A Root System Criterion for Fully Commutative and Short Braid-Avoiding Elements in Affine Weyl Groups

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Abstract. We provide simple characterizations of short-braid avoiding and fully commutative elements in an affine Weyl group W , generalizing results of Fan and Stembridge for finite Weyl groups. Our results rely on the combinatorics of the *compatible* subsets of the root system of W .

Keywords: affine Weyl group, short-braid avoiding element, fully commutative element

Introduction and basic definitions

In his paper [4] Fan introduces after Zelevinski the following notion for elements of a Coxeter system (W, S) .

Definition 1 An element $w \in W$ is short-braid avoiding if no reduced expression of w contains a substring of the form sts , $s, t \in S$.

The notion of short-braid avoiding element is strictly related to the following definitions, due to Fan [3] and Stembridge [8].

Consider $s, t \in S$ and denote by $m(s, t)$ the order of $st \in W$; we call the string $\underbrace{st \dots st}_{m(s,t)}$, the long braid of s and t .

Definition 2 For $w \in W$ we say that w is commutative if no reduced expression of w contains a substring of the form sts , s, t being non-commuting generators in S such that the simple root corresponding to t is at least as long as the simple root corresponding to s .

Definition 3 For $w \in W$ we say that w is fully commutative if no reduced expression of w contains the long braid of some pair of non-commuting generators.

Remark Denote by W_s , W_c , W_{fc} the sets of short-braid avoiding, commutative and fully commutative elements in W respectively.

It turns out that $W_s = W_c = W_{fc}$ for simply-laced Coxeter groups. The relation $W_s = W_{fc}$ holds since the only defining relations which are not commutation relations are those of type $sts = tst$, $s, t \in S$. The equality $W_c = W_{fc}$ is obvious. Moreover Fan and Stembridge [3], [5] provide the following remarkable root-theoretic characterization of these elements. Let Δ be the canonical root system of (W, S) and set

$$N(w) = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) \in -\Delta^+\}.$$

Then w is commutative if and only if $\alpha, \beta \in N(w) \Rightarrow \alpha + \beta \notin N(w)$. In the general case, the three definitions introduced differ (although the inclusion relation $W_s \subseteq W_c \subseteq W_{fc}$ holds). Let us work out explicitly the example of a Weyl group W of type G_2 . If s_2 denotes the simple reflection corresponding to the short simple root, then we have

$$\begin{aligned} W_s &= \{1, s_1, s_2, s_1s_2, s_2s_1\} \\ W_c &= \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} \\ W_{fc} &= W \setminus \{s_1s_2s_1s_2s_1s_2 (= s_2s_1s_2s_1s_2s_1)\} \end{aligned}$$

In [3], [4], [8] the types of W for which W_s, W_c, W_{fc} are finite are determined; in each of these cases, their cardinalities are also determined. Moreover, when W is a finite Weyl group, Fan provides the following remarkable criterion for $w \in W$ to be short-braid avoiding: *$w \in W$ is short-braid avoiding if and only if any reduced expression of w remains reduced when a simple reflection is deleted in any possible way.*

This result has interesting applications since it gives a simple smoothness criterion for Schubert varieties attached to braid-avoiding elements. On the other hand, as noticed in [4, 5], the criterion does not hold for affine Weyl groups (a counterexample in type $\tilde{A}_2$ is $s_1s_2s_3s_1s_2$).

In this paper we provide a combinatorial characterization of the elements in W_s, W_{fc} for affine Weyl groups W in terms of the subsets $N(w)$ which encode the elements of W .

Before stating our results we fix the notation and we give some preliminary definitions.

Let W be an irreducible Weyl group (possibly affine) and let Δ be the associated root system (lying in a real vector space V). Fix a positive system Δ^+ in Δ and let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a corresponding basis of simple roots; then we have $\Delta = W\Pi$ and $\Delta = \Delta^+ \sqcup -\Delta^+$ ($\sqcup$ denotes the disjoint union). We list some standard notation relative to these data.

$A = (a_{ij})_{i,j=1}^l$	(generalized) Cartan matrix corresponding to Δ ,
s_i	fundamental reflection relative to $\alpha_i \in \Pi$,
$S = \{s_1, \dots, s_l\}$	set of Coxeter generators for W ,
ℓ	length function w.r.t. S ,
s_α	reflection relative to $\alpha \in \Delta^+$,
$(,)$	standard W -invariant bilinear form on V ,
	positive definite if Δ is finite,
	positive semidefinite with kernel $\mathbb{R}\delta$ if Δ is affine,
$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$	$(\alpha, \beta \in \Delta)$,

$$\begin{aligned}
\text{Supp}(\alpha) & \quad \text{support of } \alpha: \\
& \quad \text{if } \alpha = \sum_{i=1}^n a_i \alpha_i, \text{ then } \text{Supp}(\alpha) = \{\alpha_i \mid a_i \neq 0\}. \\
\mathbb{N}(\alpha, \beta) &= (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta \quad (\alpha, \beta \in \Delta^+).
\end{aligned}$$

For a root $\alpha \in \Delta$ as usual $\alpha > 0$ means $\alpha \in \Delta^+$. We recall that for each $v, w \in W$ we have $\ell(v) = |N(v)|$ and $N(vw) = N(v) + vN(w)$, where $+$ denotes the symmetric difference; moreover $N(vw) = N(v) \sqcup vN(w)$ if and only if $\ell(vw) = \ell(v) + \ell(w)$.

For any $R \subseteq \Delta$ set $W(R) = \langle s_\beta \mid \beta \in R \rangle$. We say that R is a *subsystem* of Δ if it is $W(R)$ -invariant. Note that $R^+ := R \cap \Delta^+$ is a set of positive roots for R . We say that $\mathfrak{R} \subseteq \Delta^+$ is a *p-subsystem* if $\mathfrak{R} = R \cap \Delta^+$ for some subsystem R . Equivalently, $\mathfrak{R}$ is a p-subsystem if $\mathfrak{R} \subseteq \Delta^+$ and $\mathfrak{R} \cup -\mathfrak{R}$ is a subsystem.

If $R \subseteq \Delta$ is a subsystem, we denote the cardinality of a root basis for R by $\text{rk}(R)$, and we call it the *rank* of R .

