On Near Hexagons and Spreads of Generalized Quadrangles

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Abstract. The glueing-construction described in this paper makes use of two generalized quadrangles with a spread in each of them and yields a partial linear space with special properties. We study the conditions under which glueing will give a near hexagon. These near hexagons satisfy the nice property that every two points at distance 2 are contained in a quad. We characterize the class of the “glued near hexagons” and give examples, some of which are new near hexagons.

Keywords: spread, generalized quadrangle, near polygon

1. Definitions

An *incidence structure* is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ with $\mathcal{P}$ (the point set) a nonempty set and $\mathcal{L}$ (the set of lines) a (possibly empty) set and $\mathbf{I}$ a symmetric incidence relation between those sets. Although the incidence relation is symmetric, we will write, in order not to overload the notation, $\mathbf{I} \subseteq \mathcal{P} \times \mathcal{L}$ or even use “ $\in$ ” as incidence relation. The incidence structures which we will consider here are all finite. If x is a point, then $\Gamma_i(x)$ denotes the set of all points at distance i from x (in the point graph). We will denote $\Gamma(x) = \Gamma_1(x)$.

1. An incidence structure is called a *partial linear space* if the following conditions are satisfied.

- (a) Every line $L \in \mathcal{L}$ is incident with at least two points.
- (b) Two different points are incident with at most one line.

A *linear space* is a partial linear space with the property that every two points are collinear.

- 2. An incidence structure of points and lines is *connected* if its point graph is connected.
- 3. A connected partial linear space is called *degenerate* if there is a point incident with exactly one line.
- 4. A *near polygon* $\mathcal{S}$ is a connected partial linear space satisfying the following conditions.
 - (a) The diameter of the point graph Γ of $\mathcal{S}$ is finite.
 - (b) For every point p and every line L , there is a unique point q on L , nearest to p (nearest with respect to the distance $d(\cdot, \cdot)$ in Γ).

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If d is the diameter of Γ then S is called a near $2d$ -gon. A near 0-gon has only one point and no lines and a near 2-gon consists of one line with a number (≥ 2) of points on it. The near quadrangles are just the generalized quadrangles. A generalized quadrangle (GQ for short) is called degenerate if there is a point incident with exactly one line. The point-line dual of a nondegenerate GQ is again a nondegenerate GQ. If a nondegenerate GQ is neither a grid nor a dual grid, then it must have an order (s, t) .

5. A GQ is called *bad* when it is degenerate or when it is a nonsymmetrical dual grid; otherwise it is called a *good* GQ. If $\mathcal{Q}$ is a good GQ, then every point of it is incident with the same number of lines, this number being denoted by $t_{\mathcal{Q}} + 1$.
6. An *ovoid* of a generalized quadrangle $\mathcal{Q}$ is a set O of points such that every line of $\mathcal{Q}$ is incident with exactly one element of O . If $\mathcal{Q}$ has order (s, t) , then $|O| = 1 + st$. A set of $1 + st$ mutually noncollinear points of $\mathcal{Q}$ is always an ovoid of $\mathcal{Q}$. The dual notion is that of a *spread*. A spread is a set of lines of $\mathcal{Q}$ such that every point is incident with exactly one line of the set. For more details on generalized quadrangles, we refer to [6].
7. The incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is called *affine* or *embedded in the finite affine space* $\mathcal{A}$ if $\mathcal{L}$ is a set of lines of $\mathcal{A}$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$ and the incidence relation is the one induced by that of $\mathcal{A}$. If $\mathcal{A}'$ is the subspace of $\mathcal{A}$ generated by all points of $\mathcal{P}$, then we say that $\mathcal{A}'$ is the *ambient space* of $\mathcal{S}$.

A special type of affine embedding is the so-called *linear representation*. Let Π_{∞} be a projective space of dimension $n \geq 0$ embedded as a hyperplane in the projective space Π and let $\mathcal{K}$ be a nonempty subset of the point set of Π_{∞} . The linear representation $T_n^*(\mathcal{K})$ is the geometry with points the affine points of Π ($=$ the points not belonging to Π_{∞}). The lines of $T_n^*(\mathcal{K})$ are all the lines of Π which intersect Π_{∞} in a (unique) point of $\mathcal{K}$. Incidence is the one derived from Π .

8. If $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$ are two partial linear spaces, then the direct product of $\mathcal{S}_1$ and $\mathcal{S}_2$ is the partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ with $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ and $\mathcal{L} = (\mathcal{P}_1 \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times \mathcal{P}_2)$. The point (x, y) is incident with the line $(a, L) \in \mathcal{P}_1 \times \mathcal{L}_2$ if and only if $x = a$ and $y \mathcal{I}_2 L$ and it is incident with the line $(M, b) \in \mathcal{L}_1 \times \mathcal{P}_2$ if and only if $y = b$ and $x \mathcal{I}_1 M$. We denote $\mathcal{S}$ also with $\mathcal{S}_1 \times \mathcal{S}_2$. Since $\mathcal{S}_1 \times \mathcal{S}_2 \simeq \mathcal{S}_2 \times \mathcal{S}_1$ and $(\mathcal{S}_1 \times \mathcal{S}_2) \times \mathcal{S}_3 \simeq \mathcal{S}_1 \times (\mathcal{S}_2 \times \mathcal{S}_3)$, also the direct product of $k \geq 1$ partial linear spaces $\mathcal{S}_1, \dots, \mathcal{S}_k$ is well-defined. If $\mathcal{S}_i$ ($i \in \{1, 2\}$) is a near $2d_i$ -gon, then one can easily prove that $\mathcal{S}_1 \times \mathcal{S}_2$ is a near $2(d_1 + d_2)$ -gon.
9. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a partial linear space. A set $X \subseteq \mathcal{P}$ is called a *subspace* whenever all the points of a line are in X as soon as two of them are in X . Every such subspace induces a partial linear space $\mathcal{S}_X = (X, \mathcal{L}_X, \mathcal{I}')$ where $\mathcal{L}_X$ is the set of all lines of $\mathcal{L}$ which have all their points in X and $\mathcal{I}'$ is the restriction of $\mathcal{I}$ to $X \times \mathcal{L}_X$. A subspace X is called *geodetically closed* when all points of a shortest path between two points of X are also contained in X . A *quad* is a geodetically closed subset of $\mathcal{P}$ which induces a nondegenerate GQ. Since no confusion will be possible in the sequel, the GQ induced by a quad will also be called a quad. If a quad $\mathcal{Q}$ contains a unique point nearest a fixed point x , then this point is called the *projection* of x on $\mathcal{Q}$.

