) = \sum_{\mathbf{0}} F(E_h \mathbf{x}^t), \quad (6)$$

where the sum $\sum_{\mathbf{0}}$ ranges over the linear space L_0 of all Boolean functions of the form $h(\mathbf{x}) = \sum_{i=1}^N h_{\alpha_i} x_{\alpha_i} + \sum_{i < j} h_{\alpha_i, \alpha_j} x_{\alpha_i} x_{\alpha_j}$ (with $h(\mathbf{0}) = 0$). It is easy to show along the same lines as in the proof of Lemma 2, that

$$m(\mathbf{x}) = 2^{n(n+1)/2} \left(\sum_{\alpha \neq \mathbf{0}} a_{\mathbf{0}, \alpha} x_{\alpha}^2 + a_{\mathbf{0}, \mathbf{0}} x_{\mathbf{0}}^3 \right) \quad \text{and} \quad \sum_{\alpha \neq \mathbf{0}} a_{\mathbf{0}, \alpha} + 3a_{\mathbf{0}, \mathbf{0}} = 0. \quad (7)$$

Note that $m(\mathbf{x}) \neq 0$, provided $a_{\mathbf{0}, \mathbf{e}_1} \neq 0$ and the last equality in (7) holds since $m(\mathbf{x})$ is a harmonic polynomial.

Consider a matrix $B = B(D, R, R, 0)$ in Θ_n , where $D = \text{diag}(1, 1, 0, \dots, 0)$ and $R = \text{diag}(0, 0, 1, \dots, 1)$. The matrix B can be represented as $B = \text{diag}(A_2, \dots, A_2)$ with a suitable numbering of rows and columns. Here A_2 is the 4×4 Hadamard matrix $A_2 = 1/2 \| (-1)^{\langle \alpha_i, \alpha_j \rangle} \|$, $\alpha_1 = \mathbf{0}$, $\alpha_2 = \mathbf{e}_1$, $\alpha_3 = \mathbf{e}_2$, $\alpha_4 = \mathbf{e}_1 + \mathbf{e}_2$. Notice that entry $v_{\alpha, \beta}$ of the matrix $B = \| v_{\alpha, \beta} \|$ is not identically zero iff $\alpha + \beta \in L_2$, where $L_2 = L_{\mathbf{e}_1, \mathbf{e}_2}$ is the two-dimensional subspace in $\mathbb{F}_2^n$ spanned by vectors $\mathbf{e}_1, \mathbf{e}_2$. In this case $v_{\alpha, \beta} = (-1)^{\alpha D \beta^t}$.

Let $a_{\mathbf{0}} = 0$. In this case we show, that the polynomial $m(B\mathbf{x})$ has a monomial $x_{\alpha_i}x_{\alpha_j}x_{\alpha_k}$ with not identically zero coefficient and three-element set $\{\alpha_i, \alpha_j, \alpha_k\}$ of indices.

Let $\beta = \{\beta_1, \beta_2, \beta_3\}$ be a three-element subset of the set $\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\} = L_2$ and let

$$S_i = (\{\beta_1, \beta_2, \beta_3\}) = \frac{1}{8} \sum_{\gamma} (-1)^{\langle \beta_i, \gamma_2 \rangle + \langle \beta_i, \gamma_3 \rangle}, \quad i = 1, 2, 3,$$

where the sum ranges over all permutation $(\gamma_1, \gamma_2, \gamma_3)$ of the triple $(\beta_1, \beta_2, \beta_3)$. Direct calculations show that

$$m_F(B\mathbf{x}^t) 2^{-n(n+1)/2} = \sum_{\beta} (a_{\mathbf{0}, \mathbf{e}_1} S_1(\beta) + a_{\mathbf{0}, \mathbf{e}_2} S_2(\beta) + a_{\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2} S_3(\beta)) x_{\beta_1} x_{\beta_2} x_{\beta_3} + \dots, \quad (8)$$

where the sum $\sum_{\beta}$ ranges over all four distinct three-element subsets β of the set L_2 , and dots stand for monomials $x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}$ such that $\{\alpha_1, \alpha_2, \alpha_3\} \not\subset L_2$.