Moreover we say that R is *parabolic* if

$$\Delta \cap \text{Span}_{\mathbb{Q}}(R) = R.$$

As usual we say that a subsystem R of Δ is *standard parabolic* if $\Pi \cap R$ is a basis for R . Clearly a standard parabolic subsystem is parabolic. Moreover it is easily seen that a subsystem R is parabolic if and only if $R^+ = vR'^+$ for some standard parabolic subsystem R' and $v \in W$ (see [1, VI, 1.7, Prop. 24]).

If Δ is an irreducible affine root system and Δ^0 the associated finite root system, then Δ^0 is irreducible (in particular it has a unique highest root); moreover (cf. [6])

$$\Delta^+ = ((\Delta^0)^+ + \mathbb{N}\delta) \cup (-((\Delta^0)^+ + \mathbb{Z}^+\delta))$$

We call the elements of Δ^0 (resp. $(\Delta^0)^+$) *finite roots* (resp. *finite positive roots*).

Let R_0 be any subsystem of Δ^0 . Then

$$R = \{\beta + k\delta \mid \beta \in R_0, \quad k \in \mathbb{Z}\}$$

is clearly a subsystem of Δ and it is the affine root system associated to R_0 . For $\alpha \in \Delta^0$ set:

$$\underline{\alpha} := \begin{cases} \{\alpha + n\delta \mid n \in \mathbb{N}\} & \text{if } \alpha \in (\Delta^0)^+, \\ \{\alpha + m\delta \mid m \in \mathbb{Z}^+\} & \text{if } -\alpha \in (\Delta^0)^+; \end{cases}$$

we call $\underline{\alpha}$ the δ -string of α . When considering an affine root $\beta_0 + k\delta$, $\beta_0 \in \Delta^0$, we write $k \in \mathbb{N}'$ to mean $k \in \mathbb{N}$ if $\beta_0 > 0$ and $k \in \mathbb{Z}^+$ if $\beta_0 < 0$.

Moreover we say that a root β is *parallel* to α ($\beta \parallel \alpha$) if $\beta + \alpha \in \mathbb{Z}\delta$ or $\beta - \alpha \in \mathbb{Z}\delta$.

Definition 4 We say that $L \subseteq \Delta^+$ is *dependent* if there exist pairwise non-parallel roots $\alpha, \beta, \gamma \in L$ and $k \in \mathbb{Z}^+$ such that $\alpha + \beta = k\gamma$; we say that L is *independent* if it is not dependent.

Our main theorems are the following.

Theorem 1 *Assume $\Delta \not\cong \tilde{A}_1$. Then $w \in W_s$ if and only if $N(w)$ is independent.*

Remark If $\Delta \cong \tilde{A}_1$, then $w \notin W_s$ if and only if $\ell(w) \geq 3$. This is equivalent to the condition of dependence in Definition 4 without any requirement about parallelism.

Theorem 2 *Let $w \in W$. Then $w \in W_{fc}$ if and only if $N(w)$ does not contain any irreducible parabolic p -subsystem of rank 2.*

Preliminaries

We introduce now the main tools for the proof of the main theorems.

Definition 5 Let $L \subseteq \Delta^+$ and $<$ be a total order on L . We say that L is associated to $w \in W$ if $L = N(w)$. We say that $(L, <)$ is associated to the reduced expression $s_{i_1} \cdots s_{i_m} = w$ ($w \in W$) if

$$L = \{\alpha_{i_1} < s_{i_1}(\alpha_{i_2}) < \cdots < s_{i_1} \cdots s_{i_{m-1}}(\alpha_{i_m})\}$$

(in particular $L = N(w)$).

The following Proposition 1 is the easy part of a well known theorem of Dyer [2]; it holds for any Coxeter system. We prove it here for completeness.

Proposition 1 *Assume that $(L, <)$ is associated to some reduced expression of some element of W . Then, for each $\alpha, \beta \in \Delta^+$, $q, r \in \mathbb{R}^+$, the following conditions hold:*

- (I) *if $\alpha, \beta \in L$, $\alpha < \beta$, and $q\alpha + r\beta \in \Delta$, then $q\alpha + r\beta \in L$ and $\alpha < q\alpha + r\beta < \beta$.*
- (II) *if $q\alpha + r\beta \in L$ and $\beta \notin L$, then $\alpha \in L$ and $\alpha < q\alpha + r\beta$.*

In particular, if L is associated to some element of W , then, for $\alpha, \beta \in \Delta^+$, $q, r \in \mathbb{R}^+$ we have:

- (I') *if $\alpha, \beta \in L$, $q\alpha + r\beta \in \Delta$, then $q\alpha + r\beta \in L$.*
- (II') *if $q\alpha + r\beta \in L$ and $\beta \notin L$, then $\alpha \in L$.*

Proof: Assume that $(L, <) = \{\beta_1 < \cdots < \beta_n\}$ with $\beta_1 = \alpha_{i_1}$, $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$, $1 < j \leq n$, $w = s_{i_1} \cdots s_{i_n}$ reduced expression. First assume $\alpha, \beta \in L$, $q, r \in \mathbb{R}^+$, $q\alpha + r\beta \in \Delta$. Then by definition $w^{-1}(\alpha), w^{-1}(\beta) < 0$; thus $w^{-1}(q\alpha + r\beta) = qw^{-1}(\alpha) + rw^{-1}(\beta) < 0$, i.e., $q\alpha + r\beta \in L$. Assume $\alpha = \beta_j$ and $\beta = \beta_k$, with $j < k$ and put $v = s_{i_1} \cdots s_{i_{j-1}}$. Then $(L', <') := \{v^{-1}(\beta_j) < \cdots < v^{-1}(\beta_k)\}$ is associated to the reduced expression $s_{i_j} \cdots s_{i_k}$ and has $v^{-1}(\alpha)$ and $v^{-1}(\beta)$ as its first and last element, respectively. By the first part of our proof we have $v^{-1}(q\alpha + r\beta) \in L'$, and clearly $v^{-1}(\alpha) <' v^{-1}(q\alpha + r\beta) <' v^{-1}(\beta)$. It follows directly that $\alpha < q\alpha + r\beta < \beta$.