2. Some theorems

Theorem 2.1 ([7, 8]) *Let x and y be two points of a near polygon at mutual distance 2. If x and y have two common neighbours c and d such that the line xc contains at least three points, then x and y are in a unique (necessarily good) quad.*

Theorem 2.2 *Let S be a near polygon and let x be a point at distance at most 1 from a quad Q , then there exists a unique point x' of Q nearest to x and $d(x, y) = d(x, x') + d(x', y)$ for all points y of Q . Hence, if L is a line of Q , then the unique point of L nearest to x is also the unique point of L nearest to x' .*

Proof: This follows from the fact that Q is geodetically closed. $\square$

Corollary 2.3 *Let Q be a quad of a near polygon S and let x and y be two collinear points of S such that the line xy is disjoint with Q . If x , respectively y , is collinear with $x' \in Q$, respectively $y' \in Q$, then $d(x', y') = 1$.*

Proof: By Theorem 2.2, we have that $2 = d(x', y) = d(x', y') + d(y', y) = 1 + d(x', y')$. $\square$

Theorem 2.4 ([3]) *Let S be a near polygon with the property that every two points at distance 2 are contained in a good quad, then each point of S is incident with the same number of lines.*

Proof: Let x and y be two collinear points. The point x (respectively y) is incident with $t_x + 1$ (respectively $t_y + 1$) lines. Now

$$t_x + 1 = 1 + \sum t_Q = t_y + 1,$$

where the summation ranges over all quads Q through the line xy . Hence x and y are incident with the same number of lines and the result follows by connectedness of S . $\square$

Theorem 2.5 ([3]) *Let S be a near polygon satisfying the following properties:*

- (a) *every two points at distance 2 have at least two common neighbours,*
 - (b) *there are lines incident with a different number of points,*
- then S is the direct product of a number of near polygons, each of which has a constant length for the lines.*

If $S = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a near 2-gon or a good GQ, then $|\Gamma_i(p)|$ ($i \in \{0, 1, 2\}$) is independent of $p \in \mathcal{P}$. We derive a similar property for near hexagons.

Theorem 2.6 *Let $S = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a near hexagon such that every two points at distance 2 are contained in a good quad, then $|\Gamma_i(p)|$ ($i \in \{0, 1, 2, 3\}$) is independent of $p \in \mathcal{P}$.*

Proof: If not all lines of $\mathcal{S}$ are incident with the same number of points, then Theorem 2.5 implies that $\mathcal{S}$ is the direct product of a line with a good GQ. It is straightforward to check that the result is true in this case. Hence we may suppose that all lines are incident with $s + 1$ points. Theorem 2.4 implies then that $\mathcal{S}$ has an order (s, t) . Now, let $p \in \mathcal{P}$ be a fixed point and put $n_i = |\Gamma_i(p)|$. Then $n_0 = 1, n_1 = s(t + 1)$. Let V be the set of quads through p . Counting points in $\Gamma_2(p)$ we find

$$n_2 = s^2 \sum_{x \in V} t_x. \quad (1)$$

Counting edges between $\Gamma_2(p)$ and $\Gamma_3(p)$ we find that

$$n_3(t + 1) = s^3 \sum_{x \in V} t_x(t - t_x). \quad (2)$$

Finally, counting triples (L_1, L_2, Q) where L_1, L_2 are two different lines through p and Q is the quad through L_1 and L_2 , yields

$$t(t + 1) = \sum_{x \in V} t_x(t_x + 1). \quad (3)$$

Eliminating $\sum t_x$ and $\sum t_x^2$, we find that $n_3 = s(n_2 - s^2 t)$. Together with $v = n_0 + n_1 + n_2 + n_3$ this gives

$$n_2 = \frac{v}{s + 1} - 1 + st(s - 1), \quad (4)$$

$$n_3 = s \left(\frac{v}{s + 1} - st - 1 \right). \quad (5)$$

□

Corollary 2.7 *If $\mathcal{S}$ is a near hexagon satisfying the property that every two points at distance 2 are contained in a quad of order (s, t_1) or (s, t_2) , $s \geq 1$ and $1 \leq t_1 < t_2$, then for each $i \in \{1, 2\}$, the number of quads of order (s, t_i) through a point is independent of that point.*

Proof: This follows from Eqs. (1), (3) and (4). □

Remark The previous corollary was proved in [2] in the case that $s = 2, t_1 = 1, t_2 = 2$ by using the same double countings as in the proof of Theorem 2.6.

Theorem 2.8 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a partial linear space of order $(s, t) \neq (s, 0)$ satisfying*

1. *for every point p and every line L not through p , there exists at most one point on L collinear with p ,*
2. *$a = |\Gamma_2(x)|$ is independent of the point $x \in \mathcal{P}$,*
3. *$d(x, L) \leq 2$ for all $x \in \mathcal{P}$ and $L \in \mathcal{L}$,*

then $b = |\Gamma_3(x)|$ is also independent of $x \in \mathcal{P}$ and the following inequalities hold:

- $a \geq s^2t$,
- $b \leq s(a - s^2t)$.

Moreover, $\mathcal{S}$ is a generalized quadrangle if and only if $a = s^2t$ and $\mathcal{S}$ is a near hexagon if and only if $a > s^2t$ and $b = s(a - s^2t)$.

Proof: Clearly $|\Gamma_3(x)| = |\mathcal{P}| - 1 - s(t + 1) - |\Gamma_2(x)|$ is independent of $x \in \mathcal{P}$. Take an arbitrary line L and let r be a point of L . There are a points in $\Gamma_2(r)$, s^2t of these are contained in $\Gamma_1(L)$. Hence $a \geq s^2t$ and $|\Gamma_2(L)| \leq (s + 1)(a - s^2t)$. If $a = s^2t$ then $\Gamma_2(L) = \emptyset$ implies that $\mathcal{S}$ is a generalized quadrangle. So, suppose that $a \neq s^2t$, then $\mathcal{S}$ is a near hexagon if and only if $|\Gamma_2(L)| = (s + 1)(a - s^2t)$. From $|\Gamma_2(L)| = |\mathcal{P}| - (s + 1) - st(s + 1) = a + b - s^2t$, it follows that $b \leq s(a - s^2t)$ and equality appears if and only if $\mathcal{S}$ is a near hexagon. $\square$

3. A possible construction for near hexagons

Let $\mathcal{Q}_i = (\mathcal{P}_i, \mathcal{L}_i, \mathbf{I}_i)$ (for each $i \in \{1, 2\}$) be a GQ of order (s, t_i) , let $S_i = \{L_1^{(i)}, \dots, L_{1+st_i}^{(i)}\} \subset \mathcal{L}_i$ be a spread of $\mathcal{Q}_i$ and let θ be a bijection from $L_1^{(1)}$ to $L_1^{(2)}$ (here we suppose that every line is a subset of the point set).

For every $i \in \{1, 2\}$ and every $j \in \{1, \dots, 1 + st_i\}$, $\Phi_j^{(i)} : \mathcal{P}_i \mapsto L_j^{(i)}$ is defined such that $x \in \mathcal{P}_i$ is mapped to the unique point of $L_j^{(i)}$ nearest to x (in the generalized quadrangle $\mathcal{Q}_i$).

Let $\Gamma(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ (Γ for short if no confusion is possible) be the graph with vertex set $L_1^{(1)} \times S_1 \times S_2$. Two different points $(x, L_i^{(1)}, L_j^{(2)})$ and $(y, L_k^{(1)}, L_l^{(2)})$ are adjacent whenever at least one of the following two conditions are satisfied:

- (1) $j = l$ and $\Phi_i^{(1)}(x), \Phi_k^{(1)}(y)$ are collinear points in $\mathcal{Q}_1$,
- (2) $i = k$ and $\Phi_j^{(2)} \circ \theta(x), \Phi_l^{(2)} \circ \theta(y)$ are collinear points in $\mathcal{Q}_2$.

If $i = k$ and $j = l$, then both (1) and (2) are satisfied. It is clear that $\Gamma(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta) \simeq \Gamma(\mathcal{Q}_2, \mathcal{Q}_1, S_2, S_1, L_1^{(2)}, L_1^{(1)}, \theta^{-1})$. For, $\Delta : (x, L_i^{(1)}, L_j^{(2)}) \mapsto (\theta(x), L_j^{(2)}, L_i^{(1)})$ defines an isomorphism. The definition of Γ is hence symmetric in $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Remark In the sequel, we will not write the symbol “ $\circ$ ” between functions, i.e. with fg we mean the function $f \circ g$.

