A Note on Thin P-Polynomial and Dual-Thin Q-Polynomial Symmetric Association Schemes

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Abstract. Let Y denote a d -class symmetric association scheme, with $d \geq 3$. We show the following: If Y admits a P-polynomial structure with intersection numbers p_{ij}^h and Y is 1 -thin with respect to at least one vertex, then

$$p_{11}^1 = 0 \Rightarrow p_{1i}^i = 0 \quad 1 \leq i \leq d-1.$$

If Y admits a Q-polynomial structure with Krein parameters q_{ij}^h , and Y is *dual 1-thin* with respect to at least one vertex, then

$$q_{11}^1 = 0 \Rightarrow q_{1i}^i = 0 \quad 1 \leq i \leq d-1.$$

Keywords: Association scheme, distance-regular graph, intersection number, Q-polynomial

1. Introduction

Let Y denote a d -class symmetric association scheme, with $d \geq 3$. It is well-known that if Y admits a P-polynomial structure with intersection numbers p_{ij}^h , then

$$p_{11}^1 \neq 0 \Rightarrow p_{1i}^i \neq 0 \quad 1 \leq i \leq d-1 \quad (1)$$

[1, Theorem 5.5.1]. The first author shows in [3] that if Y admits a Q-polynomial structure with Krein parameters q_{ij}^h , then

$$q_{11}^1 \neq 0 \Rightarrow q_{1i}^i \neq 0 \quad 1 \leq i \leq d-1. \quad (2)$$

In the present paper we show the following: If Y admits a P-polynomial structure with intersection numbers p_{ij}^h , and Y is 1 -thin with respect to at least one vertex, then

$$p_{11}^1 = 0 \Rightarrow p_{1i}^i = 0 \quad 1 \leq i \leq d-1. \quad (3)$$

If Y admits a Q-polynomial structure with Krein parameters q_{ij}^h , and Y is *dual 1-thin* with respect to at least one vertex, then

$$q_{11}^1 = 0 \Rightarrow q_{1i}^i = 0 \quad 1 \leq i \leq d-1. \quad (4)$$

The *1-thin* and *dual 1-thin* conditions are defined in Section 1.4. Our main results are in Theorems 2.1 and 2.2.

In the following sections we introduce notation and recall basic results, following [1, Section 2.1] and [4, Section 3].

1.1. Symmetric association schemes

By a *d-class symmetric association scheme* we mean a pair $Y = (X, \{R_i\}_{0 \leq i \leq d})$, where X is a non-empty finite set, and where

- (i) $\{R_i\}_{0 \leq i \leq d}$ is a partition of $X \times X$;
- (ii) $R_0 = \{xx \mid x \in X\}$;
- (iii) $R_i = R_i^t$ for $0 \leq i \leq d$, where $R_i^t = \{yx \mid xy \in R_i\}$;
- (iv) there exist integers p_{ij}^h such that for all integers h with $0 \leq h \leq d$ and all vertices $x, y \in X$ with $xy \in R_h$,

$$p_{ij}^h = |\{z \in X \mid xz \in R_i, yz \in R_j\}| \quad 0 \leq i, j \leq d. \quad (5)$$

We refer to X as the *vertex set* of Y , and refer to the integers p_{ij}^h as the *intersection numbers* of Y . Abbreviate $k_i = p_{ii}^0$, and observe k_i is non-zero for $0 \leq i \leq d$.

1.2. The Bose-Mesner algebra

Let $Y = (X, \{R_i\}_{0 \leq i \leq d})$ denote a symmetric association scheme. Let $\text{Mat}_X(\mathbb{R})$ denote the algebra of matrices over $\mathbb{R}$ with rows and columns indexed by X . The *associate matrices* for Y are the matrices $A_0, \dots, A_d \in \text{Mat}_X(\mathbb{R})$ defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{otherwise} \end{cases} \quad x, y \in X. \quad (6)$$

From (i)–(iv) above we obtain

$$A_0 + \dots + A_d = J, \quad (7)$$

$$A_i \circ A_j = \delta_{ij} A_i \quad 0 \leq i, j \leq d, \quad (8)$$

$$A_0 = I, \quad (9)$$

$$A_i = A_i^t \quad 0 \leq i \leq d, \quad (10)$$

$$A_i A_j = \sum_{h=0}^d p_{ij}^h A_h \quad 0 \leq i, j \leq d, \quad (11)$$

where J is the all-1s matrix and $\circ$ denotes the entry-wise matrix product.

