On Mod- p Alon-Babai-Suzuki Inequality

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Received October 15, 1998; Revised April 27, 1999

Abstract. Alon, Babai and Suzuki proved the following theorem:

Let p be a prime and let K, L be two disjoint subsets of $\{0, 1, \dots, p-1\}$. Let $|K| = r$, $|L| = s$, and assume $r(s-r+1) \leq p-1$ and $n \geq s+k_r$ where k_r is the maximal element of K . Let $\mathcal{F}$ be a family of subsets of an n -element set. Suppose that

- (i) $|F| \in K \pmod{p}$ for each $F \in \mathcal{F}$;
- (ii) $|E \cap F| \in L \pmod{p}$ for each pair of distinct sets $E, F \in \mathcal{F}$.

Then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \dots + \binom{n}{s-r+1}$.

They conjectured that the condition that $r(s-r+1) \leq p-1$ in the theorem can be dropped and the same conclusion should hold. In this paper we prove that the same conclusion holds if the two conditions in the theorem, i.e. $r(s-r+1) \leq p-1$ and $n \geq s+k_r$ are replaced by a single more relaxed condition $2s-r \leq n$.

Keywords: combinatorial, inequality

1. Introduction

In this paper, we let n be a positive integer, $I_n = \{1, 2, \dots, n\}$, $X = \{x_1, x_2, \dots, x_n\}$ be an n -element set, p be a prime number and $L \subseteq I_{p-1} \cup \{0\} = \{0, 1, \dots, p-1\}$ be an s -element set for some positive integer $s < p$. We call a family $\mathcal{F}$ of subsets of X a *mod p L -intersection family* if $|E \cap F| \in L \pmod{p}$, $\forall E, F \in \mathcal{F}$ with $E \neq F$. Here $n \in L \pmod{p}$ means there exists $l \in L$ for which $n \equiv l \pmod{p}$.

For any $0 \leq i \leq j \leq n$, let $I_n(i, j)$ be the 0-1 incidence matrix of $\mathbb{P}_i(X)$ and $\mathbb{P}_j(X)$ with rows (columns) indexed by $\mathbb{P}_i(X)$ ($\mathbb{P}_j(X)$). The (A, B) -entry of $I_n(i, j)$ is 1 if $A \subseteq B$ and 0 otherwise for any $A \in \mathbb{P}_i(X)$ and $B \in \mathbb{P}_j(X)$.

Convention: Throughout the paper, unless otherwise specified, all vector spaces are assumed to be over $\mathbb{F}_p$ which we abbreviate as $\mathbb{F}$. Therefore for the sake of brevity $\text{rank}(I_n(i, j))$ will denote the rank of $I_n(i, j)$ considered as a matrix over $\mathbb{F}$.

Alon, Babai and Suzuki [1] proved the following inequality which generalizes the classic Frankl-Ray-Chaudhuri-Wilson Inequality [3].

Theorem Let p be a prime and K, L be two disjoint subsets of $\{0, 1, \dots, p-1\}$. Let $|K| = r$, $|L| = s$, and assume $r(s-r+1) \leq p-1$ and $n \geq s+k_r$ where k_r is the maximal element of K . Let $\mathcal{F}$ be a family of subsets of an n -element set. Suppose that

- (i) $|F| \in K \pmod{p}$ for each $F \in \mathcal{F}$;
 - (ii) $|E \cap F| \in L \pmod{p}$ for each pair of distinct sets $E, F \in \mathcal{F}$.
- Then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

They went on and conjectured that the condition $r(s-r+1) \leq p-1$ in the statement of the above theorem can be dropped and the conclusion of the theorem will still hold. Snevily [7] confirmed and improved the conjecture when n is sufficiently large. He showed that when n is sufficiently large, then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-2} + \binom{n}{s-4} + \cdots + \binom{n}{s-2\lfloor s/2 \rfloor}$. The main result of this paper is the following theorem which confirms the conjecture of Alon, Babai and Suzuki to a large extent.