Let $S(\beta) = (S_1(\beta), S_2(\beta), S_3(\beta))$. By easy calculations we have $S(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2) = \frac{1}{8}(6, 2, 2)$, $S(\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2) = S(\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2) = \frac{1}{8}(6, -2, -2)$, and $S(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2) = \frac{1}{8}(6, 6, 6)$. The space $\mathbf{S}, \mathbf{S} \subset \mathbb{R}^3$, spanned by these three vectors has dimension 2. The vector $(0, -1, 1)$ is a basis of one-dimensional space $\mathbf{S}^{\perp}$ orthogonal to $\mathbf{S}$.

By assumption $a_{0,e_1} \neq 0$, therefore $\mathbf{a} = (a_{0,e_1}, a_{0,e_2}, a_{0,e_1+e_2}) \notin \mathbf{S}^\perp$. It follows that in the four-element set $\mathbf{L}_2$ of vectors there is at least one, say β , such that $(S(\beta), \mathbf{a}) \neq 0$. Thus we get that the polynomial $m(B\mathbf{x}^t)$ (cf. (8)) has some monomial $x_{\beta_1}x_{\beta_2}x_{\beta_3}$, $\{\beta_1, \beta_2, \beta_3\} \subset \mathbf{L}_2$, with not identically zero coefficient and three-element set of indices $\beta_1, \beta_2, \beta_3$.

Let now $a_0 \neq 0$. Consider the subgroup Υ' of the group Υ , consisting of all substitution matrices, corresponding to linear maps $\sigma : \mathbf{x} \rightarrow Q\mathbf{x}$ in the space $(\mathbf{F}_2)^n$, where Q is a $n \times n$ nondegenerate matrix over the field $\mathbf{F}_2$. Notice, that $\mathbf{0}$ is a fixed point of the map σ , therefore the polynomials x_0, x_0^2 and x_0^3 and only they are fixed points of any transformation in Υ' .

The polynomial

$$q(\mathbf{x}) = \sum_{P \in \Upsilon'} m_F(P\mathbf{x}^t) = a \sum_{\alpha \neq \mathbf{0}} x_0 x_\alpha^2 + b x_0^3, \quad a + 3b = 0, \quad b = a_0 |\Upsilon'|, \quad (9)$$

is invariant with respect to all transformations in Υ' . Therefore under the hypothesis $a_0 \neq 0$ the factor a in this expression is not identically zero.

Next we show, that a nondegenerate polynomial $q(B\mathbf{x}^t)$ has monomial $x_{\beta_1}x_{\beta_2}x_{\beta_3}$, $\{\beta_1, \beta_2, \beta_3\} \subset \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\} = \mathbf{L}_2$, with three-element set of indices and not identically zero coefficient. The proof of this statement goes along the same lines as the proof of the previous case $a_0 = 0$. Namely, first we explicitly calculate a vector of coefficients $S(\beta) = (S_0(\beta), S_1(\beta), S_2(\beta), S_3(\beta))$, $\beta = \{\beta_1, \beta_2, \beta_3\}$, to be used in the expression for the coefficient corresponding to the monomial $x_{\beta_1}x_{\beta_2}x_{\beta_3}$ (cf. (8)). Notice, that now we have extended a set of coefficients $S_i(\beta)$, $i = 0, 1, 2, 3$, by a new one $S_0(\beta)$, determined by the relation $u(2B\mathbf{x}^t) = \sum_{\beta} S_0(\beta) x_{\beta_1} x_{\beta_2} x_{\beta_3}$, where $u(\mathbf{x}) = x_0^3$.

Again, just as in the case $a_0 = 0$ the space $\mathbf{S}$, spanned by vectors from $S(\beta)$ has dimension 2. The space $\mathbf{S}^\perp$ of vectors $\mathbf{a} = (a_0, a_{0,e_1}, a_{0,e_2}, a_{0,e_1+e_2})$ such that $(\mathbf{S}, \mathbf{a}) = 0$ also has dimension 2 and its basis is $(-1, 1, 0, 0), (0, 0, -1, 1)$.