By the *Bose-Mesner algebra* of Y we mean the subalgebra M of $\text{Mat}_X(\mathbb{R})$ generated by the associate matrices $A_0, \dots, A_d$. Observe by (8) and (11) that the associate matrices form a basis for M . In particular, M is symmetric and closed under $\circ$.

The algebra M has a second basis $E_0, \dots, E_d$ such that

$$E_0 + \dots + E_d = I, \quad (12)$$

$$E_i E_j = \delta_{ij} E_i \quad 0 \leq i, j \leq d, \quad (13)$$

$$E_0 = \frac{1}{|X|} J, \quad (14)$$

$$E_i = E_i^t \quad 0 \leq i \leq d, \quad (15)$$

[1, Theorem 2.6.1]. We refer to $E_0, \dots, E_d$ as the *primitive idempotents* of Y . Since M is closed under $\circ$, there exist real numbers q_{ij}^h satisfying

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^d q_{ij}^h E_h \quad 0 \leq i, j \leq d. \quad (16)$$

The numbers q_{ij}^h are the *Krein parameters* for Y . Abbreviate $k_i^* = q_{ii}^0$ for $0 \leq i \leq d$.

By (8), (9), and the fact that $A_0, \dots, A_d$ is a basis for M , the primitive idempotents have constant diagonal; in fact

$$(E_i)_{xx} = \frac{k_i^*}{|X|} \quad 0 \leq i \leq d, \quad x \in X \quad (17)$$

and $k_i^* \neq 0$ [1, p. 45]. We apply (17) in the proof of Lemma 4.1.

1.3. The dual Bose-Mesner algebra

Let Y denote a d -class symmetric association scheme with vertex set X , associate matrices $A_0, \dots, A_d$, primitive idempotents $E_0, \dots, E_d$, and Bose-Mesner algebra M . Fix a vertex $x \in X$.

For each integer i with $0 \leq i \leq d$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{R})$ defined by

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad y \in X. \quad (18)$$

We refer to $A_0^*, \dots, A_d^*$ as the *dual associate matrices* for Y with respect to x . Let $M^* = M^*(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{R})$ generated by the dual associate matrices. We refer to M^* as the *dual Bose-Mesner algebra* for Y with respect to x . From (16) we obtain

$$A_i^* A_j^* = \sum_{h=0}^d q_{ij}^h A_h^* \quad 0 \leq i, j \leq d. \quad (19)$$

In particular, the dual associate matrices form a basis for M^* .

For each integer i with $0 \leq i \leq d$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{R})$ defined by

$$(E_i^*)_{yy} = (A_i)_{xy} \quad y \in X. \quad (20)$$

From (7), (8) we obtain

$$E_0^* + \cdots + E_d^* = I, \quad (21)$$

$$E_i^* E_j^* = \delta_{ij} E_i^* \quad 0 \leq i, j \leq d. \quad (22)$$

We refer to $E_0^*, \dots, E_d^*$ as the *dual idempotents* for Y with respect to x . Note that the dual idempotents form a second basis for M^* .

1.4. The thin and dual-thin conditions

Let Y denote a d -class symmetric association scheme with vertex set X . Fix a vertex $x \in X$, and write $M^* = M^*(x)$.

Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{R})$ generated by M and M^* . We refer to T as the *subconstituent algebra* for Y with respect to x . By a T -module we mean a subspace of the standard module $V = \mathbb{R}^X$ which is closed under multiplication by T . A T -module is said to be *irreducible* if it properly contains no T -modules other than 0. Recall that T is semi-simple, so that V may be decomposed as a direct sum of irreducible T -modules [4, Lemma 3.4].

An irreducible T -module W is said to be *thin* if

$$\dim E_i^* W \leq 1 \quad 0 \leq i \leq d, \quad (23)$$

and *dual thin* if

$$\dim E_i W \leq 1 \quad 0 \leq i \leq d. \quad (24)$$

We say Y is *i-thin* with respect to x if every irreducible T -module W with $E_i^* W \neq 0$ is thin. We say Y is *dual i-thin* with respect to x if every irreducible T -module W with $E_i W \neq 0$ is dual thin.