Theorem 1 *Let p be a prime number, r, s be two positive integers with $2s-r \leq n$, L be an s -subset of $I_{p-1} \cup \{0\}$ and K be an r -subset of $I_{p-1} \cup \{0\}$ with $L \cap K = \emptyset$. If $\mathcal{F}$ is a mod p L -intersection family and $|E| \in K \pmod{p}$, $\forall E \in \mathcal{F}$, then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

We note that in some instances the condition $2s-r \leq n$ holds but Alon, Babai and Suzuki's condition $n \geq s + k_r$ does not. For instance, if $n = 9$, $p = 7$, $K = \{2, 5, 6\}$ and $L = \{0, 1, 3, 4\}$, then it is clear that $2s-r = 2 \cdot 4 - 3 = 5 \leq 9 = n$, but $k_r + s = 6 + 4 > 9 = n$. In some other instances, however, the Alon, Babai and Suzuki's condition holds but the condition $2s-r \leq n$ does not. For example, $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $p = 7$, $K = \{1\}$, $L = \{0, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\{9\}, \{1, 2, 3, 4, 5, 6, 7, 8\}\}$. It is clear that $k_r + s = 7 < 9$ but $2s-r = 11 > 9$.

2. Proof of Theorem 1

For the proof of the theorem we need the following lemma which is mentioned by Frankl in [2].

Lemma 1 *If $0 \leq a \leq b < p$ and $a+b \leq n$, then $\text{rank}_p(I_n(a, b)) = \binom{n}{a}$.*

Proof: We may assume $a \neq 0$. The proof is by induction on $a+b+n$. Note that $a+b+n \geq 4$. It is clear that the lemma holds when $a+b+n = 4$.

Suppose it holds when $a+b+n < l$. Now we consider the case $a+b+n = l$. We distinguish two cases.

Case 1 $a+b = n$. In this case, it is easy to verify that $\mathbb{P}_b(X)$ is an L' -intersection family with $L' = \{n-2a, n-2a+1, \dots, n-a-1\}$ and $b = n-a$ and $b \notin L' \pmod{p}$. Now we use the following result of Frankl and Wilson [3]:

If $\mathcal{G} \subseteq \mathbb{P}_k(X)$ is a mod p L -intersection family for some set L consisting of non-negative integers with $k \notin L \pmod{p}$ and $\binom{k-i}{l-i} \not\equiv 0 \pmod{p}$ for $i = 0, 1, \dots, l$, then $|\mathcal{G}| \leq \text{rank}(I_n(l, \mathcal{G}))$, where $l = |L|$ and $I_n(l, \mathcal{G})$ is a 0-1 incidence matrix whose rows and

columns are indexed by $\mathbb{P}_l(X)$ and $\mathcal{G}$ respectively and the (A, F) -entry of $I_n(l, \mathcal{G})$ is 1 if $A \subseteq F$ and 0 otherwise for any $A \in \mathbb{P}_l(X)$ and $F \in \mathcal{G}$.

Notice that if we take $\mathcal{G} = \mathbb{P}_b(X)$, then $I_n(a, \mathcal{G}) = I_n(a, b)$. So by the above result we have $\binom{n}{b} = |\mathbb{P}_b(X)| \leq \text{rank}(I_n(a, b))$. On the other hand, it is clear that $\text{rank}(I_n(a, b)) \leq \binom{n}{b}$. So $\text{rank}(I_n(a, b)) = \binom{n}{b}$, which implies $\text{rank}(I_n(a, b)) = \binom{n}{a}$ since $b = n - a$ in this case. This proves the lemma in the first case.

Case 2 $a + b < n$. In this case, we partition $\mathbb{P}_b(X)$ into two families: one consists of all those s -subsets of X not containing x_n , the other one consists of all those containing x_n . We do the same thing to $\mathbb{P}_a(X)$. It is clear that

$$I_n(a, b) = \begin{pmatrix} I_{n-1}(a, b) & B \\ 0 & I_{n-1}(a-1, b-1) \end{pmatrix} \quad \text{for some matrix } B.$$

We observe that in this case $a + b \leq n - 1$ and $a - 1 + b - 1 \leq n - 1$. By the induction hypothesis, $\text{rank}(I_{n-1}(a, b)) = \binom{n-1}{a-1}$ and $\text{rank}(I_{n-1}(a-1, b-1)) = \binom{n-1}{a-1}$, i.e. both the rows of $I_{n-1}(a, b)$ and the rows of $I_{n-1}(a-1, b-1)$ are linearly independent. So the rows of $I_n(a, b)$ are linearly independent, which implies that $\text{rank}(I_n(a, b)) = \binom{n}{a}$ and hence the proof of the lemma is complete. $\square$

Remark By Lemma 1, it is clear that the row vectors of $I_n(a, b)$ can be expanded into a basis of $\mathbb{F}^{\binom{n}{b}}$ by adding some other $\binom{n}{b} - \binom{n}{a}$ vectors in $\mathbb{F}^{\binom{n}{b}}$.