Next assume $q\alpha + r\beta \in L$; $q\alpha + r\beta = \beta_m$, $m \leq n$. Then $\{\beta_1 < \cdots < \beta_m\}$ is associated to the reduced expression $u = s_{i_1} \cdots s_{i_m}$ and has $q\alpha + r\beta$ as its last element. Assume towards a

contradiction that neither $\alpha \in N(u)$ nor $\beta \in N(u)$. Then, by definition, $u^{-1}(\alpha), u^{-1}(\beta) > 0$; thus $u^{-1}(q\alpha + r\beta) = qu^{-1}(\alpha) + ru^{-1}(\beta) > 0$, against the assumption, so that we have necessarily $\alpha \in N(u)$ or $\beta \in N(u)$. Moreover, clearly, if α or β belongs to $N(u)$, then it precedes $q\alpha + r\beta = \beta_m$. This implies the assertion for L . $\square$

Remark For $k \in \mathbb{N}$ we have $\alpha + k\delta = \frac{1}{k+1}\alpha + \frac{k}{k+1}(\alpha + (k+1)\delta)$; if $\alpha \in \Delta^+$ and $\alpha + k\delta \in L$ for some $k \in \mathbb{N}$, then since L is finite, by condition (II) of Proposition 1 we obtain that $\alpha \in L$. Moreover, by condition (I), $\alpha + h\delta \in L$ if $0 \leq h \leq k, h \in \mathbb{N}$. Similarly, since for each k $\alpha + k\delta = (k+1)\alpha + k(-\alpha + \delta)$, if $\alpha \in L$, then $-\alpha + \delta \notin L$. In fact if $\underline{\alpha} \cap L \neq \emptyset$, then $\underline{-\alpha} \cap L = \emptyset$.

The conditions of Proposition 1 are also sufficient for $(L, <)$ to be associated to some reduced expression of some $w \in W$ [2]. Indeed, for the root system of an (affine) Weyl group they can be weakened [7], as we shall see below. We denote by $\bar{\Delta}$ (resp. $\bar{\Delta}^+$) the generalized root system (resp. positive root system) [8] associated to the Cartan matrix A of Δ ,

$$\bar{\Delta} = \Delta \sqcup \pm\mathbb{Z}^+\delta, \quad \bar{\Delta}^+ = \Delta^+ \sqcup \mathbb{Z}^+\delta.$$

Theorem A [7] *Let $L \subseteq \Delta^+$ be finite and $<$ be a total order on L . $(L, <)$ is associated to some reduced expression of some element of W if and only if, for each $\alpha, \beta \in \bar{\Delta}^+$, the following conditions hold:*

- (1) *if $\alpha, \beta \in L$, $\alpha < \beta$, and $\alpha + \beta \in \bar{\Delta}$, then $\alpha + \beta \in L$ and $\alpha < \alpha + \beta < \beta$.*
- (2) *If $\alpha + \beta \in L$ and $\beta \notin L$, then $\alpha \in L$ and $\alpha < \alpha + \beta$.*

L is associated to some element of W if and only if for each $\alpha, \beta \in \bar{\Delta}^+$:

- (1') *if $\alpha, \beta \in L$, $\alpha + \beta \in \bar{\Delta}$, then $\alpha + \beta \in L$*
- (2') *if $\alpha + \beta \in L$ and $\beta \notin L$, then $\alpha \in L$.*

Corollary 1 *Let $L \subseteq \Delta^+$ be finite. Then the following are equivalent:*

- i) *L is associated to some element of W ;*
- ii) *L satisfies conditions (I') and (II') of Proposition 1;*
- iii) *L satisfies conditions (1') and (2') of Theorem A.*

Corollary 2 *Let $L \subseteq \Delta^+$ be finite and $<$ be a total order on L . Then the following are equivalent:*

- i) *$(L, <)$ is associated to some reduced expression of some element of W ;*
- ii) *$(L, <)$ satisfies conditions (I) and (II) of Proposition 1;*
- iii) *$(L, <)$ satisfies conditions (1) and (2) of Theorem A.*

Definition 6 Let $L \subseteq \Delta^+$ and $<$ be a total order on L . L is called compatible if it satisfies one of the three equivalent conditions of Corollary 1. $(L, <)$ is called compatible, if it satisfies one of the three equivalent conditions of Corollary 2 (in particular L is compatible). In such a case we also say that $<$ is a compatible order.

Note that $N(w)$ determines $w \in W$, thus Theorem A establishes a bijection between W and the compatible finite subsets of Δ^+ ; moreover it gives a bijection between the compatible orders on $N(w)$ and the reduced expressions of w , for any fixed $w \in W$.

Proofs of the main theorems

If $<$ is a compatible order on $N(w)$, then, by condition (I) of Proposition 1, or (1) of Theorem A, we get that $\min(N(w), <)$ is a simple root. Indeed any simple root in $N(w)$ can be taken as the least root for some compatible order on $N(w)$; in the following lemma we state this and other basic properties of compatible sets and orders in a convenient form for our next developments.

Lemma 1 *Let L be a finite compatible set, $\alpha \in L$ be a simple root, and $L' = s_\alpha(L \setminus \{\alpha\})$. Then:*

- i) L' is compatible.
- ii) *If $<'$ is a compatible order on L' , then the total order defined on L by:
 $\alpha = \min(L, <)$ and $\beta < \beta'$ if and only if $s_\alpha(\beta) <' s_\alpha(\beta')$ for $\beta, \beta' \in L \setminus \{\alpha\}$ (*)
is compatible. In particular there exists a compatible order $<$ on L such that $\alpha = \min(L, <)$.*
- iii) *Conversely, if $<$ is a compatible order on L such that $\alpha = \min(L, <)$, then the total order $<'$ defined on L' by (*) is compatible. In particular if β is the successor of α in $(L, <)$, then $s_\alpha(\beta)$ is a simple root.*

Proof:

- i) By assumption there exists $w \in W$ such that $L = N(w)$. Set $w' = s_\alpha w$; then $\ell(w') = \ell(w) - 1$, hence $N(w) = N(s_\alpha) \sqcup s_\alpha N(w') = \{\alpha\} \sqcup s_\alpha N(w')$ and therefore $N(w') = s_\alpha(N(w) \setminus \{\alpha\}) = L'$. It follows that L' is compatible.
- ii) If $(L', <')$ is associated to the reduced expression $s_{i_1} \cdots s_{i_k}$, then $(L, <)$ is associated to the reduced expression $s_\alpha s_{i_1} \cdots s_{i_k}$, therefore it is compatible.
- iii) $(L, <)$ is associated to the some reduced expression starting with s_α , say $s_\alpha s_{i_1} \cdots s_{i_k}$; then $(L', <')$ is associated to the reduced expression $s_{i_1} \cdots s_{i_k}$, therefore it is compatible. $\square$