1.5. P - and Q -polynomial structures

Let Y denote a d -class symmetric association scheme, with vertex set X , intersection numbers p_{ij}^h , and Krein parameters q_{ij}^h . We say that an ordering $A_0, \dots, A_d$ of the associate matrices is a *P -polynomial structure* for Y whenever

$$p_{ij}^h = 0 \quad \text{if one of } h, i, j \text{ is greater than the sum of the other two,} \quad (25)$$

$$p_{ij}^h \neq 0 \quad \text{if one of } h, i, j \text{ is equal to the sum of the other two} \quad (26)$$

for $0 \leq h, i, j \leq d$. Recall that if $A_0, \dots, A_d$ is a P-polynomial structure for Y , then A_1 generates M [4, Lemma 3.8].

We say that an ordering $E_0, \dots, E_d$ of the primitive idempotents is a *Q-polynomial structure* for Y whenever

$$q_{ij}^h = 0 \quad \text{if one of } h, i, j \text{ is greater than the sum of the other two,} \quad (27)$$

$$q_{ij}^h \neq 0 \quad \text{if one of } h, i, j \text{ is equal to the sum of the other two} \quad (28)$$

for $0 \leq h, i, j \leq d$. Recall that if $E_0, \dots, E_d$ is a Q-polynomial structure for Y , then for each $x \in X$ the dual associate matrix $A_1^*(x)$ generates $M^*(x)$ [4, Lemma 3.11].

2. Results

Our main results are the following:

Theorem 2.1 *Let Y denote a d -class symmetric association scheme, with $d \geq 3$. Suppose $A_0, \dots, A_d$ is a P-polynomial structure for Y with intersection numbers p_{ij}^h , and suppose Y is 1-thin with respect to at least one vertex. Then*

$$p_{11}^1 = 0 \Rightarrow p_{1i}^i = 0 \quad 1 \leq i \leq d-1. \quad (29)$$

We prove Theorem 2.1 in Section 3.

Theorem 2.2 *Let Y denote a d -class symmetric association scheme, with $d \geq 3$. Suppose $E_0, \dots, E_d$ is a Q-polynomial structure for Y with Krein parameters q_{ij}^h , and suppose Y is dual 1-thin with respect to at least one vertex. Then*

$$q_{11}^1 = 0 \Rightarrow q_{1i}^i = 0 \quad 1 \leq i \leq d-1. \quad (30)$$

We prove Theorem 2.2 in Section 4.

3. Proof of Theorem 2.1

Define a symmetric bilinear form on $\text{Mat}_X(\mathbb{R})$ (where X is any set) by

$$\langle B, C \rangle = \text{tr}(B^t C) \quad B, C \in \text{Mat}_X(\mathbb{R}). \quad (31)$$

Observe that $\langle B, C \rangle$ is just the sum of the entries of $B \circ C$. In particular, the form is positive definite.

Lemma 3.1 (Terwilliger [4]) *Let $Y = (X, \{R_i\}_{0 \leq i \leq d})$ denote a symmetric association scheme with associate matrices $A_0, \dots, A_d$ and intersection numbers p_{ij}^h . Fix a vertex $x \in X$, and write $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Then:*

(i) for $0 \leq h, h', i, i', j, j' \leq d$,

$$\langle E_i^* A_h E_j^*, E_{i'}^* A_{h'} E_{j'}^* \rangle = \delta_{hh'} \delta_{ii'} \delta_{jj'} k_h p_{ij}^h; \quad (32)$$

(ii) for $0 \leq h, i, j \leq d$,

$$E_h^* A_i E_j^* = 0 \Leftrightarrow p_{ij}^h = 0. \quad (33)$$

Proof of (i): Observe

$$(E_i^* A_h E_j^*)_{yz} = (E_i^*)_{yy} (A_h)_{yz} (E_j^*)_{zz} \quad (34)$$

$$= (A_i)_{xy} (A_h)_{yz} (A_j)_{xz}, \quad (35)$$

so that $(E_i^* A_h E_j^*)_{yz} \neq 0$ if and only if $xy \in R_i$, $yz \in R_h$, and $xz \in R_j$. Since the relations $R_0, \dots, R_d$ are disjoint, the matrices $E_i^* A_h E_j^*$ and $E_{i'}^* A_{h'} E_{j'}^*$ have no non-zero entries in common unless $h = h', i = i', j = j'$. In this case there are precisely $k_h p_{ij}^h$ non-zero entries, each equal to 1. The result follows.