Following the idea of Ramanan [6], we associate a variable x_F for each $F \in \mathcal{F}$. For $I \subseteq X$, we define the linear form L_I by

$$L_I = \sum_{F \in \mathcal{F}, I \subseteq F} x_F.$$

Now let us prove a lemma which is useful in the proof of the theorem.

Lemma 2 For any positive integers u, v with $u < v < p$ and $u + v \leq n$, we have

$$\dim \left(\frac{\langle L_J : J \in \mathbb{P}_v(X) \rangle}{\langle \sum_{J \in \mathbb{P}_v(X), I \subseteq J} L_J : I \in \mathbb{P}_u(X) \rangle} \right) \leq \binom{n}{v} - \binom{n}{u}.$$

Here $\frac{A}{B}$ is the quotient space of two vector spaces A and B with $B \subseteq A$ and $\langle L_J : J \in \mathbb{P}_v(X) \rangle$ is the vector space spanned by $\{L_J : J \in \mathbb{P}_v(X)\}$.

Proof: Let $V = \langle L_J : J \in \mathbb{P}_v(X) \rangle$. We define the following linear mapping $f : \mathbb{F}^{\binom{n}{v}} \rightarrow V$ as follows. We view a vector $\underline{w}$ in $\mathbb{F}^{\binom{n}{v}}$ as a mapping from $\mathbb{P}_v(X)$ to $\mathbb{F}$. For each vector $\underline{w} \in \mathbb{F}^{\binom{n}{v}}$ whose J 'th component is a_J , we define $f(\underline{w}) = \sum_{J \in \mathbb{P}_v(X)} a_J L_J$.

Let W be the vector space generated by the rows of $I_n(u, v)$. It is clear that f is a surjective map that maps W to $\langle \sum_{J \in \mathbb{P}_v(X), I \subseteq J} L_J : I \in \mathbb{P}_u(X) \rangle$. By linear algebra

$$\begin{aligned} & \dim \left(\frac{\langle L_J : J \in \mathbb{P}_v(X) \rangle}{\langle \sum_{J \in \mathbb{P}_v(X), I \subseteq J} L_J : I \in \mathbb{P}_u(X) \rangle} \right) \\ & \leq \dim \left(\frac{f(\mathbb{F}^{\binom{n}{v}})}{f(W)} \right) \\ & \leq \dim \left(\frac{\mathbb{F}^{\binom{n}{v}}}{W} \right) \\ & \leq \binom{n}{v} - \binom{n}{v} \quad \text{by the above remark.} \end{aligned}$$

This proves Lemma 2. □

Consider the system of linear equations:

$$\left\{ L_I = 0, \quad \text{where } I \text{ runs through } \bigcup_{i=0}^s \mathbb{P}_i(X) \right\}. \quad (*)$$

By the method employed in Qian and Ray-Chaudhuri [4] or [5], we have the following proposition.

Proposition *Assume that $L \cap K = \emptyset$. If $\mathcal{F}$ is an mod p L -intersection family with $|E| \in K \pmod{p}$ for any $E \in \mathcal{F}$, then the only solution of the above system of linear equations is the trivial solution.*

Proof: Let (v_E) be a solution of (*). We need to show that (v_E) is the all-zero solution. Suppose on the contrary that not all of v_E 's are 0. Let E_0 be an element in $\mathcal{F}$ with $v_{E_0} \neq 0$. Let $\mathbb{F}$ be the finite field containing p elements. Since $\binom{x}{0}, \binom{x}{1}, \dots, \binom{x}{s}$ form a basis for the vector space spanned by all the polynomials in $\mathbb{F}(X)$ of degrees at most s , there exist $a_0, a_1, \dots, a_s \in \mathbb{F}$ with

$$\sum_{i=0}^s a_i \binom{x}{i} = \prod_{j=1}^s (x - l_j).$$

We denote $\prod_{j=1}^s (x - l_j)$ by $g(x)$. Next we prove the following identity,

$$\sum_{i=0}^s a_i \sum_{I \in \mathbb{P}_i(X), I \subseteq E_0} L_I = \sum_{F \in \mathcal{F}} g(|F \cap E_0|) x_F.$$