Lemma 3.1 *Through every two adjacent vertices of Γ , there is a unique maximal clique. This clique has size $s + 1$.*

Proof: Let $a = (x, L_i^{(1)}, L_j^{(2)})$ and $b = (y, L_k^{(1)}, L_l^{(2)})$ be two fixed adjacent vertices; we determine what the common neighbours $(z, L_m^{(1)}, L_n^{(2)})$ look like. If $i = k \neq m$, then $j = n = l$ and $\Phi_i^{(1)}(x) \sim \Phi_m^{(1)}(z) \sim \Phi_i^{(1)}(y)$ implies that $x = y$ and hence $a = b$, a contradiction. Similarly, $j = l \neq n$ is impossible. If $i = k = m$, then $\Phi_j^{(2)}\theta(x) \sim \Phi_n^{(2)}\theta(z) \sim \Phi_l^{(2)}\theta(y)$ implies that $\Phi_n^{(2)}\theta(z)$ is an element of the line of $\mathcal{Q}_2$ through $\Phi_j^{(2)}\theta(x)$ and $\Phi_l^{(2)}\theta(y)$. This

yields $s - 1$ common neighbours of a and b and they are all mutually adjacent. Together with the vertices a and b , they yield a clique of size $s + 1$. A similar reasoning holds in the case $j = l = n$. $\square$

Let $\mathcal{S}(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$ be the partial linear space with points the vertices of Γ and with lines the maximal cliques of Γ . The incidence is the natural one. Again, we will write $\mathcal{S}$ when no confusion is possible.

Definition 3.2 A line L is said to be of *type I*, if there exists a fixed j , such that every point of L is of the form $(x, L_i^{(1)}, L_j^{(2)})$. A line M is said to be of *type II*, if there exists a fixed i , such that every point of M is of the form $(x, L_i^{(1)}, L_j^{(2)})$. Remark that there are lines which are of both types, namely the lines $\{(x, L_i^{(1)}, L_j^{(2)}) \mid x \in L_1^{(1)}\}$, where i and j are fixed. These lines partition the point set of $\mathcal{S}$ (hence they form a spread of $\mathcal{S}$).

Lemma 3.3

- (a) For a fixed $j \in \{1, \dots, 1 + st_2\}$, the set $\{(x, L_i^{(1)}, L_j^{(2)}) \mid x \in L_1^{(1)}, 1 \leq i \leq 1 + st_1\}$ is a quad isomorphic to $\mathcal{Q}_1$.
- (b) For a fixed $i \in \{1, \dots, 1 + st_1\}$, the set $\{(x, L_i^{(1)}, L_j^{(2)}) \mid x \in L_1^{(1)}, 1 \leq j \leq 1 + st_2\}$ is a quad isomorphic to $\mathcal{Q}_2$.

Proof: The isomorphisms are given by $\Delta_1 : (x, L_i^{(1)}, L_j^{(2)}) \mapsto \Phi_i^{(1)}(x)$ for (a) and $\Delta_2 : (x, L_i^{(1)}, L_j^{(2)}) \mapsto \Phi_j^{(2)}\theta(x)$ for (b). $\square$

Definition 3.4

- (1) The previous lemma shows that several GQ's (isomorphic to $\mathcal{Q}_1$ or $\mathcal{Q}_2$) are glued together to form the geometry $\mathcal{S}$. For this reason the above construction is called *glueing* and $\mathcal{S}$ will be called a *glued geometry*.
- (2) A *quad of type I, respectively II* is a quad that arises like in (a), respectively (b) of the previous lemma. The following properties hold then.
 - Every line contained in a quad of type A ($\in \{I, II\}$) is also of type A .
 - Two quads of the same type are equal or disjoint.
 - Two quads of different type meet each other in a line which is of both types.
 - Through every point of $\mathcal{S}$, there is a unique quad of each type.
 - Every line of type A ($\in \{I, II\}$) is contained in a unique quad of type A .

Lemma 3.5 $\mathcal{S}$ has order $(s, t_1 + t_2)$ and satisfies properties 1 and 3 of Theorem 2.8.

Proof: Let p be an arbitrary point of $\mathcal{S}$. The quad of type I (respectively type II) through p contains $t_1 + 1$ (respectively $t_2 + 1$) lines through p and both quads have exactly one line in common. Hence $\mathcal{S}$ has order $(s, t_1 + t_2)$.

Property 1 clearly holds by Lemma 3.1, so let x and M be a point and a line of $\mathcal{S}$, both arbitrarily chosen. Through M , there is a quad $\mathcal{R}_1$ of type $A \in \{I, II\}$. Take the unique quad

$\mathcal{R}_2$ through p of type B such that $\{A, B\} = \{I, II\}$. On the intersection line of $\mathcal{R}_1$ and $\mathcal{R}_2$ there is a unique point nearest to x . This point has distance at most 1 to x and M . This proves the lemma. $\square$

Definition 3.6

- For all $i, j \in \{1, \dots, 1 + st_1\}$, $\phi_{i,j}^{(1)}$ is the permutation of $L_1^{(1)}$ equal to the restriction of $\Phi_1^{(1)}\Phi_j^{(1)}\Phi_i^{(1)}$ to $L_1^{(1)}$. The group of permutations of $L_1^{(1)}$ generated by the elements $\phi_{i,j}^{(1)}$ is denoted by G_1 .
- For all $i, j \in \{1, \dots, 1 + st_2\}$, $\phi_{i,j}^{(2)}$ is the permutation of $L_1^{(2)}$ equal to the restriction of $\Phi_1^{(2)}\Phi_j^{(2)}\Phi_i^{(2)}$ to $L_1^{(2)}$. The group of permutations of $L_1^{(2)}$ generated by the elements $\phi_{i,j}^{(2)}$ is denoted by G_2 .

Remark

- $\phi_{i,i}^{(1)}, \phi_{i,i}^{(2)}$ are identity permutations,
- $\phi_{i,j}^{(k)}$ and $\phi_{j,i}^{(k)}$ ($k \in \{1, 2\}$) are inverse permutations.

Theorem 3.7 $\mathcal{S}$ is a near hexagon if and only if $[G_1, \theta^{-1}G_2\theta] = 0$. (Here 0 stands for the trivial group and $[G_1, \theta^{-1}G_2\theta]$ is the group generated by all commutators $[g_1, \theta^{-1}g_2\theta]$ with $g_1 \in G_1$ and $g_2 \in G_2$.)