Proof of (ii): Immediate from (i). □

Let Y denote a d -class symmetric association scheme, with vertex set X . Suppose $A_0, \dots, A_d$ is a P-polynomial structure for Y , with intersection numbers p_{ij}^h . Fix a vertex $x \in X$, and write $T = T(x)$, $M^* = M^*(x)$, and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$.

There are three matrices in T which are of particular interest to us (their duals will be used in Section 4). These are the *lowering* matrix $L = L(x)$, the *flat* matrix $F = F(x)$, and the *raising* matrix $R = R(x)$, defined by

$$L = \sum_{i=1}^d E_{i-1}^* A_1 E_i^*, \quad (36)$$

$$F = \sum_{i=0}^d E_i^* A_1 E_i^*, \quad (37)$$

$$R = \sum_{i=0}^{d-1} E_{i+1}^* A_1 E_i^*. \quad (38)$$

It is easily shown using (25), (21), and (33) that

$$A_1 = L + F + R. \quad (39)$$

Recall that A_1 generates the Bose-Mesner algebra M , so that A_1 and $E_0^*, \dots, E_d^*$ generate T . In particular, L, F, R , and $E_0^*, \dots, E_d^*$ generate T by (39).

Lemma 3.2 *Let Y denote a d -class symmetric association scheme, with vertex set X . Suppose $A_0, \dots, A_d$ is a P-polynomial structure for Y , with intersection numbers p_{ij}^h . Fix*

a vertex $x \in X$, and write $T = T(x)$, $L = L(x)$, and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. If Y is l -thin with respect to x , then:

(i) for any irreducible T -module W with $E_1^*W \neq 0$,

$$LE_i^*W = 0 \Rightarrow E_i^*W = 0 \quad 2 \leq i \leq d; \quad (40)$$

(ii) for $w \in TE_1^*V$,

$$LE_i^*w = 0 \Rightarrow E_i^*w = 0 \quad 2 \leq i \leq d; \quad (41)$$

(iii) for $B \in TE_1^*$,

$$LE_i^*B = 0 \Rightarrow E_i^*B = 0 \quad 2 \leq i \leq d. \quad (42)$$

Proof of (i): Let W be given. Fix an integer i with $2 \leq i \leq d$, and suppose $LE_i^*W = 0$. Let W' denote the subspace of W defined by

$$W' = E_i^*W + \cdots + E_d^*W. \quad (43)$$

Observe by (36)–(38) and (13) that W' is closed under multiplication by L , F , R , and $E_0^*, \dots, E_d^*$. Since T is generated by these matrices, W' is a T -module. Since $E_1^*W' = 0$ and $E_1^*W \neq 0$, W' is a proper submodule of W . Since W is irreducible, we now have $W' = 0$, and $E_i^*W \subseteq W'$ is zero as desired.

Proof of (ii): Since V may be decomposed into a direct sum of irreducible T -modules, it suffices to show that the result holds for $w \in TE_1^*W$ where W is an irreducible T -module. Fix an integer i with $2 \leq i \leq d$ and an irreducible T -module W , and suppose $w \in TE_1^*W$ has $LE_i^*w = 0$.

Suppose $E_i^*w \neq 0$. Observe $E_1^*W \neq 0$, since $0 \neq E_i^*w \in E_i^*TE_1^*W$. Since Y is l -thin with respect to x , W is thin and $\dim E_i^*W \leq 1$. In particular, $E_i^*w \in E_i^*W$ spans E_i^*W , and $LE_i^*W = 0$. By (i) we have $E_i^*W = 0$, and $E_i^*w = 0$ for a contradiction. Thus $E_i^*w = 0$ as desired.