We prove it by comparing the coefficients of both sides. For any $F \in \mathcal{F}$, the coefficient of x_F in the left hand side is

$$\sum_{i=0}^s a_i |\{I \in \mathbb{P}_i(X) : I \subseteq E_0, I \subseteq F\}| = \sum_{i=0}^s a_i \binom{|F \wedge E_0|}{i},$$

which is equal to $g(|F \wedge E_0|)$ by the definition of $g(x)$. This proves the above identity.

Specializing $x_E = v_E$ for all $E \in \mathcal{F}$ in the above identity, we have

$$\sum_{i=0}^s a_i \sum_{I \in \mathbb{P}_i(X), I \subseteq E_0} L_I((v_E)) = \sum_{F \in \mathcal{F}} g(|F \wedge E_0|) v_F.$$

It is clear that left hand side is 0 since (v_E) is a solution of (*). For $F \in \mathcal{F}$ with $F \neq E_0$, $|F \wedge E_0| \in L \pmod{p}$ and so $g(|F \wedge E_0|) = 0$. So the right hand side of the above identity is equal to $g(|E_0|)v_{E_0}$. So $0 = g(|E_0|)v_{E_0}$. Since $L \cap K = \emptyset$, We have $g(|E_0|) \neq 0$ and so $v_{E_0} = 0$. This is a contradiction to the definition of E_0 and thus it proves the proposition. $\square$

As a result of this proposition, we have:

$$|\mathcal{F}| \leq \dim \left(\left\{ L_I : I \in \bigcup_{i=0}^s \mathbb{P}_i(X) \right\} \right). \quad (1)$$

where $\dim(\{L_I : I \in \bigcup_{i=0}^s \mathbb{P}_i(X)\})$ is defined to be the dimension of the space spanned by $\{L_I : I \in \bigcup_{i=0}^s \mathbb{P}_i(X)\}$.

The following lemma is of critical importance in the proof of the theorem.

Lemma 3 *For any $i \in \{0, 1, \dots, s - r + 1\}$ and every $I \in \mathbb{P}_i(X)$, the linear form $\sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H$ is linearly dependent on the set of linear forms $\{L_H : i \leq |H| \leq i + r - 1, H \subseteq X\}$ over $\mathbb{F}$.*

Proof of Lemma 3: We distinguish two cases.

Case 1 $i \notin K \pmod{p}$. In this case $\forall k_j \in K, k_j - i \neq 0$ in $\mathbb{F}$ and so $c = (-1)^{r+1}(k_1 - i)(k_2 - i) \cdots (k_r - i) \neq 0$ in $\mathbb{F}$. It is clear that there exist $a_1, a_2, \dots, a_{r-1} \in \mathbb{F}, a_r = r! \in \mathbb{F} - \{0\}$ such that

$$\begin{aligned} & a_1 \binom{x}{1} + a_2 \binom{x}{2} + \cdots + a_r \binom{x}{r} \\ &= (x - (k_1 - i))(x - (k_2 - i)) \cdots (x - (k_r - i)) + c, \end{aligned}$$

since the polynomial in the right hand side has constant term equal to 0.

Next we show that

$$\sum_{j=1}^r a_j \sum_{H \in \mathbb{P}_{i+j}(X), I \subseteq H} L_H = c \cdot L_I. \quad (2)$$

In fact both sides are linear forms in x_E 's, $E \in \mathcal{F}$. The coefficient of x_E in the left hand side is $\sum_{j=1}^r a_j |\{H \mid I \subseteq H \subseteq E, |H| = i + j\}|$. So it is equal to 0 if $I \not\subseteq E$ and $a_1 \binom{|E|-i}{1} + a_2 \binom{|E|-i}{2} + \cdots + a_r \binom{|E|-i}{r}$ if $I \subseteq E$. By the above polynomial identity,

$$\begin{aligned} & a_1 \binom{|E|-i}{1} + a_2 \binom{|E|-i}{2} + \cdots + a_r \binom{|E|-i}{r} \\ &= (|E| - i - (k_1 - i))(|E| - i - (k_2 - i)) \cdots (|E| - i - (k_r - i)) + c \\ &= c \quad \text{since } |E| \in K \pmod{p}. \end{aligned}$$

The coefficient of x_E 's in the right hand side is obviously the same. This proves (2).