Proof: Suppose that $\mathcal{S}$ is a near hexagon. It suffices to prove that $\phi_{i,j}^{(1)}$ commutes with $\theta^{-1}\phi_{k,l}^{(2)}\theta$ for all possible i, j, k, l with $i \neq j$ and $k \neq l$. If $x \in L_1^{(1)}$, then we have the following adjacencies:

$$\begin{aligned}
 & (\Phi_1^{(1)}\Phi_j^{(1)}\Phi_i^{(1)}\theta^{-1}\Phi_1^{(2)}\Phi_l^{(2)}\Phi_k^{(2)}\theta(x), L_j^{(1)}, L_l^{(2)}) \\
 & \sim (\theta^{-1}\Phi_1^{(2)}\Phi_l^{(2)}\Phi_k^{(2)}\theta(x), L_i^{(1)}, L_l^{(2)}) \\
 & \sim (x, L_i^{(1)}, L_k^{(2)}) \\
 & \sim (\Phi_1^{(1)}\Phi_j^{(1)}\Phi_i^{(1)}(x), L_j^{(1)}, L_k^{(2)}) \\
 & \sim (\theta^{-1}\Phi_1^{(2)}\Phi_l^{(2)}\Phi_k^{(2)}\theta\Phi_1^{(1)}\Phi_j^{(1)}\Phi_i^{(1)}(x), L_j^{(1)}, L_l^{(2)}).
 \end{aligned}$$

Let p be the point $(x, L_i^{(1)}, L_k^{(2)})$ and L be the line $\{(x, L_j^{(1)}, L_l^{(2)}) \mid x \in L_1^{(1)}\}$ (this is a line of type I and of type II). Since there is only one point of L at distance 2 from p , it follows that

$$\theta^{-1}\phi_{k,l}^{(2)}\theta\phi_{i,j}^{(1)} = \phi_{i,j}^{(1)}\theta^{-1}\phi_{k,l}^{(2)}\theta.$$

Conversely, suppose that $[G_1, \theta^{-1}G_2\theta]$ is the trivial group. Let x be an arbitrary point of $\mathcal{S}$. Through x , there is a unique quad $\mathcal{R}_1$ of type I and a unique quad $\mathcal{R}_2$ of type II. In $\mathcal{R}_1 \cup \mathcal{R}_2$, there are $s^2(t_1 + t_2)$ points of $\Gamma_2(x)$. The points of $\mathcal{S}$ not in $\mathcal{R}_1 \cup \mathcal{R}_2$ are partitioned

by $s^2 t_1 t_2$ lines which have both types. The previous reasoning shows that each of these lines contains a unique point at distance 2 from x . Hence $a = |\Gamma_2(x)| = s^2(t_1 t_2 + t_1 + t_2)$ is independent of the point x . From this it follows that $b = |\Gamma_3(x)| = (s+1)(st_1+1)(st_2+1) - 1 - |\Gamma_1(x)| - |\Gamma_2(x)| = s^3 t_1 t_2$. Since $a > s^2(t_1 + t_2)$ and $b = s(a - s^2(t_1 + t_2))$, it follows from Theorem 2.8 that $\mathcal{S}$ is a near hexagon. $\square$

Above, we defined $\mathcal{S} = \mathcal{S}(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$. Take now an arbitrary line $L_i^{(1)}$ in S_1 and an arbitrary line $L_j^{(2)}$ in S_2 . If we define $\theta_{i,j}$ as the restriction of $\Phi_j^{(2)} \theta \Phi_1^{(1)}$ to $L_i^{(1)}$, then we can define

$$\mathcal{S}_{i,j} = \mathcal{S}(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_i^{(1)}, L_j^{(2)}, \theta_{i,j}).$$

Theorem 3.8 *If $\mathcal{S}$ is a near hexagon, then $\mathcal{S}_{i,j}$ is isomorphic to $\mathcal{S}$ for all $i \in \{1, \dots, 1+st_1\}$ and all $j \in \{1, \dots, 1+st_2\}$.*

Proof: We prove that $\Delta : L_1^{(1)} \times S_1 \times S_2 \mapsto L_i^{(1)} \times S_1 \times S_2, (x, L_k^{(1)}, L_l^{(2)}) \mapsto (\Phi_i^{(1)} \phi_{k,i}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(x), L_k^{(1)}, L_l^{(2)})$ is an isomorphism between $\mathcal{S}$ and $\mathcal{S}_{i,j}$. This map is clearly a bijection and it suffices to prove that adjacency is preserved in the point graph of the geometries. Consider the two adjacent vertices $a = (x, L_k^{(1)}, L_l^{(2)})$ and $b = (y, L_k^{(1)}, L_m^{(2)})$ in $\mathcal{S}$, then $y = \theta^{-1} \phi_{l,m}^{(2)} \theta(x)$ and

$$\begin{aligned} \Delta(a) &= (\Phi_i^{(1)} \phi_{k,i}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(x), L_k^{(1)}, L_l^{(2)}), \\ \Delta(b) &= (\Phi_i^{(1)} \phi_{k,i}^{(1)} \theta^{-1} \phi_{m,j}^{(2)} \theta(y), L_k^{(1)}, L_m^{(2)}). \end{aligned}$$

Now, $\Delta(a) \sim \Delta(b)$ (in $\mathcal{S}_{i,j}$) if and only if

$$\begin{aligned} \Phi_l^{(2)} \Phi_j^{(2)} \theta \Phi_1^{(1)} \Phi_i^{(1)} \phi_{k,i}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(x) &\sim \Phi_m^{(2)} \Phi_j^{(2)} \theta \Phi_1^{(1)} \Phi_i^{(1)} \phi_{k,i}^{(1)} \theta^{-1} \phi_{m,j}^{(2)} \theta(y) \\ \Phi_l^{(2)} \Phi_j^{(2)} \theta \phi_{k,i}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(x) &\sim \Phi_m^{(2)} \Phi_j^{(2)} \theta \phi_{k,i}^{(1)} \theta^{-1} \phi_{m,j}^{(2)} \theta(y) \\ \Phi_l^{(2)} \Phi_j^{(2)} \phi_{l,j}^{(2)} \theta \phi_{k,i}^{(1)}(x) &\sim \Phi_m^{(2)} \Phi_j^{(2)} \phi_{m,j}^{(2)} \theta \phi_{k,i}^{(1)}(y) \\ \Phi_l^{(2)} \theta \phi_{k,i}^{(1)}(x) &\sim \Phi_m^{(2)} \theta \phi_{k,i}^{(1)}(y) \\ \theta^{-1} \phi_{l,m}^{(2)} \theta \phi_{k,i}^{(1)}(x) &= \phi_{k,i}^{(1)}(y) \\ \theta^{-1} \phi_{l,m}^{(2)} \theta \phi_{k,i}^{(1)}(x) &= \phi_{k,i}^{(1)} \theta^{-1} \phi_{l,m}^{(2)} \theta(x). \end{aligned}$$

Consider the two adjacent vertices $a = (x, L_k^{(1)}, L_l^{(2)})$ and $b = (y, L_m^{(1)}, L_l^{(2)})$ in $\mathcal{S}$, then $y = \phi_{k,m}^{(1)}(x)$ and

$$\begin{aligned} \Delta(a) &= (\Phi_i^{(1)} \phi_{k,i}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(x), L_k^{(1)}, L_l^{(2)}), \\ \Delta(b) &= (\Phi_i^{(1)} \phi_{m,i}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(y), L_m^{(1)}, L_l^{(2)}). \end{aligned}$$

Now, $\Delta(a) \sim \Delta(b)$ (in $\mathcal{S}_{i,j}$) if and only if

$$\begin{aligned}\Phi_k^{(1)} \Phi_i^{(1)} \phi_{k,i}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(x) &\sim \Phi_m^{(1)} \Phi_i^{(1)} \phi_{m,i}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(y) \\ \Phi_k^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(x) &\sim \Phi_m^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(y) \\ \phi_{k,m}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(x) &= \theta^{-1} \phi_{l,j}^{(2)} \theta(y) \\ \phi_{k,m}^{(1)} \theta^{-1} \phi_{l,j}^{(2)} \theta(x) &= \theta^{-1} \phi_{l,j}^{(2)} \theta \phi_{k,m}^{(1)}(x).\end{aligned}$$

□

Theorem 3.9 *If $\mathcal{S}$ is a near hexagon, then any two points at distance 2 are contained in a quad.*

Proof: Let $p_1 = (x, L_i^{(1)}, L_j^{(2)})$ and $p_2 = (y, L_k^{(1)}, L_l^{(2)})$ denote the two points at distance 2. If $i = k$ (respectively $j = l$), then p_1 and p_2 are contained in a quad of type II (respectively I). If $i \neq k$ and $j \neq l$, then the adjacencies of Theorem 3.7 show that p_1 and p_2 have two common neighbours p_3 and p_4 . If $s \geq 2$, then Theorem 2.1 implies that p_1 and p_2 are contained in a quad (which is a $(s+1) \times (s+1)$ -grid in this case). If $s = 1$, then p_1 and p_2 are contained in a quad, since $\{p_1, p_2, p_3, p_4\}$ is geodetically closed and induces a (2×2) -grid. □

Definition 3.10 Suppose $\mathcal{S}$ is a near hexagon. The quads in $\mathcal{S}$, different from the above defined quads of type I and II are called the quads of type III. These quads are $(s+1) \times (s+1)$ -grids.