Proof of (iii): Immediate from (ii). □

Lemma 3.3 *Let Y denote a d -class symmetric association scheme, with vertex set X . Suppose $A_0, \dots, A_d$ is a P -polynomial structure for Y , with intersection numbers p_{ij}^h . Fix a vertex $x \in X$, and write $L = L(x)$ and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Then:*

(i) for $1 \leq i \leq d-1$,

$$LE_i^*A_{i+1}E_1^* = p_{1,i+1}^i E_{i-1}^*A_iE_1^*; \quad (44)$$

(ii) for $1 \leq i \leq d$, if $p_{1,i-1}^{i-1} = 0$ then

$$LE_i^*A_iE_1^* = p_{1i}^i E_{i-1}^*A_iE_1^*. \quad (45)$$

Proof of (i): Let i be given. Observe by (22), (25), (33), (21), and (11) that

$$LE_i^* A_{i+1} E_1^* = E_{i-1}^* A E_i^* A_{i+1} E_1^* \quad (46)$$

$$= E_{i-1}^* A \left(\sum_{h=0}^d E_h^* \right) A_{i+1} E_1^* \quad (47)$$

$$= E_{i-1}^* A A_{i+1} E_1^* \quad (48)$$

$$= E_{i-1}^* \left(\sum_{h=0}^d p_{1,i+1}^h A_h \right) E_1^* \quad (49)$$

$$= p_{1,i+1}^i E_{i-1}^* A_i E_1^*, \quad (50)$$

as desired.

Proof of (ii): Let i be given, with $p_{1,i-1}^{i-1} = 0$. Observe as in (i) that

$$LE_i^* A_i E_1^* = E_{i-1}^* A E_i^* A_i E_1^* \quad (51)$$

$$= E_{i-1}^* A \left(\sum_{h=0}^d E_h^* \right) A_i E_1^* \quad (52)$$

$$= E_{i-1}^* A A_i E_1^* \quad (53)$$

$$= E_{i-1}^* \left(\sum_{h=0}^d p_{1i}^h A_h \right) E_1^* \quad (54)$$

$$= p_{1i}^i E_{i-1}^* A_i E_1^*, \quad (55)$$

as desired. $\square$

Proof of Theorem 2.1: Suppose Y is l -thin with respect to x , and write $L = L(x)$ and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Suppose $p_{11}^1 = 0$, and suppose for a contradiction that $p_{1i}^i \neq 0$ for some i with $2 \leq i \leq d-1$. Fix $i \geq 2$ minimal with $p_{1i}^i \neq 0$. Then by Lemma 3.3,

$$0 = L(p_{1i}^i E_i^* A_{i+1} E_1^* - p_{1,i+1}^i E_i^* A_i E_1^*), \quad (56)$$

and by Lemma 3.2(iii),

$$0 = p_{1i}^i E_i^* A_{i+1} E_1^* - p_{1,i+1}^i E_i^* A_i E_1^*. \quad (57)$$

The summands in (57) are nonzero by (33) and orthogonal by (32), for a contradiction. Thus $p_{1i}^i = 0$ for $2 \leq i \leq d-1$, as desired. $\square$

4. Proof of Theorem 2.2

Our proof is based upon the following result:

Lemma 4.1 (Cameron, Goethals, Seidel [2]) *Let Y denote a d -class symmetric association scheme, with vertex set X , primitive idempotents $E_0, \dots, E_d$, and Krein parameters q_{ij}^h . Fix a vertex $x \in X$, and write $A_i^* = A_i^*(x)$ for $0 \leq i \leq d$. Then:*

(i) *for $0 \leq h, h', i, i', j, j' \leq d$,*

$$\langle E_i A_h^* E_j, E_{i'} A_{h'}^* E_{j'} \rangle = \delta_{hh'} \delta_{ii'} \delta_{jj'} k_h^* q_{ij}^h; \quad (58)$$

(ii) *for $0 \leq h, i, j \leq d$,*

$$E_h A_i^* E_j = 0 \Leftrightarrow q_{ij}^h = 0. \quad (59)$$

Proof of (i): Recall $\text{tr}(BC) = \text{tr}(CB)$, and observe by (15), (13), (18), (16), and (17) that

$$\langle E_i A_h^* E_j, E_{i'} A_{h'}^* E_{j'} \rangle = \text{tr}(E_j A_h^* E_i E_{i'} A_{h'}^* E_{j'}) \quad (60)$$

$$= \text{tr}(E_{j'} E_j A_h^* E_i E_{i'} A_{h'}^*) \quad (61)$$

$$= \delta_{ii'} \delta_{jj'} \text{tr}(E_j A_h^* E_i A_{h'}^*) \quad (62)$$

$$= \delta_{ii'} \delta_{jj'} \sum_{y, z \in X} (E_j)_{yz} (A_h^*)_{zz} (E_i)_{zy} (A_{h'}^*)_{yy} \quad (63)$$