Writing (2) in a different way, we have

$$\sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H = \frac{1}{r!} \left(c L_I - \sum_{j=1}^{r-1} a_j \sum_{H \in \mathbb{P}_{i+j}(X), I \subseteq H} L_H \right)$$

This proves the lemma in case 1.

Case 2 $i \in K \pmod{p}$. In this case, the constant term of $(x - (k_1 - i))(x - (k_2 - i)) \cdots (x - (k_r - i))$ is 0 in $\mathbb{F}$. So there exists $a_1, a_2, \dots, a_{r-1} \in \mathbb{F}, a_r = r! \in \mathbb{F} - \{0\}$ such that

$$a_1 \binom{x}{1} + a_2 \binom{x}{2} + \cdots + a_r \binom{x}{r} = (x - (k_1 - i))(x - (k_2 - i)) \cdots (x - (k_r - i)).$$

As a consequence we have

$$\sum_{j=1}^r a_j \sum_{H \in \mathbb{P}_{i+j}(X), I \subseteq H} L_H = 0 \quad \forall I \in \mathbb{P}_i(X),$$

i.e. we have

$$\sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H = -\frac{1}{r!} \left(\sum_{j=1}^{r-1} a_j \sum_{H \in \mathbb{P}_{i+j}(X), I \subseteq H} L_H \right) \quad \forall I \in \mathbb{P}_i(X).$$

This finishes the proof of this lemma. $\square$

From the above lemma, we easily deduce the following corollary.

Corollary *With the same condition as in Lemma 3, we have*

$$\begin{aligned} & \left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_H(X) \right\rangle \\ &= \left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_H(X) \right\rangle + \left\langle \sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H : I \in \mathbb{P}_i(X) \right\rangle. \end{aligned}$$

Next we prove our last lemma.

Lemma 4 For any $i \in \{0, 1, \dots, s-r+1\}$,

$$\begin{aligned} & \binom{n}{i} + \binom{n}{i+1} + \binom{n}{i+r-1} + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i}^s \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X) \rangle} \right) \\ & \leq \binom{n}{s-r+1} + \binom{n}{s-r+2} + \dots + \binom{n}{s}. \end{aligned}$$

Proof of Lemma 4: We induct on $s-r+1-i$. It is clearly true when $s-r+1-i=0$, i.e. $i=s-r+1$. Suppose the lemma holds for $s-r+1-i < l$ for some positive integer l . Now we want to show that it holds for $s-r+1-i=l$.

Let us recall two well-known linear algebra facts:

Fact 1. Let A, B, C be vector spaces with $C \subseteq B$. Then $\dim(\frac{A+B}{A+C}) \leq \dim(\frac{B}{C})$.

Fact 2. Let $C \subseteq B \subseteq A$ be three vector spaces. Then $\dim(\frac{A}{C}) = \dim(\frac{B}{C}) + \dim(\frac{A}{B})$.

We observe that $i+i+r \leq (s-r) + (s-r) + r \leq n$ by the condition in the theorem. By the above corollary, we have

$$\begin{aligned} & \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X) \rangle} \right) \\ & = \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X) \rangle + \langle L_H : H \in \mathbb{P}_{i+r}(X) \rangle}{\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X) \rangle + \langle \sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H : I \in \mathbb{P}_i(X) \rangle} \right) \\ & \leq \dim \left(\frac{\langle L_H : H \in \mathbb{P}_{i+r}(X) \rangle}{\langle \sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H : I \in \mathbb{P}_i(X) \rangle} \right) \quad \text{by fact 1 above} \\ & \leq \binom{n}{i+r} - \binom{n}{i} \quad \text{by Lemma 2 with } u=i \text{ and } v=i+r. \end{aligned}$$

In summary, we have

$$\dim \left(\frac{\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X) \rangle} \right) \leq \binom{n}{i+r} - \binom{n}{i}. \quad (3)$$

Now we are ready for the key part of the proof of the lemma.