Lemma 2 *Suppose that $N(w)$ is endowed with a compatible order $<$. If R^+ is a finite p -subsystem, then $M = R^+ \cap N(w)$ is compatible as a subset of $R = R^+ \cup -R^+$, and the restriction of $<$ to M is compatible.*

Proof: Suppose $\alpha, \beta \in M$, $\alpha + \beta \in R^+ = \bar{\Delta}^+ \cap R = \Delta^+ \cap R \subseteq \Delta^+$; then the compatibility of $N(w)$ implies $\alpha + \beta \in N(w)$ and in turn $\alpha + \beta \in M$. If now $\alpha + \beta \in M$, then as above the compatibility of $N(w)$ and the relation $R^+ = \Delta^+ \cap R$ imply $\alpha \in M$ or $\beta \in M$ as desired. The claim regarding the order is proved in the same way. $\square$

Lemma 3 *Let ξ be a positive root in an affine root system Δ ; then $\xi, \xi + \delta$ can be consecutive in a compatible order on some compatible subset of Δ^+ if and only if Δ is of type $\tilde{A}_1$.*

Proof: Assume that $\xi, \xi + \delta$ are consecutive in a compatible order. By Lemma 1 iii) there exists $w \in W$ such that both $\alpha = w(\xi)$ and $s_\alpha w(\xi + \delta) = -\alpha + \delta$ are simple roots; this clearly implies $\Delta \cong \tilde{A}_1$. The converse is also clear. $\square$

Theorem 1. Assume $\Delta \not\cong \tilde{A}_1$. Then $w \in W_s$ if and only if $N(w)$ is independent.

Proof: Set $N = N(w)$. Assume $w \notin W_s$; we have to prove that N is dependent. By hypothesis, w can be written in reduced form as $w = us_i s_j s_i v$ where $s_i, s_j \in S$, $u, v \in W$. Then in the ordering induced by such a reduced expression, $u(\alpha_i), u(s_i(\alpha_j)), u(s_i s_j(\alpha_i))$ are consecutive. But $u(\alpha_i) + u(s_i s_j(\alpha_i)) = -a_{ij} u(s_i(\alpha_j))$ and moreover, since $\Delta \not\cong \tilde{A}_1$, α_i and α_j are not parallel; therefore N is dependent.

Conversely, assume that N is dependent. Endow N with an arbitrary compatible order $<$ and consider the set

$$I_< = \{(\alpha, \gamma, \beta) \in (N)^{\times 3} \mid \alpha < \gamma < \beta, \alpha \nparallel \beta, \exists k \in \mathbb{Z}^+ \alpha + \beta = k\gamma\}.$$

For a triple $(\alpha, \gamma, \beta) \in I_<$ set

$$\rho_<(\alpha, \gamma, \beta) = |\{x \in N \mid \alpha < x < \beta\}|$$

Take any triple (α, γ, β) such that $\rho_<(\alpha, \gamma, \beta)$ is minimal. By repeated applications of Lemma 1 we may assume $\alpha = \min N$, so that α is simple. Since α, β , hence γ , are not mutually parallel, they are contained in a uniquely determined finite parabolic subsystem R of rank 2. Let β' be the only root in Δ^+ which completes α to a root basis for R^+ , so that $R^+ = \mathbb{N}(\alpha, \beta')$. By Lemma 2, $(R^+ \cap N, <)$ is compatible in R . Since it has α as its first element, it is associated to some expression of type $s_\alpha s_{\beta'} s_\alpha \dots$, so that $(R^+ \cap N, <) = \{\alpha < s_\alpha(\beta') < s_\alpha s_{\beta'}(\alpha) < \dots\}$. Therefore, by the minimality of $\rho_<$ we have $\gamma = s_\alpha(\beta')$ and $\beta = s_\alpha s_{\beta'}(\alpha)$. If γ is the successor of α in N , then by Lemma 1 $s_\alpha(\gamma) = \beta'$ is simple in Δ . Also $s_{\beta'} s_\alpha(\beta) = \alpha$ is simple, hence, again by Lemma 1, there exists a compatible order on N starting with $\alpha < \gamma < \beta$, which corresponds to a reduced expression of w starting with the braid $s_\alpha s_{\beta'} s_\alpha$. Assume that there exists $x \in N$, such that $\alpha < x < \gamma$. We define $\gamma_0 = \gamma$ and, for $i \geq 1$, $\gamma_i = \max_{<} \{\eta \in N \mid \eta < \gamma_{i-1}, (\eta, \gamma_{i-1}) \neq 0\}$ if $\{\eta \in N \mid \eta < \gamma_{i-1}, (\eta, \gamma_{i-1}) \neq 0\}$ is non empty. Let γ_n be the last element we can define in such a way. If $\gamma_n \neq \alpha$, then we can replace $<$ with a suitable compatible order in which γ_n precedes α ; therefore without loss of generality we may assume that $\gamma_n = \alpha$. Moreover, if $n = 1$, then for each $x \in N$ such that $\alpha < x < \gamma$ we have $x \perp \gamma$; thus we can bring γ adjacent to α and we are done. Assume by contradiction $n \geq 2$. We claim that $\gamma_i \nparallel \gamma_{i+1} \forall i = 0, \dots, n-1$. Otherwise, since N is compatible, $\gamma_i - \gamma_{i+1} = \delta$. Moreover, by the definition of γ_{i+1} , there exists a compatible order in which γ_{i+1}, γ_i appear in consecutive positions: this contradicts Lemma 3. Now remark that, since $\gamma_i + \gamma_{i+1}$ is not a root by the minimality of $\rho_<$, the definition of the γ_i 's forces $\gamma_i - \gamma_{i+1}$ to be a root. Such a root must be positive, otherwise we get $\gamma_{i+1} - \gamma_i \in N$ and $\gamma_{i+1} - \gamma_i < \gamma_{i+1} < \gamma_i$, against the minimality of $\rho_<$. Then we define $k_i = \max\{h \in \mathbb{Z}^+ \mid h\gamma_i - \gamma_{i+1} \in \Delta^+\}$. Since we are assuming $n \geq 2$, we have $w^{-1}(k_i \gamma_i - \gamma_{i+1}) > 0$: this follows from the minimality of $\rho_<$ (if $\gamma_{i+1} = \alpha$ then $\gamma_i \neq \gamma$ and by our previous remarks $k_i \gamma_i - \gamma_{i+1} \neq \beta$). Adding up such