$$= \delta_{ii'} \delta_{jj'} |X|^2 \sum_{y, z \in X} (E_j)_{yz} (E_h)_{xz} (E_i)_{zy} (E_{h'})_{xy} \quad (64)$$

$$= \delta_{ii'} \delta_{jj'} |X|^2 \sum_{y \in X} ((E_i \circ E_j) E_h)_{yx} (E_{h'})_{xy} \quad (65)$$

$$= \delta_{ii'} \delta_{jj'} |X| q_{ij}^h \sum_{y \in X} (E_h)_{yx} (E_{h'})_{xy} \quad (66)$$

$$= \delta_{ii'} \delta_{jj'} |X| q_{ij}^h (E_{h'} E_h)_{xx} \quad (67)$$

$$= \delta_{hh'} \delta_{ii'} \delta_{jj'} |X| q_{ij}^h (E_h)_{xx} \quad (68)$$

$$= \delta_{hh'} \delta_{ii'} \delta_{jj'} k_h^* q_{ij}^h, \quad (69)$$

as desired.

Proof of (ii): Immediate from (i). □

Let Y denote a d -class symmetric association scheme, with vertex set X . Suppose $E_0, \dots, E_d$ is a Q-polynomial structure for Y , with Krein parameters q_{ij}^h . Fix a vertex $x \in X$, and write $T = T(x)$, $M^* = M^*(x)$, and $A_i^* = A_i^*(x)$ for $0 \leq i \leq d$.

The dual lowering matrix $L^* = L^*(x)$, the dual flat matrix $F^* = F^*(x)$, and the dual raising matrix $R^* = R^*(x)$ are defined by

$$L^* = \sum_{i=1}^d E_{i-1} A_i^* E_i, \quad (70)$$

$$F^* = \sum_{i=0}^d E_i A_1^* E_i, \quad (71)$$

$$R^* = \sum_{i=0}^{d-1} E_{i+1} A_1^* E_i. \quad (72)$$

It is easily shown using (27), (12), and (59) that

$$A_1^* = L^* + F^* + R^*. \quad (73)$$

Recall that A_1^* generates the dual Bose-Mesner algebra M^* , so that A_1^* and $E_0, \dots, E_d$ generate T . In particular, L^*, F^*, R^* , and $E_0^*, \dots, E_d^*$ generate T by (73).

Lemma 4.2 *Let Y denote a d -class symmetric association scheme, with vertex set X . Suppose $E_0, \dots, E_d$ is a Q -polynomial structure for Y , with Krein parameters q_{ij}^h . Fix a vertex $x \in X$, and write $T = T(x)$, $L^* = L^*(x)$, and $A_i^* = A_i^*(x)$ for $0 \leq i \leq d$. If Y is dual 1-thin with respect to x , then:*

(i) *for any irreducible T -module W with $E_1 W \neq 0$,*

$$L^* E_i W = 0 \Rightarrow E_i W = 0 \quad 2 \leq i \leq d; \quad (74)$$

(ii) *for $w \in T E_1 V$,*

$$L^* E_i w = 0 \Rightarrow E_i w = 0 \quad 2 \leq i \leq d; \quad (75)$$

(iii) *for $B \in T E_1$,*

$$L^* E_i B = 0 \Rightarrow E_i B = 0 \quad 2 \leq i \leq d. \quad (76)$$

Proof: Similar to the proof of Lemma 3.2. □

Lemma 4.3 *Let Y denote a d -class symmetric association scheme, with vertex set X . Suppose $E_0, \dots, E_d$ is a Q -polynomial structure for Y , with Krein parameters q_{ij}^h . Fix a vertex $x \in X$, and write $L^* = L^*(x)$ and $A_i^* = A_i^*(x)$ for $0 \leq i \leq d$. Then:*

(i) *for $1 \leq i \leq d$,*

$$L^* E_i A_{i+1}^* E_1 = q_{1,i+1}^i E_{i-1} A_i^* E_1; \quad (77)$$

(ii) *for $1 \leq i \leq d$, if $q_{1,i-1}^{i-1} = 0$ then*

$$L^* E_i A_i^* E_1 = q_{1i}^i E_{i-1} A_i^* E_1. \quad (78)$$

Proof: Similar to the proof of Lemma 3.3. □

Proof of Theorem 2.2: Similar to the proof of Theorem 2.1. □

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