Remarks

- (a) For $i \in \{1, 2\}$ fixed, let $\mathcal{Q}_i$ be an $(s+1) \times (s+1)$ -grid and S_i be one of the two spreads of $\mathcal{Q}_i$. Since $\phi_{j,k}^{(i)}$ is the identity permutation for all $j, k \in \{1, \dots, 1+s\}$, one has that $[G_1, \theta^{-1}G_2\theta] = 0$, hence $\mathcal{S}$ is a near hexagon. It is straightforward to check that $\mathcal{S}$ is the direct product of $\mathcal{Q}_{3-i}$ with a line of size $s+1$.
- (b) For every $t \in \mathbb{N} \setminus \{0\}$, there is a unique GQ of order $(1, t)$. This GQ contains several spreads which are all equivalent. Since G_2 is a commutative group, the above construction with $s = 1, t_1, t_2 \geq 1$ will yield a thin near hexagon.
- (c) In the next sections we will construct near hexagons using two generalized quadrangles ($\mathcal{Q}_1$ and $\mathcal{Q}_2$) and certain spreads in them (S_1 and S_2 respectively). In the definition of $\mathcal{S}$, we took in each spread two special lines (namely $L_1^{(1)}$ and $L_1^{(2)}$). Theorem 3.8 says (in the case that $\mathcal{S}$ is a near hexagon) that those special lines are in fact not so special. One can obtain the same near hexagon starting with two arbitrary lines (one in each spread) by taking a suitable θ .
- (d) We will not study the problem of determining suitable spreads and suitable maps θ . Also, the above construction can be generalized to obtain other near polygons (e.g. near octagons). These two problems will be considered in forthcoming papers.

4. A new construction for $(T_2^*(O_1), T_2^*(O_2))$

4.1. The generalized quadrangle $T_2^*(O)$

Consider a hyperoval O in $\text{PG}(2, q)$ with q even. Embed $\text{PG}(2, q)$ as a hyperplane in $\text{PG}(3, q)$, then $T_2^*(O)$ is a generalized quadrangle of order $(q-1, q+1)$, see [1, 5, 6]. Let p be a fixed point of O , then the set of lines of $\text{PG}(3, q)$ intersecting O in p defines a spread S of $T_2^*(O)$. Consider now the model of $\text{PG}(3, q)$ where the points are the 1-dimensional subspaces of $V(4, q)$ and let L_1, L_2, L_3 denote three arbitrary (but different) lines of S . The plane $\langle L_i, L_j \rangle$ ($i \neq j$ and $i, j \in \{1, 2, 3\}$) intersects $\text{PG}(2, q)$ in a line through p . Let $\langle \bar{c}_{ij} \rangle$ denote the second point of O on that line. Take $\bar{a}, \bar{b} \in V(4, q)$ such that $p = \langle \bar{a} \rangle$ and $L_1 = \langle \bar{a}, \bar{b} \rangle$ and let $x = \langle \alpha \bar{a} + \bar{b} \rangle$ with $\alpha \in \mathbb{F}_q$ be an arbitrary point of L_1 . The projection (in $T_2^*(O)$) of x on L_2 is equal to $\Phi_2(x) = \langle \alpha \bar{a} + \bar{b} + \beta \bar{c}_{12} \rangle$ where $\beta \in \mathbb{F}_q$ is independent of α . In the same way, we will find that $\Phi_1 \Phi_3 \Phi_2(x) = \langle \alpha \bar{a} + \bar{b} + \beta \bar{c}_{12} + \gamma \bar{c}_{23} + \delta \bar{c}_{31} \rangle$ where γ, δ are independent of α . Now $\beta \bar{c}_{12} + \gamma \bar{c}_{23} + \delta \bar{c}_{31} = \mu \bar{a}$ where μ is independent of α . Hence the map $\phi_{2,3}$ (which is equal to the restriction of $\Phi_1 \Phi_3 \Phi_2$ to L_1) maps the point $\langle \alpha \bar{a} + \bar{b} \rangle$ to $\langle (\alpha + \mu) \bar{a} + \bar{b} \rangle$ where μ is independent of $\alpha \in \mathbb{F}_q$.

4.2. The near hexagon $(T_2^*(O_1), T_2^*(O_2))$

In [4] the following near hexagon was described. Let $\prod_\infty$ be a $\text{PG}(4, q)$, with q even, embedded as a hyperplane in the 5-dimensional space $\prod$. Consider in $\prod_\infty$ two planes α_1 and α_2 meeting each other in a point p and consider in α_i ($i = 1, 2$) a hyperoval O_i containing p . It was proved in [4] that $T_4^*(O_1 \cup O_2)$ is a near hexagon and it was denoted there by $(T_2^*(O_1), T_2^*(O_2))$.

Theorem 4.1 *The near hexagon $(T_2^*(O_1), T_2^*(O_2))$ is glued.*

Proof: Let a be a fixed affine point of $\prod$ and put $A_i = \langle a, \alpha_i \rangle$ ($i \in \{1, 2\}$). For every affine point $x \in \prod$, we define $\mathcal{Q}_i(x)$ ($i \in \{1, 2\}$) as the GQ with the affine points of $\langle x, \alpha_i \rangle$ as points, two points are collinear in the GQ whenever they are collinear in $T_4^*(O_1 \cup O_2)$. These GQ's are quads of $T_4^*(O_1 \cup O_2)$ and each point of $T_4^*(O_1 \cup O_2)$ has distance at most one to each such quad. For $i = \{1, 2\}$, let $\mathcal{Q}_i = \mathcal{Q}_i(a)$, let S_i be the set of lines of A_i intersecting $\prod_\infty$ in p , let $L_1^{(1)} = L_1^{(2)} = pa$ and finally let θ be the identity map. In the previous paragraph we determined what $\phi_{i,j}^{(1)}$ and $\phi_{i,j}^{(2)}$ look like. We can conclude that $[G_1, G_2] = 0$, hence we can define a near hexagon $\mathcal{S} = \mathcal{S}(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, pa, pa, \theta)$. We will construct now an isomorphism Δ between $T_4^*(O_1 \cup O_2)$ and $\mathcal{S}$. Let x be an arbitrary affine point of $\prod$. The quad $\mathcal{Q}_1(x)$ (respectively $\mathcal{Q}_2(x)$) intersects $\mathcal{Q}_2$ (respectively $\mathcal{Q}_1$) in a line $\delta_2(x)$ (respectively $\delta_1(x)$) of S_2 (respectively S_1). We put $\gamma(x)$ equal to the unique point of pa nearest to x (in $T_4^*(O_1 \cup O_2)$). The point of $\mathcal{Q}_i$ nearest to x is then equal to the projection (in $\mathcal{Q}_i$) of $\gamma(x)$ on the line $\delta_i(x) \in S_i$, see Theorem 2.2. If we put $\Delta(x) = (\gamma(x), \delta_1(x), \delta_2(x))$, then we will prove that Δ is an isomorphism. Let $(a, L_1, L_2) = (\gamma(x), \delta_1(x), \delta_2(x))$ and put a_i ($i \in \{1, 2\}$) equal to the projection of a on the line L_i of $\mathcal{Q}_i$. If $L_1 = pa$, then $x = a_2$; if $L_2 = pa$, then $x = a_1$; if $L_1 \neq pa \neq L_2$, then x is the common neighbour