relations we get that $w^{-1}(k_{n-1} \cdots k_0 \gamma - \alpha)$ is a sum of positive roots with non-negative coefficients. This is a contradiction if $k_{n-1} \cdots k_0 \geq k$, since $w^{-1}(k\gamma - \alpha) = w^{-1}(\beta) < 0$ and $w^{-1}(\gamma) < 0$; in particular we get a contradiction if $k = 1$. So we assume $k > 1$. We remark that if $\Delta \not\cong G_2, \tilde{G}_2$, any finite rank 2 indecomposable subsystem of Δ is of type A_2 or B_2 . If $\Delta \cong G_2, \tilde{G}_2$, any finite rank 2 subsystem of Δ is of type A_2 or G_2 . Therefore, if $k > 1$ then $k = 2$ if α, γ, β are included in a subsystem of type B_2 , and $k = 3$ if α, γ, β are included in a subsystem of type G_2 . It follows that for each $k_i > 1$, we have $k_i = k$. Moreover, if η, η' are non parallel and non orthogonal roots in Δ , then $\langle \eta, \eta' \rangle = \pm 1$ if η and η' have the same length or η is short; $\langle \eta, \eta' \rangle = \pm k$ if η is long and η' is short. Thus if $k > 1$, we get in particular that α is long and γ is short. Now we remark that $\langle \gamma_i, \gamma_{i+1} \rangle > 0$. If $k_i = 1$ for each $0 \leq i \leq n-1$, then $\gamma_0, \dots, \gamma_n$ all have the same length: this is impossible since $\gamma_0 = \gamma$ is short and $\gamma_n = \alpha$ is long. Thus for some i we have $k_i = k$ and thus $k_{n-1} \cdots k_0 \geq k$. $\square$

Lemma 4 Assume $\Delta \not\cong G_2, \tilde{G}_2$ and $\alpha, \beta \in \Delta^+$. If $\langle \alpha, \beta \rangle < 0$ then either $\alpha \parallel \beta$ or $\mathbb{N}(\alpha, \beta)$ is a p -subsystem. In the latter case $\mathbb{N}(\alpha, \beta) \cup -\mathbb{N}(\alpha, \beta)$ is an irreducible parabolic subsystem having $\{\alpha, \beta\}$ as a basis.

Proof: Assume that α, β are not parallel and set $R = (\mathbb{Q}\alpha + \mathbb{Q}\beta) \cap \Delta$. Then R is clearly a finite parabolic rank 2 subsystem of Δ . Since $\Delta \not\cong G_2, \tilde{G}_2$ the type of R is one of $A_1 \times A_1, A_2, B_2$. Indeed it cannot be $A_1 \times A_1$, since R contains α and β which are not orthogonal, hence it is A_2 or B_2 . But if R' is a root system of type A_2 or B_2 , then any two roots with negative scalar product are a basis of R' , thus $\{\alpha, \beta\}$ is a basis for R and $R^+ = \mathbb{N}(\alpha, \beta)$. $\square$

Lemma 5 Assume that the simple roots α_i, α_j belong to $N(w)$. Then some reduced expression of w starts with the long braid of s_i and s_j .

Proof: Set $X_{ij} = \{w \in W \mid w^{-1}(\alpha_i) > 0 \text{ and } w^{-1}(\alpha_j) > 0\}$ and $W_{ij} = \langle s_i, s_j \rangle$. By [1, IV, ex. 1.3] there exist unique $u \in W_{ij}$ and $v \in X_{ij}$ such that $w = uv$; moreover $\ell(w) = \ell(u) + \ell(v)$ so that $N(w) = N(u) \sqcup uN(v)$. By definition $\alpha_i, \alpha_j \notin N(v)$ hence $\alpha \notin N(v)$ for each $\alpha \in R(\alpha_i, \alpha_j)$. Now W_{ij} permutes $R(\alpha_i, \alpha_j)$ and therefore it permutes $\Delta \setminus R(\alpha_i, \alpha_j)$. Since s_k permutes $\Delta^+ \setminus \{\alpha_k\}$ for $k = i, j$, it follows that W_{ij} permutes the positive roots out of $R(\alpha_i, \alpha_j)$. Therefore we have $\alpha_i, \alpha_j \notin uN(v)$. On the other hand $\alpha_i, \alpha_j \in N(w)$, thus $\alpha_i, \alpha_j \in N(u)$. Therefore u is the longest element in W_{ij} and its reduced expressions are the long braids of s_i and s_j . $\square$

Given any finite parabolic subsystem R in Δ , there always exists a compatible pair $(L, <)$ with L finite, in which the roots of the p -subsystem R^+ are consecutive. In fact, there exist a standard parabolic p -subsystem R' and $w \in W$ such that $wR'^+ = R^+$. Let u be the longest element of $W(R')$ and consider wu . Then $N(wu) = N(w) + wN(u) = N(w) + w(R'^+) = N(w) + R^+$; since $w^{-1}(R^+) = R'^+$, we have indeed $N(wu) = N(w) \sqcup R^+$. Therefore the join of a reduced expression of w and a reduced expression of u is a reduced expression of wu ; in the order induced on $N(wu)$ by any such reduced expression R^+ appears as a final section.