$$\begin{aligned} & \binom{n}{i} + \binom{n}{i+1} + \dots + \binom{n}{i+r-1} + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i}^s \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X) \rangle} \right) \\ & = \binom{n}{i} + \binom{n}{i+1} + \dots + \binom{n}{i+r-1} + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X) \rangle} \right) \\ & \quad + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i}^s \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X) \rangle} \right) \quad \text{by fact 2 above} \end{aligned}$$

$$\begin{aligned}
&= \binom{n}{i} + \binom{n}{i+1} + \cdots + \binom{n}{i+r-1} + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X) \rangle} \right) \\
&\quad + \dim \left(\frac{\langle L_H : H \in \mathbb{P}_i(X) \rangle + \langle L_H : H \in \bigcup_{j=i+1}^s \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \mathbb{P}_i(X) \rangle + \langle L_H : H \in \bigcup_{j=i+1}^{i+r} \mathbb{P}_j(X) \rangle} \right) \\
&\leq \binom{n}{i} + \binom{n}{i+1} + \cdots + \binom{n}{i+r-1} + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X) \rangle} \right) \\
&\quad + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i+1}^s \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i+1}^{i+r} \mathbb{P}_j(X) \rangle} \right) \quad \text{by fact 1 above} \\
&\leq \binom{n}{i} + \binom{n}{i+1} + \cdots + \binom{n}{i+r-1} + \binom{n}{i+r} - \binom{n}{i} \quad \text{by (3) above} \\
&\quad + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i+1}^s \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i+1}^{i+r} \mathbb{P}_j(X) \rangle} \right) \\
&= \binom{n}{i+1} + \cdots + \binom{n}{i+r} + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=i+1}^s \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=i+1}^{i+r} \mathbb{P}_j(X) \rangle} \right) \\
&\leq \binom{n}{s-r+1} + \cdots + \binom{n}{s},
\end{aligned}$$

where the last step is by the induction hypothesis since $s-r+1-(i+1)(s-r+1-i)=l$. This completes the proof of the Lemma 4.

Now it is easy to prove Theorem 1. By (1) we have

$$\begin{aligned}
|\mathcal{F}| &\leq \dim \left(\left\langle L_H : H \in \bigcup_{i=0}^s \mathbb{P}_i(X) \right\rangle \right) \\
&\leq \dim \left(\left\langle L_H : H \in \bigcup_{i=0}^{r-1} \mathbb{P}_i(X) \right\rangle \right) \\
&\quad + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=0}^s \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=0}^{r-1} \mathbb{P}_j(X) \rangle} \right) \quad \text{by fact 2 above} \\
&\leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r-1} + \dim \left(\frac{\langle L_H : H \in \bigcup_{j=0}^s \mathbb{P}_j(X) \rangle}{\langle L_H : H \in \bigcup_{j=0}^{r-1} \mathbb{P}_j(X) \rangle} \right) \\
&\leq \binom{n}{s-r+1} + \binom{n}{s-r+2} + \cdots + \binom{n}{s} \quad \text{by taking } i=0 \text{ in Lemma 4,}
\end{aligned}$$

which completes the proof of the theorem. $\square$

References

1. N. Alon, L. Babai, and H. Suzuki, "Multilinear polynomials and Frankl-Ray-Chaudhuri-Wilson type intersection theorems," *J. Combinatorial Theory (A)* **58** (1991), 165–180.
2. P. Frankl, "Intersection theorems and mod p rank inclusion matrices," *J. Combinatorial Theory (A)* **54** (1990), 85–94.
3. P. Frankl and R.M. Wilson, "Intersection theorem with geometric consequences," *Combinatorica* **1**(4) (1981), 357–368.
4. J. Qian and D.K. Ray-Chaudhuri, "Frankl-Füredi type inequalities for polynomial semi-lattices," *Electronic Journal of Combinatorics* **4** (1997), 28.
5. J. Qian and D.K. Ray-Chaudhuri, "Extremal case of Frankl-Ray-Chaudhuri-Wilson inequality," *J. Statist. Plann. Inference*, to be published.
6. G.V. Ramanan, "Proof of a conjecture of Frankl and Füredi," *J. Combin. Theory Ser. A* **79**(1) (1997), 53–67.
7. H.S. Snevily, "On Generalizations of the deBruijn-Erdos Theorem," *JCT-A* **68**(1) (1994), 237–238.