of a_1 and a_2 (in $T_4^*(O_1 \cup O_2)$) different from a . This proves that Δ is a bijection. Since both geometries have the same order, it suffices to prove that Δ preserves adjacency in the point graph of the geometries. Let r and r' be two adjacent points of $T_4^*(O_1 \cup O_2)$. If the line rr' intersects $\prod_{\infty}$ in a point of O_i , then $\delta_{3-i}(r) = \delta_{3-i}(r')$ and the result follows from Corollary 2.3 by considering the projection on the quad Q_i . $\square$

5. New example related to $Q(5, q)$

The generalized quadrangle $Q(5, q)$ is the GQ of the points and the lines of a nonsingular elliptic quadric in $\text{PG}(5, q)$. Its order is (q, q^2) . The corresponding dual generalized quadrangle is the GQ of the points and the lines of a nonsingular Hermitian variety $H(3, q^2)$ in $\text{PG}(3, q^2)$, see [6]. If we intersect this variety with a nontangent plane, then we get a set O of $q^3 + 1$ mutually noncollinear points in $H(3, q^2)$, hence O is an ovoid of $H(3, q^2)$. This ovoid O dualizes to a spread S of $Q(5, q)$.

Take now $Q = Q(5, q)$ and let L be an arbitrary line of S . The following theorem holds then (1_L denotes the identity permutation of the set of points of L).

Theorem 5.1 $S = S(Q, Q, S, S, L, L, 1_L)$ is a near hexagon.

Proof: We determine the permutations $\phi_{i,j}^{(1)} = \phi_{i,j}^{(2)}$ while reasoning in the dual GQ. The points of $H(3, q^2)$ are 1-dimensional subspaces of $V(4, q^2)$. Consider a nonsingular Hermitian form $(\cdot, \cdot)$ in $V(4, q^2)$, i.e. $(\sum_i \lambda_i v_i, \sum_j \mu_j w_j) = \sum_i \sum_j \lambda_i \mu_j^q (v_i, w_j)$, and let ζ be the corresponding polarity of $\text{PG}(3, q^2)$. Take now a nontangent plane π and let $\pi^\zeta = \langle \bar{u} \rangle$. Take three arbitrary (but different) points $\langle \bar{a} \rangle, \langle \bar{b} \rangle, \langle \bar{c} \rangle$ of $O = \pi \cap H(3, q^2)$. The tangent plane at $\langle \bar{a} \rangle$ intersects π in a line $\langle \bar{a}, \bar{v} \rangle$. Let $L = \langle \bar{a}, \bar{u} + \lambda \bar{v} \rangle$ be an arbitrary line of $H(3, q^2)$ through $\langle \bar{a} \rangle$. Since $(\bar{u} + \lambda \bar{v}, \bar{u} + \lambda \bar{v}) = 0$, one finds that $\lambda^{q+1} = -\frac{(\bar{u}, \bar{u})}{(\bar{v}, \bar{v})}$. We determine the line L' of $H(3, q^2)$ through $\langle \bar{b} \rangle$ intersecting L . This line looks like $\langle \bar{b}, \bar{u} + \lambda \bar{v} + \beta \bar{a} \rangle$. An easy calculation yields $\beta = -\lambda \frac{(\bar{v}, \bar{b})}{(\bar{a}, \bar{b})}$. Hence $L' = \langle \bar{b}, \bar{u} + \lambda \bar{v}' \rangle$ with $\langle \bar{v}' \rangle \in \pi \cap \langle \bar{b} \rangle^\zeta$ independent of λ . Similarly, if we project L' to a line L'' through $\langle \bar{c} \rangle$ and finally L'' to a line L''' through $\langle \bar{a} \rangle$, we will find that $L''' = \langle \bar{a}, \bar{u} + \lambda \bar{v}''' \rangle$ with $\langle \bar{v}''' \rangle \in \pi \cap \langle \bar{a} \rangle^\zeta$ independent of λ . Now $\bar{v}''' = \gamma_1 \bar{a} + \gamma_2 \bar{v}$, hence $L''' = \langle \bar{a}, \bar{u} + \lambda \gamma_2 \bar{v} \rangle$ where γ_2 is independent of λ . Just like before, one has that $(\lambda \gamma_2)^{q+1} = -\frac{(\bar{u}, \bar{u})}{(\bar{v}, \bar{v})}$ or $\gamma_2^{q+1} = 1$. It is now clear that $[G_1, G_2] = 0$, hence S is a near hexagon. $\square$

6. New example related to $AS(q)$

For every odd prime power q , there exists a generalized quadrangle of order $(q-1, q+1)$ denoted by $AS(q)$, see [1, 6]. The points of $AS(q)$ are the points of the affine space $\text{AG}(3, q)$. The lines of $AS(q)$ are the following curves of $\text{AG}(3, q)$:

- (1) $x = \sigma, \quad y = a, \quad z = b;$
- (2) $x = a, \quad y = \sigma, \quad z = b;$
- (3) $x = c\sigma^2 - b\sigma + a, \quad y = -2c\sigma + b, \quad z = \sigma.$

Here, the parameter σ ranges over $\text{GF}(q)$ and a, b, c are arbitrary elements of $\text{GF}(q)$. The incidence is the natural one. The set S which consists of all lines of type (1) is a spread of $AS(q)$. If L is an arbitrary line of S , then we have the following theorem.

Theorem 6.1 $\mathcal{S} = \mathcal{S}(AS(q), AS(q), S, S, L, L, 1_L)$ is a near hexagon.

Proof: Let $a, b, c, d \in \text{GF}(q)$ be fixed. Consider then the lines $M = \{(\sigma, a, b) \mid \sigma \in \text{GF}(q)\}$ and $N = \{(\sigma, c, d) \mid \sigma \in \text{GF}(q)\}$. Let $p = (\alpha, a, b)$ be an arbitrary point of M and let $p' = (\beta, c, d)$ be its projection on N . If $b = d$, then $\beta = \alpha$ and there is a line of type (2) through p and p' . If $b \neq d$, then the line through p and p' must necessarily be of type (3). Let $x = m\sigma^2 - l\sigma + k$, $y = -2m\sigma + l$, $z = \sigma$ be that line. Then we get the following equations:

$$\begin{aligned}\alpha &= mb^2 - lb + k, \\ \beta &= md^2 - ld + k, \\ a &= -2mb + l, \\ c &= -2md + l.\end{aligned}$$

Since $b \neq d$, m and l are completely determined by a, b, c and d . We have that $\beta = \alpha + m(d^2 - b^2) + l(b - d)$.