The vector $\mathbf{a} = \mathbf{a}(q) = (a_0, a_{0,e_1}, a_{0,e_2}, a_{0,e_1+e_2})$ of coefficients of the polynomial $q(\mathbf{x})$ is equal to (b, a, a, a) , $b \neq 0, a \neq 0$, and does not belong to $\mathbf{S}^\perp$. Therefore among four vectors $S(\beta)$, $\beta \in \mathbf{L}_2$, there must be at least one, such that $(S(\beta), \mathbf{a}(q)) \neq 0$, i.e. $q(B\mathbf{x}^t)$ has monomial $x_{\beta_1}x_{\beta_2}x_{\beta_3}$ with not identically zero coefficient and three-element set $\{\beta_i, \beta_j, \beta_k\}$ of indices.

Thus, in any nontrivial invariant subspace H there is a polynomial $g(\mathbf{x})$ having at least one monomial $x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}$ with not identically zero coefficient and three-element set $\{\alpha_1, \alpha_2, \alpha_3\}$ of indices. In the group Υ there is a matrix Y , which transforms the monomial $x_{\alpha_1}x_{\alpha_2}x_{\alpha_3}$ into monomial $x_0x_{e_1}x_{e_2}$. Therefore in any subspace H there is a polynomial which contains the monomial $x_0x_{e_1}x_{e_2}$ with not identically zero coefficient. This and Lemma C imply, that H contains the monomial $x_0x_{e_1}x_{e_2}$. This proves the theorem in the case of representation ρ_3 .

The proof of irreducibility of the representations ρ_1 and ρ_2 is easier than the proof of irreducibility of the representation ρ_3 and goes along the same lines. In particular, in the case of the space Harm(2) (respectively Harm(1)), we prove, that any not identically zero invariant subspace contains the monomial $x_0x_{e_1}$ (respectively x_0). The details are left to reader. The theorem is proved. $\square$

Theorem F For any $\mathbf{x} \in S_{N-1}$ the set $\mathcal{K}(\mathbf{x}) = \Theta_n \mathbf{x}^t$ (the orbit of the group Θ_n with the initial point $\mathbf{x}$ or the orbit code) is a 7-design.

Proof: It follows from the Theorem A, Theorem E and the identity

$$\sum_{B \in \Theta_n} F(B\mathbf{x}^t) = 0,$$

which holds for any polynomial $F(\mathbf{x})$ in $\text{Hom}(2k+1)$, since the group Θ_n contains a matrix $-E$, where E is the identity matrix.

One natural question is how one can select an initial point $\mathbf{x}$ in such a way that the number of elements of the design $\mathcal{K}_n(\mathbf{x})$ is minimal. Obviously $|\mathcal{K}_n(\mathbf{x})| = |\Theta_n|/|\Omega(\mathbf{x})|$, where $\Omega(\mathbf{x})$ is the stabilizer of the point $\mathbf{x}$ in the group Θ_n . As shown in [7] (Lemma 6)

$$|\Omega(\mathbf{o}_1)| = (2^n - 1) \cdots (2^n - 2^{n-1}) \tau_{n,2}/2,$$

where $\mathbf{o}_1 = (1, 0, \dots, 0)$ and $\tau_{n,2} = 2^{n(n+1)/2+1}$ is the number of n -variable polynomials of the second degree over the field $\mathbf{F}_2$.

This implies that

$$|\mathcal{K}(\mathbf{o}_1)| = \sum_{m=0}^n 2^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_2 \tau_{m,2}$$

It should be noted that the cardinalities $|\mathcal{K}_n(\mathbf{o}_1)|$ of the designs $\mathcal{K}_n(\mathbf{o}_1)$ for dimensions $N = 8, 16, 32$ ($n = 3, 4, 5$) coincide with the cardinalities of well-known designs derived from Barnes-Wall lattices [8].

Other natural question is whether there exists an initial point $\mathbf{x}$, for which the design $\mathcal{K}(\mathbf{x})$ has the strength larger than 7. This question will be studied by the author in forthcoming papers. $\square$

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