On the other hand if we fix a compatible set L including a parabolic p -subsystem R^+ , then it may happen that there is no compatible order on L of which R^+ is a section. For instance in type D_4 , consider $w = s_2 s_1 s_3 s_2 s_4 s_2 s_3 s_1 s_2$. Then $\alpha_2, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in N(w)$ and they form a parabolic p -subsystem of type A_2 , but they can be consecutive in no compatible order of $N(w)$. Nonetheless for the case A_n we have the following “strong” result:

Proposition 2 *Suppose $W = S_{n+1}$, the symmetric group on $n + 1$ letters (so that Δ is a root system of type A_n); consider $w \in W$. Then, for any triple of roots $\{\alpha, \alpha + \beta, \beta\}$ in $N(w)$, there exists a compatible order in $N(w)$ in which these elements are consecutive.*

Proof: We proceed by induction on $\ell(w)$. Consider a triple $\{\alpha, \alpha + \beta, \beta\} \subseteq N(w)$. If there exists a simple root $\gamma \in N(w)$ different from α and β then we consider the triple $\{s_\gamma(\alpha), s_\gamma(\alpha + \beta), s_\gamma(\beta)\} \subseteq N(s_\gamma w)$. Since $\ell(s_\gamma w) < \ell(w)$, by induction there exists a compatible order on $N(s_\gamma w)$ in which $s_\gamma(\alpha), s_\gamma(\alpha + \beta), s_\gamma(\beta)$ are consecutive; by Lemma 1 this order comes from a compatible order on $N(w)$ in which $\alpha, \alpha + \beta, \beta$ are consecutive. We have two more cases to consider: either both α and β are simple roots or one of the two—say α —is the only simple root in $N(w)$. In the first case we are done by Lemma 5; in the other case we get a contradiction, since by compatibility β should contain α in its support but this would imply $\alpha + \beta \notin \Delta$. $\square$

Theorem 2. *Let $w \in W$. Then $w \in W_{fc}$ if and only if $N(w)$ does not contain any irreducible parabolic p -subsystem of rank 2.*

Proof: We show that $w \notin W_{fc}$ if and only if $N(w)$ contains an irreducible p -subsystem of rank 2; if $w \notin W_{fc}$, then for some $i, j \in \{1, \dots, l\}$, $w = u \underbrace{s_i s_j \dots}_{m(i,j)} v$, $u, v \in W$, $\ell(w) = \ell(u) + m(i, j) + \ell(v)$.

Then

$$N(w) = N(u) \sqcup u(N(s_i s_j \dots)) \sqcup u s_i s_j \dots (N(v)).$$

But $N(s_i s_j \dots) = \mathbb{N}(\alpha_i, \alpha_j)$ is clearly an irreducible p -subsystem of rank 2, hence $u(\mathbb{N}(\alpha_i, \alpha_j)) = \mathbb{N}(u(\alpha_i), u(\alpha_j))$ is too.

Next assume that the set of irreducible rank 2 p -subsystems contained in $N(w)$ is non-empty; this implies in particular $\Delta \not\cong \tilde{A}_1$. Set $N = N(w)$. Fix any compatible order $<$ on N . For any parabolic p -subsystem $R^+ \subseteq N$ set $R = R^+ \cup -R^+$ and

$$d_{<}(R) = |\{x \in N \mid \min R^+ < x < \max R^+\}|,$$

where the maximum and minimum are taken with respect to the restriction of $<$ to R^+ . Choose a finite parabolic irreducible p -subsystem of rank 2, $R^+ \subseteq N$ such that $d_{<}(R)$ is minimal. Then set $\alpha = \min R^+$, $\beta = \max R^+$, and $\gamma = \min(R^+ \setminus \{\alpha\})$, the successor of α in R^+ . Consider the set $\{x \in N \mid \alpha < x < \gamma\}$: if it is empty, then by Lemma 1 iii), $\beta = s_\alpha(\gamma)$ is simple and we conclude using Lemma 5. We shall prove that if $x \in N$ and $\alpha < x < \gamma$, then x is orthogonal to α . From this, by Lemma 1 iii), it follows that we can

bring α adjacent to γ , still obtaining a compatible order on N , and we can conclude by Lemma 5. It is enough to prove it for x the successor of α in N since if $x \perp \alpha$, then we can exchange α and x in $(N, <)$, still obtaining a compatible order $<'$ on N in which $d_{<'}(R)$ is minimal. As in the proof of theorem 1 we may assume that $\alpha = \min N$ and that α is simple. Moreover we may assume $\beta = \max N$. So let $x = \min(N \setminus \{\alpha\})$, $x \neq \gamma$.

Remark that a subsystem of type A_2 is parabolic unless it is contained in a (sub)system of type G_2 ; moreover, if Δ contains a subsystem of type G_2 , then Δ is of type G_2 or $\tilde{G}_2$. A subsystem of type B_2 or G_2 is always parabolic. These remarks lead us to consider separately the $G_2, \tilde{G}_2$ cases.

First case: $\Delta \not\cong G_2, \tilde{G}_2$

We first prove that if Δ is an affine system, then for each $\xi \in R^+$, ξ is the least root (w.r.t. $<$) in its δ -string. Since α is simple, we have $\alpha \in \Delta^0$ or $\alpha = -\theta + \delta$, θ being the highest root in Δ^0 . Assume $\beta = \beta_0 + k\delta$ with $\beta_0 \in \Delta^0$ and $k \in \mathbb{N}'$. Since N is compatible, if $\beta_0 > 0$ then $\beta_0 \in N$ and if $\beta_0 < 0$ then $\beta_0 + \delta \in N$. We have $\langle \alpha, \beta \rangle = \langle \alpha, \beta_0 + k\delta \rangle = \langle \alpha, \beta_0 \rangle$ and $\langle \beta, \alpha \rangle = \langle \beta_0 + k\delta, \alpha \rangle = \langle \beta_0, \alpha \rangle$ for each $k \in \mathbb{Z}$; therefore, since $\Delta \not\cong \tilde{G}_2$, $\{\alpha, \beta_0 + k\delta\}$ is a basis for a parabolic irreducible subsystem of rank 2 in Δ if and only if $\{\alpha, \beta_0\}$ is. Thus, by the minimality of $d_{<}(R)$, if $\beta_0 > 0$ then $\beta = \beta_0$, and if $\beta_0 < 0$ then $\beta = \beta_0 + \delta$. Since by assumption $\alpha + \beta$ is a root, if $\beta = \beta_0 + \delta$ with $\beta_0 < 0$, then α is a finite simple root (recall that $\theta + \eta \notin \Delta \ \forall \eta \in (\Delta^0)^+$); similarly, if $\alpha = -\theta + \delta$ then β is finite positive. In both cases if $\xi \in R^+$, $\xi \neq \alpha, \beta$ then $\xi = \xi_0 + \delta$ with $-\xi_0 \in (\Delta^0)^+$. If α and β are positive finite, then the same holds for any $\xi \in R^+$. In any case, each $\xi \in R^+$ has the required minimality condition.