It is now clear that the maps $\phi_{i,j}^{(1)} = \phi_{i,j}^{(2)}$ are translations of the line L . This proves that $[G_1, G_2] = 0$, hence $\mathcal{S}$ is a near hexagon. $\square$

Remark All the near hexagons with lines of size 3 and quads through every two points at distance 2 were classified in [2]. The near hexagons derived here from $AS(3)$ and $Q(5, 2)$ are both isomorphic to example (vi) of [2]. (Notice that $AS(3) \simeq Q(5, 2)$.)

7. Characterizations

7.1. The local space

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \text{I})$ be a near hexagon satisfying the property that every two points at distance 2 are contained in a unique quad. For $x \in P$, we define the following incidence structure $\mathcal{S}_x$.

- The points of $\mathcal{S}_x$ are the lines of $\mathcal{S}$ through x .
- A line of $\mathcal{S}_x$ is the set of lines of $\mathcal{S}$ through x in a quad on x .
- Incidence is the symmetrized containment.

The space $\mathcal{S}_x$ is linear and is called *the local space at x* .

For $u, v \in \mathbb{N} \setminus \{0\}$, let $\mathcal{S}_{u,v} = (\mathcal{P}_{u,v}, \mathcal{L}_{u,v}, \text{I}_{u,v})$ be the following linear space:

- $\mathcal{P}_{u,v} = \{\alpha, \beta_1, \dots, \beta_u, \gamma_1, \dots, \gamma_v\}$,
- $\mathcal{L}_{u,v} = \{\{\alpha, \beta_1, \dots, \beta_u\}, \{\alpha, \gamma_1, \dots, \gamma_v\}\} \cup \{\{\beta_i, \gamma_j\} \mid 1 \leq i \leq u \text{ and } 1 \leq j \leq v\}$,
- $\text{I}_{u,v}$ is the symmetrized containment.

$\mathcal{S}_{u,v}$ is a linear space with a thin point (namely α). Conversely, every linear space with a thin point is obtained in this way. If $\mathcal{S}$ is a glued near hexagon, then $\mathcal{S}_x \simeq \mathcal{S}_{t_1,t_2}$ for all points x of $\mathcal{S}$.

Theorem 7.1 *Let $\mathcal{S}$ be a near hexagon satisfying the following properties:*

- every two points at distance 2 are contained in a quad,
- if all lines of $\mathcal{S}$ are thin, then all quads are good,
- there exists a point x of $\mathcal{S}$ such that $\mathcal{S}_x \simeq \mathcal{S}_{1,r}$ for some $r \in \mathbb{N} \setminus \{0\}$,

then $\mathcal{S}$ is the direct product of a line with a nondegenerate GQ.

Proof: If not all lines of $\mathcal{S}$ have the same number of points, then $\mathcal{S}$ is the direct product of a line with a GQ, see Theorem 2.5. Hence, by Theorem 2.4, we may assume that $\mathcal{S}$ has order (s, t) with $t = r + 1$. Consider through x a quad R_x containing t lines through x and let L_x be the remaining line through x . Every point z of R_x is incident with exactly one line L_z which is not in R_x . Let $y \in L_x \setminus \{x\}$ be fixed. Let M_1 and M_2 be two lines through y different from L_x and let R_y be the quad through M_1 and M_2 . The quad through M_i ($i \in \{1, 2\}$) and L_x intersects R_x in a line M'_i . Now, let u be one of the $s^2(t-1)$ points of R_x at distance 2 from x . Let u_i ($i \in \{1, 2\}$) be the unique point on M'_i collinear with u . The quad through uu_i and L_{u_i} is a grid. Let u'_i be the intersection of L_{u_i} with M_i and let v_i be the unique neighbour of u'_i and u different from u_i . The point v_i is then the unique point of L_u at distance 2 from y . This implies that $v = v_1 = v_2$. Since v is collinear with the points u'_1 and u'_2 of R_y , v is itself contained in R_y . Hence $|\Gamma_2(y) \cap R_y| \geq s^2(t-1)$. This implies that R_y is a GQ of order $(s, t-1)$ containing all lines through y , except the line L_x and that $R_y \simeq R_x$. The result follows now immediately. $\square$

7.2. Characterizations of the new class of near polygons

Theorem 7.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a near hexagon satisfying the following properties:*

- every two points at distance 2 are contained in a quad,
- if all lines of $\mathcal{S}$ are thin, then all quads are good,
- there exists a point x such that $\mathcal{S}_y$ has a thin point for all $y \in \Gamma(x)$,

then $\mathcal{S}$ is the direct product of a line with a nondegenerate GQ or $\mathcal{S}$ is a glued near hexagon.

Proof: If not all lines of $\mathcal{S}$ have the same number of points, then $\mathcal{S}$ is the direct product of a line with a nondegenerate GQ. Hence, by Theorem 2.4 we may assume that $\mathcal{S}$ has an order (s, t) . If $\mathcal{S}_y$ (with $y \in \mathcal{P}$) is a linear space with a thin point, then we may suppose that $\mathcal{S}_y$ contains a unique thin point which we denote by L_y , otherwise the result would follow from Theorem 7.1. The line L_y is then contained in exactly two quads. The following properties hold now.

- (a) If y is a point for which $\mathcal{S}_y$ is a linear space with a thin point, then $\mathcal{S}_{y'} \simeq \mathcal{S}_y$ and $L_{y'} = L_y$ for all points $y' \in L_y$.

Proof: Suppose $\mathcal{S}_y \simeq \mathcal{S}_{t_1,t_2}$ with $t_1, t_2 > 1$ and $t = t_1 + t_2$. The point L_y of $\mathcal{S}_{y'}$ is contained in exactly two lines of $\mathcal{S}_{y'}$, one line has $t_1 + 1$ points, the other $t_2 + 1$ points. Since there are exactly $t_1 + t_2 + 1$ points in $\mathcal{S}_{y'}$, it follows that $\mathcal{S}_{y'} \simeq \mathcal{S}_{t_1,t_2}$. $\square$

- (b) If y_1, y_2 are points such that $\mathcal{S}_{y_1}, \mathcal{S}_{y_2}$ are linear spaces with a thin point, then L_{y_1} and L_{y_2} are equal or disjoint.

Proof: This follows immediately from (a). $\square$

- (c) There exists a point $y \in \Gamma(x)$ such that $x \in L_y$.

Proof: Suppose that this is not true. Let $y \in \Gamma(x)$ be fixed. Let $\mathcal{Q}$ be the quad of order (s, t') through xy and L_y . There are $s(t' + 1)$ points $z_i \in \mathcal{Q}$ collinear with x . These give rise to $s(t' + 1)$ lines L_{z_i} and all these lines are different and hence disjoint by (b). Suppose that L_z is not contained in $\mathcal{Q}$ for a certain $z \in \Gamma(x) \cap \mathcal{Q}$, then $\mathcal{S}_z$ contains at least three thick lines (namely the line defined by $\mathcal{Q}$ and the two lines of $\mathcal{S}_z$ through L_z), a contradiction since $\mathcal{S}_z$ is a linear space with a unique thin point. Hence, all lines L_z are contained in $\mathcal{Q}$ and there are at least $(s + 1)(st' + s)$ points in $\mathcal{Q}$, but this is again impossible. $\square$

Let $y \in \Gamma(x)$ such that $x \in L_y$. Hence $\mathcal{S}_x$ is also a linear space with a unique thin point L_x . Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the two quads through L_x with respective orders (s, t_1) and (s, t_2) . In $\mathcal{Q}_i$, there are st_i points z collinear with x and not on L_x . These give rise to st_i disjoint lines L_z which together with L_x form a spread S_i of $\mathcal{Q}_i$. Put $S_i = \{L_1^{(i)}, \dots, L_{1+st_i}^{(i)}\}$ with $L_1^{(i)} = L_x$. Finally, let θ be the identity permutation of L_x . We prove now that $\mathcal{S} \simeq \mathcal{S}(\mathcal{Q}_1, \mathcal{Q}_2, S_1, S_2, L_1^{(1)}, L_1^{(2)}, \theta)$.

First we prove that every point u of $\mathcal{S}$ has distance at most 1 to each $\mathcal{Q}_i$ ($i \in \{1, 2\}$). Let u' be the unique point of L_x nearest to u ; we may suppose that $d(u, u') = 2$. Since $\mathcal{S}_{u'} \simeq \mathcal{S}_x$, it follows that the quad through u and u' intersects each $\mathcal{Q}_i$ in a line. This proves that each $\mathcal{Q}_i$ contains a point collinear with u . For $i \in \{1, 2\}$ and $u \in \mathcal{P}$, let $p_i(u)$ denote the unique point of $\mathcal{Q}_i$ nearest to u .

Next we prove that all the local spaces $\mathcal{S}_u$ are isomorphic to $\mathcal{S}_{t_1, t_2}$. Since for all $u \in \mathcal{Q}_i$, L_u is contained in exactly two quads ($\mathcal{Q}_i$ and another quad), we have that $G_u \simeq \mathcal{S}_{t_1, t_2}$. Let u be a point of $\mathcal{S}$ not contained in $\mathcal{Q}_1 \cup \mathcal{Q}_2$. Let $u' = p_1(u)$ and $u'' = p_2(u)$. The local space $\mathcal{S}_u$ contains $t_1 + t_2 + 1$ points, a line with $t_1 + 1$ points (determined by the quad through uu'' and $L_{u''}$) and a line with $t_2 + 1$ points (determined by the quad through uu' and $L_{u'}$). From this it follows that $\mathcal{S}_u \simeq \mathcal{S}_{t_1, t_2}$. Hence L_u is defined for all $u \in \mathcal{P}$ and all these lines determine a spread of $\mathcal{S}$. Each L_u is contained in exactly two quads. One quad intersects $\mathcal{Q}_2$ in a line and is isomorphic to $\mathcal{Q}_1$. The other quad intersects $\mathcal{Q}_1$ and is isomorphic to $\mathcal{Q}_2$. Note that the isomorphisms are defined by the projections $p_i, i \in \{1, 2\}$.

We consider now the following map $\Delta : \mathcal{P} \mapsto L_x \times S_1 \times S_2$, $\Delta(u) = (\gamma(u), \delta_1(u), \delta_2(u))$, where $\gamma(u)$ is the unique point of L_x nearest to u and $\delta_i(u)$ ($i \in \{1, 2\}$) is the unique line of S_i incident with $p_i(u)$. By Theorem 2.2, it follows that $p_i(u)$ is the projection (in $\mathcal{Q}_i$) of $\gamma(u)$ on the line $\delta_i(u)$. Let $(a, L_1, L_2) = (\gamma(u), \delta_1(u), \delta_2(u))$ and put a_i ($i \in \{1, 2\}$) equal to the projection of a on the line L_i of $\mathcal{Q}_i$. If $L_1 = L_x$, then $u = a_2$; if $L_2 = L_x$, then $u = a_1$; if $L_1 \neq L_x \neq L_2$, then u is the common neighbour of a_1 and a_2 different from a . This proves that Δ is a bijection. Since both geometries have the same order, it suffices to prove that Δ preserves adjacency in the point graph of the geometries. Let x and x' be two adjacent points. If x and x' are contained in a quad intersecting $\mathcal{Q}_2$, then

$\delta_2(x) = \delta_2(x')$ and the result follows from Corollary 2.3 by projection on the quad $\mathcal{Q}_1$. If x and x' are contained in a quad intersecting $\mathcal{Q}_1$, then $\delta_1(x) = \delta_1(x')$ and the result follows from Corollary 2.3 by projection on the quad $\mathcal{Q}_2$. $\square$

Theorem 7.3 *Let $\mathcal{S}$ be a near hexagon satisfying the following properties:*

- *every two points at distance 2 are contained in a quad,*
- *if all lines of $\mathcal{S}$ are thin, then all quads are good,*
- *there exists a point x such that $\mathcal{S}_x$ has a thin point and such that $\mathcal{S}_y$ contains the same number of lines for all $y \in \Gamma(x)$,*

then $\mathcal{S}$ is the direct product of a line with a nondegenerate GQ or $\mathcal{S}$ is a glued near hexagon.

Proof: Just like before, we may suppose that $\mathcal{S}$ has an order (s, t) . Theorem 2.6 implies that the number of points in $\Gamma_2(y)$ is independent of the point y of $\mathcal{S}$. For $y \in \Gamma(x)$, let V_y denote the set of quads through y . Now,

$$\sum_{\mathcal{Q} \in V_y} 1, \quad \sum_{\mathcal{Q} \in V_y} s^2 t_{\mathcal{Q}}, \quad \sum_{\mathcal{Q} \in V_y} t_{\mathcal{Q}}(t_{\mathcal{Q}} + 1),$$

are respectively equal to the number of quads through y , the number of points in $\Gamma_2(y)$ and $t(t+1)$, hence these quantities are independent of $y \in \Gamma(x)$. Let L_x be a thin point of $\mathcal{S}_x$ and let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the two quads through L_x with respective orders (s, t_1) and (s, t_2) . One has that $t = t_1 + t_2$. Let $z \neq x$ be a second point of L_x . If $y \in \mathcal{Q}_1 \cap \Gamma(x)$, then

$$\sum_{\mathcal{Q} \in V_y} (t_{\mathcal{Q}} - 1)(t_2 - t_{\mathcal{Q}}) = \sum_{\mathcal{Q} \in V_z} (t_{\mathcal{Q}} - 1)(t_2 - t_{\mathcal{Q}}) = (t_1 - 1)(t_2 - t_1).$$

Let $V'_y = V_y \setminus \{\mathcal{Q}_1\}$, then

$$\sum_{\mathcal{Q} \in V'_y} (t_{\mathcal{Q}} - 1)(t_2 - t_{\mathcal{Q}}) = 0.$$

Since there are only $t+1$ lines through y and $\mathcal{Q}_1$ has t_1+1 lines through y , one has that $1 \leq t_{\mathcal{Q}} \leq t_2$ for all $\mathcal{Q} \in V'_y$. This implies that $t_{\mathcal{Q}} = 1$ or $t_{\mathcal{Q}} = t_2$ for all $\mathcal{Q} \in V'_y$. By Theorem 7.1, we may suppose that $t_1, t_2 \neq 1$. From

$$\sum_{\mathcal{Q} \in V_y} 1 = \sum_{\mathcal{Q} \in V_z} 1,$$

and

$$\sum_{\mathcal{Q} \in V_y} t_{\mathcal{Q}} = \sum_{\mathcal{Q} \in V_z} t_{\mathcal{Q}},$$

it follows now that the number of quads $\mathcal{Q}$ of V'_y with $t_{\mathcal{Q}} = t_2$ is equal to 1. This implies that $\mathcal{S}_y \simeq \mathcal{S}_{t_1, t_2}$ for all $y \in \Gamma(x) \cap \mathcal{Q}_1$. A similar reasoning shows that this is also true for $y \in \Gamma(x) \cap \mathcal{Q}_2$. The result follows now from the previous theorem. $\square$

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