From the above result we get that x does not belong to the same δ -string of any root in R^+ other than α . Indeed, by compatibility, x is not parallel to any root in R^+ other than α ; moreover, by Lemma 3, x is not parallel to α .

Henceforth Δ may be finite or not. We distinguish several cases.

I. $R \cong A_2$. Then $R^+ = \{\alpha, \gamma, \beta\}$, $\gamma = \alpha + \beta$.

First suppose that x has the same length as α, β, γ . Remark that $(x, \alpha) \geq 0$: otherwise $x + \alpha \in \Delta$ and $\alpha < \alpha + x < x$ against the choice of x . Thus $\langle x, \alpha \rangle = \langle \alpha, x \rangle = 1$. If $\langle x, \gamma \rangle = 0$, then $\langle x, \beta \rangle = -1$ and we get a contradiction by Lemma 4. Similarly $\langle x, \gamma \rangle \neq -1$, therefore $\langle x, \gamma \rangle = \langle \gamma, x \rangle = 1$. It follows that $x - \alpha$ and $\gamma - x$ are roots. The compatibility of the order forces both $x - \alpha$ and $\gamma - x$ to be positive. Now $\{\alpha, x, x - \alpha\}$ and $\{x, \gamma, \gamma - x\}$ are parabolic p -subsystems of Δ of type A_2 , thus, by the minimality of $d_{<}(R^+)$, $x - \alpha, \gamma - x \notin N$. But $\beta = (x - \alpha) + (\gamma - x) \in N$, against the compatibility of N .

Next assume that α, β, γ are long and x is short. Then $(\langle \alpha, x \rangle, \langle x, \alpha \rangle) = (2, 1)$. As above $x \not\perp \gamma$, thus $(\langle \gamma, x \rangle, \langle x, \gamma \rangle) = (2, 1)$. Then $\alpha, x, 2x - \alpha, x - \alpha$ are roots and the compatibility of $<$ forces them to be positive; thus they form a parabolic p -subsystem of Δ of type B_2 . Similarly, $x, \gamma, \gamma - x, \gamma - 2x$ form a parabolic p -subsystem of Δ . By our choice of minimality we have $x - \alpha, \gamma - 2x \notin N$.

But $\beta = (\gamma - x) + (x - \alpha) = (2x - \alpha) + (\gamma - 2x)$, therefore by compatibility, $\gamma - x, 2x - \alpha \in N$. Now $\langle \gamma - 2x, \alpha \rangle = -1$, thus $\gamma - 2x + \alpha$ is a (positive) root. We have $\gamma = (\gamma - 2x + \alpha) + (2x - \alpha)$; it is easily seen that $\gamma, \gamma - 2x + \alpha$, and

$2x - \alpha$ are all long, thus they form a parabolic p-subsystem of Δ of type A_2 . Since $\gamma, 2x - \alpha \in N$, by minimality we get $\gamma - 2x + \alpha \notin N$. But then the decomposition $\gamma - x = (\gamma - 2x + \alpha) + (x - \alpha)$ contradicts the compatibility of N .

Finally assume that α, β, γ are short and x is long. Then we have $(\langle \alpha, x \rangle, \langle x, \alpha \rangle) = (\langle \gamma, x \rangle, \langle x, \gamma \rangle) = (1, 2)$. As above we get that $\{\alpha, x, x - \alpha, x - 2\alpha\}$ and $\{x, \gamma, 2\gamma - x, \gamma - x\}$ are parabolic p-subsystems of Δ of type B_2 . By minimality, $x - 2\alpha, \gamma - x \notin N$ and by compatibility, $x - \alpha, 2\gamma - x \in N$. Now $\langle \gamma - x, \alpha \rangle = -1$, thus $\gamma - x + \alpha$ is a (positive) root. Now $\gamma - x + \alpha, x - \alpha$, and γ are all short and $(\gamma - x + \alpha) + (x - \alpha) = \gamma$, thus $\gamma - x + \alpha, x - \alpha$, and γ form a parabolic p-subsystem of Δ of type A_2 ; since $\gamma, x - \alpha \in N$, by minimality $\gamma - x + \alpha \notin N$. But then we get a contradiction: $\beta = (x - 2\alpha) + (\gamma - x + \alpha) \in N$ and $x - 2\alpha, \gamma - x + \alpha \notin N$.

II. a) $R \cong B_2$ and α is long. Then $R^+ = \{\alpha, \alpha + \beta, \alpha + 2\beta, \beta\}$. Set $\gamma = \alpha + \beta$ and $\gamma' = \alpha + 2\beta$.

Assume that x is long. Then $\langle x, \alpha \rangle = \langle \alpha, x \rangle = 1$; as above $\langle x, \gamma \rangle \neq 0$, otherwise $\langle x, \beta \rangle < 0$. Since also $\langle x, \gamma \rangle \neq 0$, we get thus $(\langle x, \gamma \rangle, \langle \gamma, x \rangle) = (2, 1)$. It follows that $\{\alpha, x, x - \alpha\}$ is a parabolic p-subsystem of Δ of type A_2 and $\{x, \gamma, 2\gamma - x, \gamma - x\}$ is a parabolic p-subsystem of type B_2 . By minimality, $x - \alpha, \gamma - x \notin N$, whereas $(x - \alpha) + (\gamma - x) = \beta \in N$: a contradiction.

Next assume that x is short. Then $(\langle x, \alpha \rangle, \langle \alpha, x \rangle) = (1, 2)$ and $\langle x, \gamma \rangle = \langle \gamma, x \rangle = 1$; $\{\alpha, x, 2x - \alpha, x - \alpha\}$ is a parabolic p-subsystem of type B_2 and $\{x, \gamma, \gamma - x\}$ is a parabolic p-subsystem of type A_2 . As above we get a contradiction since by minimality $x - \alpha, \gamma - x \notin N$.

II. b) $R \cong B_2$ and α is short. Then $R^+ = \{\alpha, 2\alpha + \beta, \alpha + \beta, \beta\}$. Set $\gamma = 2\alpha + \beta$ and $\gamma' = \alpha + \beta$.

Assume that x is short. Arguing as above we get $\langle x, \alpha \rangle = \langle \alpha, x \rangle = 1$ and $\langle x, \gamma' \rangle = \langle \gamma', x \rangle = 1$. Thus $\{\alpha, x, x - \alpha\}$ and $\{x, \gamma', \gamma' - x\}$ are parabolic p-subsystems of Δ of type A_2 . By minimality $\gamma' - x, x - \alpha \notin N$ and as above we get a contradiction.

Finally assume that x is long. Then $(\langle x, \alpha \rangle, \langle \alpha, x \rangle) = (\langle x, \gamma' \rangle, \langle \gamma', x \rangle) = (2, 1)$; it follows $\langle x, \beta \rangle = 0$ and thus $\langle x, \gamma \rangle = \langle \gamma, x \rangle = 1$. Then $\{\alpha, x, x - \alpha, x - 2\alpha\}$ is a parabolic p-subsystem of type B_2 and $\{x, \gamma, \gamma - x\}$ is a parabolic p-subsystem of type A_2 . Since $\beta = (x - 2\alpha) + (\gamma - x)$, we get a contradiction arguing as in the previous cases. This concludes the proof for all types of Δ other than $G_2, \tilde{G}_2$.

Second case: $\Delta \cong G_2, \tilde{G}_2$

The case $\Delta = G_2$ is trivial, since there are no irreducible proper parabolic p-subsystems of rank 2. So we assume $\Delta \cong \tilde{G}_2$.

First assume that $R \cong G_2$. We can argue as in the general case and get that each element in $\mathbb{N}^+(\alpha, \beta)$ is minimal in its δ -string, with respect to $<$. Clearly x must be parallel to some root in $\mathbb{N}^+(\alpha, \beta)$; but it is not parallel to α , being consecutive to α , and it cannot be parallel to any other root in R^+ , since each element in such a set is minimal in its δ -string. Therefore we must have $d_<(R) = 0$.

Next assume $R \cong A_2$. First we prove the following criterion.

Suppose $a = a_0 + h\delta, b = b_0 + k\delta$, with $a_0, b_0 \in \Delta^0, h, k \in \mathbb{Z}$. $\{a, b\}$ is a basis for a parabolic subsystem of type A_2 if and only if a_0 and b_0 are long, $(a_0, b_0) < 0$, and $3 \nmid (2h + k), (h + 2k)$.

Assume that $a, b \in \Delta$ are a basis for a parabolic subsystem $R \cong A_2$. Then $R = (\mathbb{Z}a + \mathbb{Z}b) \cap \Delta = (\mathbb{Q}a + \mathbb{Q}b) \cap \Delta$. Clearly we have $\langle a_0, b_0 \rangle = \langle b_0, a_0 \rangle = -1$; moreover, from the Dynkin diagram, we see that a, b , hence a_0, b_0 must be long. Then we have $\frac{1}{3}(2\alpha_0 + b_0) \in \Delta^0$, therefore, if $3 \mid (2h + k)$, also $\frac{1}{3}(2a + b) \in \Delta$: this would imply $(\mathbb{Z}a + \mathbb{Z}b) \cap \Delta \neq (\mathbb{Q}a + \mathbb{Q}b) \cap \Delta$ against the assumption. Therefore $3 \nmid (2h + k)$ and similarly $3 \nmid (h + 2k)$. Conversely, assume that a, b are long roots such that $\langle a, b \rangle = \langle b, a \rangle = -1$. If $\mathbb{N}(a, b)$ is not parabolic, a, b are included in a parabolic subsystem of type G_2 . Then $\frac{1}{3}(2a + b), \frac{1}{3}(a + 2b) \in \Delta$; therefore $3 \mid (2h + k), (h + 2k)$.

Now we go on by a direct inspection.

- I. $\alpha = \alpha_1$. By the above criterion, together with our choice of minimality, we get either $\beta = (\alpha_1 + 3\alpha_2) + \delta$, or $\beta = -\theta + \delta$; we distinguish the two cases.
 - a) $\beta = (\alpha_1 + 3\alpha_2) + \delta$. By compatibility, $\alpha_1 + 3\alpha_2 \in N$; moreover, since $\beta = (\alpha_1 + \delta) + 3\alpha_2$, at least one of $\alpha_2, \alpha_1 + \delta$ belongs to N . In the first case we get a contradiction since α_1, α_2 clearly generate a parabolic p-subsystem; in the latter case we get a contradiction since $\alpha_1 + 3\alpha_2, \alpha_1 + \delta$ generate a parabolic p-subsystem of type A_2 .
 - b) $\beta = -\theta + \delta$. Then $\gamma = -\alpha_1 - 3\alpha_2 + \delta$. In this case β, γ are minimal in their δ -string, thus x cannot be parallel to any of α, β, γ and therefore it cannot be long. Moreover $\langle \alpha_1, x \rangle > 0$ and x must be minimal in its δ string, therefore we have either $x = \alpha_1 + \alpha_2$ or $x = -\alpha_2 + \delta$. In the first case we get a contradiction since $\{x, \beta\}$ is a basis for a parabolic subsystem of type G_2 ; the second case is not possible, since $x = \beta + 2(\alpha_1 + \alpha_2)$ and neither β , nor $\alpha_1 + \alpha_2$ precede x in N .
- II. $\alpha = -\theta + \delta$. Then $\beta = \alpha_1$ or $\beta = \alpha_1 + 3\alpha_2$. In both subcases, β, γ are minimal in their δ -string and therefore x is short; since $\langle x, \alpha \rangle > 0$, we get $x = -\alpha_1 - \alpha_2 + \delta$ or $x = -\alpha_1 - 2\alpha_2 + \delta$.
 - a) $\beta = \alpha_1$. Then $\gamma = -\alpha_1 - 3\alpha_2 + \delta$. If $x = -\alpha_1 - \alpha_2 + \delta$ we get a contradiction since $\{x, \beta\}$ is a basis for a G_2 subsystem. The case $x = -\alpha_1 - 2\alpha_2 + \delta$ is not possible since $x = \gamma + \alpha_2$ and neither γ , nor α_2 precede x in N .
 - b) $\beta = \alpha_1 + 3\alpha_2$. Then $\gamma = -\alpha_1 + \delta$. We get $x \neq -\alpha_1 - 2\alpha_2 + \delta$, otherwise x, β would be a basis for a G_2 subsystem. Finally the case $x = -\alpha_1 - \alpha_2 + \delta$ is not possible since $2x = (-\theta + 2\delta) + \alpha_2$ and neither $-\theta + 2\delta$, nor α_2 precede x in N . $\square$

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